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Erdős, Paul; Hall, R.R.; Tenenbaum, G.
On the densities of sets of multiples. (In English)
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Let A denote a strictly increasing sequence of integers greater than 1, and let $M(A) = \{ma : m \geq 1, a \in A\}$. The authors call A a Besicovitch sequence if $M(A)$ has an asymptotic density; if this density equals 1, then A is a Behrend sequence. It was shown by Besicovitch in 1934 that there are sequences A for which $M(A)$ does not have a density. In 1948, Erdős obtained a criterion for A to be a Besicovitch sequence, and a short proof of his result is included in this paper.

The authors prove several theorems concerning Besicovitch sequences. For example, Theorem 3 states that $A = \{a_1, a_2, \dots\}$ is a Besicovitch sequence if, for some fixed positive integer k, every $\gcd(a_i, a_j)$, $i \neq j$, has at most k distinct prime factors.

Let $\tau(n, A)$ denote the number of divisors of n belonging to A, so $M(A) = \{n : \tau(n, A) > 0\}$, and let $A^{(k)}$ denote the k-th derived sequence of A, so $M(A^{(k)}) = \{n : \tau(n, A) > k\}$. The remaining theorems provide quantitative forms of the result that $\tau(n, A) \to \infty$ p.p. whenever A is Behrend, and these are stated in terms of the logarithmic density $t_k(A)$ of $\{n : \tau(n, A) \leq k\}$. For example, the authors prove in Theorem 5 that

$$\inf\{t_0(A) : |A| \leq k\} = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)$$

where p_j denotes the j-th prime.

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11B75 Combinatorial number theory

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