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Chain conditions on set mappings and free sets. (In English)

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Given an infinite set E, a function f mapping E into $\mathcal{P}(E)$, the set of all subsets of E, is called a set mapping if $x \notin f(x)$ holds for any $x \in E$. A subset X of E is called free (with respect to f) if $X \cap f(x) = 0$ holds for any $x \in X$. *A. Hajnal* [Fundam. Math. 50, 123-128 (1961; Zbl 100.28003)] showed that if $|f(x)| < \mu < |E|$ ($|A|$ denotes the cardinality of A) holds with some cardinal μ for any $x \in E$, then there is a free set of cardinality $|E|$. The aim of the present paper is to weaken the assumptions in Hajnal's theorem. To this end, say that a set S satisfies the η-chain condition for some ordinal η if there is no sequence $\langle s_\alpha : \alpha < \eta \rangle$ of elements of S such that $s_\alpha \subset s_\beta$ whenever $\alpha < \beta < \eta$ ($\subset$ means strict inclusion here). Consider the following conditions imposed on f : $|E| = \kappa$ is a regular cardinal, $|f(x)| < \kappa$ for any $x \in E$, and, for any $\tau < \kappa$ and any decomposition $E = \bigcup_{\alpha<\tau} E_\alpha$ of E into pairwise disjoint sets E_α of cardinality κ, there is an ordinal $\gamma < \tau$ and a set $F \subseteq E_\gamma$ of cardinality κ such that the set $\{f(x) \cap F : x \in E\}$ satisfies the κ-chain condition. Under these assumptions it is proved by a tree argument that (i) there exists an infinite free set, (ii) if μ is a cardinal $< \kappa$ such that for every $\nu < \kappa$ we have $\nu^\mu < \kappa$, then there exists a free set of cardinality μ, and (iii) if κ is inaccessible and weakly compact, then there exists a free set of cardinality κ. (iv) If there is a κ-Souslin tree, or if (v) $\kappa = 2^\lambda = \lambda^+$, then it is shown that the above conditions do not imply the existence of a free set of cardinality κ. Several stronger negative results are announced without proof.

Classification:
04A20 Combinatorial set theory
03E35 Consistency and independence results (set theory)
03E15 Descriptive set theory (logic)
03E55 Large cardinals