Zbl 013.39003
Erdős, Paul
On the representation of an integer as the sum of k k-th powers. (In English)
J. London Math. Soc. 11, 133-136 (1936).

Let $f(m)$ denote the number of representations of m as the sum of k k-th powers of non-negative integers. The well-known "Hypothesis K" of *Hardy* and *Littlewood* [Math. Z. 23, 1-37 (1925)] is a conjecture that $f(m) = O(m^\varepsilon)$ for every $\varepsilon > 0$. The author proves a result in the opposite direction, namely $f(m) > \exp\left\{\frac{c1 \log m}{\log\log m}\right\}$ for an infinity of m, where c_1 is a positive number depending only on k. Of course, this does not disprove Hypothesis K. The method of proof is slightly different for odd and even k. Let us suppose that k is odd and take $p_1, ..., p_r$ to be consecutive primes greater than k for which $(p-1, k) = 1$. Let $A = p_1 ... p_k$, $n = A^k$, B/A, $A = BC$. Let S_B denote the number of solutions of $x_i \le n$, $x_i \equiv 0 \pmod B$, $x_1^k + \cdots + x_k^k \equiv 0 \pmod n$, $(x_1^k + \cdots + x_{k-1}^k, C) = 1$ in non-negative integers $x_1, ..., x_k$. Then $S_B > \frac{c_2^{n^{k-1}}}{\log p_r}$ and the number of solutions of $x_i \le n$, $x_1^k + \cdots + x_k^k \equiv 0 \pmod n$ is at least $\frac{c_3 2^r n^{k-1}}{\log p_r}$. Hence there is an $m \le kn^k$ which is a multiple of n and for which $f(m) \ge \frac{c_3 2^r}{k \log p}$. Since $r > \frac{c_4 \log n}{\log\log n}$, the result follows. — The proof for even k depends on the lemma: If C is a product of different primes, each of which satisfies $p + k$, $p \equiv 3 \pmod 4$, $(p-1, k) = 2$, then the number of solutions of $x^k + y^k \equiv a \pmod{C^k}$, where $(a, C) = 1$, is $C^k \prod_{p/c}(1 + p^{-1})$.
The author states that his method enables him to prove that, if $a_1, a_2, ...$ are integers and $\frac{1}{k_1} + \cdots + \frac{1}{k_l} = 1$, then there is an infinity of m with more than $\exp\left\{\frac{c \log m}{\log\log m}\right\}$ representation in the form

$$a_1 x_1^k + \cdots + a_l x_l^k \qquad (x_i \ge 0).$$

Wright (Aberdeen)

Classification:
11P05 Waring's problem and variants
11P55 Appl. of the Hardy-Littlewood method
11D85 Representation problems of integers