Zbl 114.01301

Czipszer, J.; Erdős, Pál; Hajnal, András

Some extremal problems on infinite graphs (In English)

Publ. Math. Inst. Hung. Acad. Sci., Ser. A 7, 441-457 (1962).

Let the vertices of the infinite graph $G^{(\infty)}$ be the integers $1, 2, ..., n....$ Let $G^{(n)}$ be the subgraph of $G^{(\infty)}$ consisting of the vertices $1, 2, ..., n$ and the edges joining them, and let $g(n)$ denote the number of edges in $G^{(n)}$. If $G^{(n)}$ contains k vertices such that (i_j, i_{j+1}) is an edge for $j = 1, 2, ..., k$, then we say $G^{(k)}$ contains an I_k path. An I_∞ path may be defined in similar manner. The authors proved in this paper the following theorems:
Theorem I: Let $G^{(\infty)}$ be a graph and assume for all $n > n_0$ an $\varepsilon > 0$ $g(n) > (\frac{1}{4} - \frac{1}{4}k^{-1} + \varepsilon)n^2$ where $k = 2$ or $k = 3$. Then $G^{(\infty)}$ contains infinitely many I_k-paths.
Theorem II. Let $G^{(\infty)}$ be a graph for which

$$g(n) > \frac{1}{8}n^2 + \left(\frac{1}{32} + \varepsilon\right) n^2/\log^2 n \text{ if } n > n_0.$$

Then $G^{(\infty)}$ contains infinitely many I_2-paths. The result is the best possible since there exists a $G^{(\infty)}$ for which $g(n) > \frac{1}{8}n^2 + \frac{1}{32}n^2/\log^2 n + o(n^2/\log^2 n)$ and which does not contain any I_2-path.
Theorem III. Let $G^{(\infty)}$ be such that $g(n) \geq \frac{1}{4}n^2 - Cn$. Then $G^{(\infty)}$ contains an infinite path. This result is best possible in the sense that C can not be replaced by $A(n)$ where $A(n) \to \infty$.
Theorem IV. There exists a $G^{(\infty)}$ with $\liminf[g(n)/n^2] > \frac{1}{4}$ which does not contain an I_∞-path. But there exists a constant $\alpha > 0$ such that every $G^{(\infty)}$ with $\liminf[g(n)/n^2] > \frac{1}{2} - \alpha$ contains an I_∞-path. Theorem V. If $g(n) > \frac{1}{2}n^2 - Cn$ for infinitely many n then $G^{(\infty)}$ contains an infinite complete subgraph. But if we only assume that $g(n) > \frac{1}{2}n^2 - f(n)n$ for all n where $f(n)$ tends to infinity as slowly as we please then $G^{(\infty)}$ does not have to contain an infinite complete graph.

J.Dénes

Classification:
05C35 Extremal problems (graph theory)