Zbl 079.07401
Erdős, Pál
On the irrationality of certain series. (In English)
Nederl. Akad. Wet., Proc., Ser. A 60, 212-219 (1957).

The author shows that the series

$$\sum_{u=1}^{\infty} t^{-\varphi(n)} \text{ and } \sum_{n=1}^{\infty} t^{-\sigma(n)}$$

($t = 2, 3, 4, ...$) are irrational; here φ and σ are Euler's function, and the sum of divisors. The proof depends on a general Lemma 1 on irrational series, and on these properties: (1) There are only $O(x)$ integers n satisfying $\varphi(n) \leq x$ (or $\sigma(n) \leq x$). (2) There are only $o(x)$ integers $n \leq x$ for which $\varphi(k) = n$ (or $\sigma(k) = n$) has a solution k.

Lemma 1 is obtained as a special case of the more general Lemma 4: Let $\{a_k\}$ and $\{b_k\}$ be two infinite sequences of integers ≥ 0 such that $a_k \leq k^s$ and $b_k \leq k^s$ (s a constant). Let $f(n)$ and $g(n)$ denote the number of positive a_k and b_k with $1 \leq k \leq n$; assume that $f(n) \to \infty$ as $n \to \infty$. Let there exist an infinite sequence of integers $\{m_i\}$ such that

$$\sum_{k=1}^{m_i}(a_k + b_k) < c_1 m_i, \qquad f(m_i) = o(m_i), \qquad g(m_i) = o\left(\frac{m_i}{\log m_i}\right).$$

Finally assume there exists a constant $c > 0$ as follows: When i_1 and i_2 are consecutive suffixes with $b_{i_1} b_{i_2} > 0$, and when $i_1 + cx < i_2$, then there is a k with $i_1 + x < k < i_2 + cx$ such that $a_k > 0$. Under these conditions all series

$$\sum_{k=1}^{\infty} \frac{a_k \mp b_k}{t^k} \qquad (t = 2, 3, 4, ...)$$

are irrational.

From Lemma 4, the author deduces Theorem 2: Let $\{n_k\}$ be a strictly increasing sequence of positive integers such that $\limsup_{n\to\infty} \frac{n_k}{k^l} = \infty$ ($l > 0$ a constant). If $t \geq 2$ is an integer, then the series $\sum_{k=1}^{\infty} t^{-n_k}$ cannot have as its sum an algebraic number of degree ≤ 1.

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Classification:
11J72 Irrationality