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METRIC DISTORTION AND TRIANGLE MAPS

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Abstract. Suppose T is a geodesic triangle with respect to the spherical or the hyperbolic metric. Let $f\colon T \to E$ be a triangle map onto a euclidean triangle E, and assume that the angle measures of two of the vertices are preserved by f. We prove that the metric distortion of f is extremal on the side of T opposite the angle whose measure is not preserved. As applications we determine the minimum points of the hyperbolic and spherical densities on some symmetric regions.

1. Introduction

We identify the Riemann sphere with $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, and we consider two non-degenerate circular-arc triangles T and E. We think of T and E as closed subsets of the Riemann sphere. A *triangle map* $f\colon T \to E$ is a homeomorphism f from T onto E which is a conformal map from the interior of T onto the interior of E and maps the vertices of T onto the vertices of E. We will assume that T is a geodesic triangle, either on the Riemann sphere equipped with the spherical metric or in the unit disc equipped with the hyperbolic metric. According to these two cases we call T a spherical or a hyperbolic triangle. The triangle E will always be a euclidean triangle, meaning the edges of E will be straight line segments in the complex plane.

The *metric distortion* associated to a triangle map $f\colon T \to E$ at $z \in T$ is the ratio of the euclidean length element $|d\zeta|$ at $\zeta = f(z)$ to the spherical length element $2|dz|/(1+|z|^2)$ at z, or to the hyperbolic length element $2|dz|/(1-|z|^2)$ at z, depending on whether T is a spherical or hyperbolic triangle. The main result of this paper is the following theorem on metric distortion.

Theorem 1.1. *Suppose $f\colon T \to E$ is a triangle map of a spherical* (*respectively hyperbolic*) *triangle T onto a euclidean triangle E such that f preserves the angle measure at two vertices of T. Any point where the metric distortion of f attains its minimum* (*respectively its maximum*) *over T lies on the side of T that is opposite the angle whose measure is changed by f.*

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The sum of the angle measures of the vertices of a triangle is larger than π, equal to π, or less than π according to whether the triangle is spherical, euclidean, or hyperbolic, respectively. Under the assumptions of the theorem, the triangle map f preserves the angle measure at two vertices of T, which we will call A and B, say. It follows that the angle measure of the third vertex of T, which we will call C, is decreased by f if T is spherical and increased if T is hyperbolic. At C the metric distortion of f becomes singular, *i.e.*, it tends to infinity as we approach C if T is spherical and vanishes at C if T is hyperbolic. One might then expect that the other extreme would occur on the edge of T opposite C, which is exactly what Theorem 1.1 says.

In general, there is no obvious point on the side opposite C where the extremum of the metric distortion is attained. In special cases, however, the location of the extremum can be precisely determined (cf. Corollary 2.9).

Using well-known expressions for the triangle functions considered here in terms of hypergeometric functions, the metric distortion can be written down more or less explicitly. Thus, at first sight, the reduction of the statement of Theorem 1.1 to a straightforward calculus problem seems plausible. However, because of the highly involved nature of these explicit formulas, this approach seems not to be feasible, and so we take a different approach.

Rather than attempting to use explicit formulas, we use the differential equation for triangle maps together with geometric arguments to prove our theorem. The proof runs along the following lines. We consider the function on T which is the logarithm of the ratio of the two length elements. This is a real-valued function, so to find its critical points, we set its z-derivative equal to zero, where z is a holomorphic coordinate on the geodesic triangle T. We get an equation involving z and $\bar{z}$. This equation can be solved for $\bar{z}$ in terms of a function H that is meromorphic in T. Surprisingly, the function H has remarkable mapping properties. Namely, the image of T under H is also a circular-arc triangle. In fact, the image of T under $\overline{H}$, the mapping H post-composed by complex conjugation, is a circular-arc triangle T' which is complementary to T in a sense we will precisely describe before the statement of Theorem 2.5. From these mapping properties of $\overline{H}$, Theorem 1.1 will follow as an easy corollary (cf. Corollary 2.8). The fact that H has the stated mapping property is the deepest and most striking result of the present paper. The proof, if appropriately organized, can be reduced to a direct computation checking a differential equation for H.

In Section 3 we give two applications of Theorem 1.1. Our first application is a new proof of a theorem of A. Baernstein II ([B], [BV]) and H. Montgomery [M], which says that if one considers the complex plane minus a symmetric lattice, then the ratio of the hyperbolic to the euclidean length elements is smallest at the natural points of symmetry that are farthest from the lattice points. Our second application is to determine sharp upper bounds on the spherical derivative at the origin of a map from the unit disc into a symmetric region in the Riemann sphere, as in our previous paper [BC]. In particular we are able to obtain precise upper bounds on the spherical derivative at the origin of a map from the unit disc into the Riemann sphere minus the vertices of a regular inscribed dodecahedron or icosahedron, two cases we were not able to handle in [BC].

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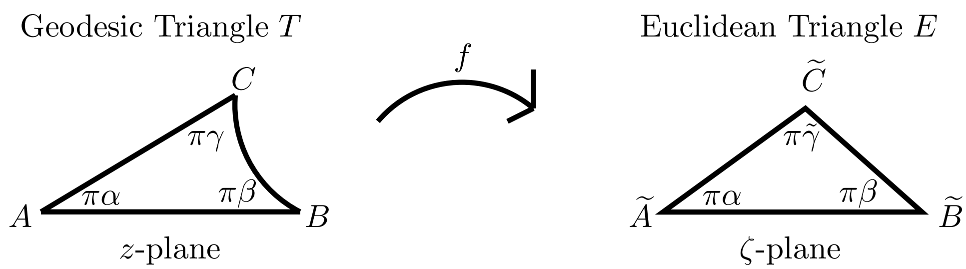

Figure 1.

2. Maps from geodesic triangles onto euclidean triangles

Suppose $f\colon T \to E$ is a triangle map. In this section we make the assumptions of Theorem 1.1: namely, that T is a hyperbolic or spherical triangle, that E is a euclidean triangle, and that f preserves the angle measure of two vertices. We continue to denote the vertices of T by A, B, and C, and we denote the corresponding angle measures by $\pi\alpha$, $\pi\beta$, and $\pi\gamma$. The vertices of E we will denote by $\tilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, with corresponding angle measures $\pi\tilde{\alpha}$, $\pi\tilde{\beta}$, and $\pi\tilde{\gamma}$. Under our assumptions on f, we may assume $f(A) = \tilde{A}$, $f(B) = \widetilde{B}, f(C) = \widetilde{C}$, $\alpha = \tilde{\alpha}$, and $\beta = \tilde{\beta}$. With this notation, C is the vertex of T whose angle measure is changed by f and so $\gamma \neq \tilde{\gamma}$. Since E is a euclidean triangle, we must also have that $\alpha + \beta < 1$. We let z denote a complex variable on the domain triangle T, and we let ζ denote a complex variable on the image triangle E. The length element in the domain of f is given by

$$\frac{2|dz|}{1 \pm |z|^2},$$

where here and henceforth, we make the following sign convention: *When we use the symbols $\pm$ and $\mp$, the upper sign corresponds to the spherical case and the lower sign to the hyperbolic case.*

The sense preserving isometries of the Riemann sphere with the spherical metric or the unit disc with the hyperbolic metric are given by the Möbius transformations

$$U\colon \tilde{z} \mapsto z = e^{i\theta}\frac{\tilde{z}-\tilde{z}_0}{1 \pm \overline{\tilde{z}}_0\tilde{z}}.$$

Here $\theta \in [0, 2\pi]$. Moreover, $|\tilde{z}_0| < 1$ in the hyperbolic case. In the spherical case, $\tilde{z}_0 \in \mathbf{C} \cup \{\infty\}$. If $\tilde{z}_0 = \infty$, the definition of U needs to be adjusted appropriately, *i.e.*, $U\colon \tilde{z} \mapsto -e^{i\theta}/\tilde{z}$. Similar adjustments have to be made in the following to include the case of infinity in formulas, and we will usually not mention this explicitly. The sense preserving isometries of the euclidean ζ-plane are the maps

$$V\colon \zeta \mapsto \tilde{\zeta} = e^{i\theta}\zeta + \zeta_0.$$

Here $\theta \in [0, 2\pi]$ and $\zeta_0 \in \mathbf{C}$.

We use I to denote the mapping

$$I(z) = \mp\frac{1}{\bar{z}}.$$

Note that I is the map to the antipodal point on the Riemann sphere in the spherical case or inversion through the unit circle in the hyperbolic case.

We want to find the extrema of the metric distortion by the map f. It is technically a little easier and amounts to the same to consider the logarithm of (twice) this quantity. Explicitly, it is given by

$$\mu(z) = \log(1 \pm |z|^2) + \tfrac{1}{2}\log\bigl(|f'(z)|^2\bigr).$$

Let x and y be the real and imaginary-parts of the holomorphic coordinate $z = x + iy$. If $\phi(z)$ is a function with values in $\overline{\mathbf{C}}$, we denote by $\bar{\phi}$ the function $z \mapsto \overline{\phi(z)}$, *i.e.*, the composition of complex conjugation and ϕ. For differentiable ϕ defined on an open subregion of the z-plane, we define

$$\frac{\partial\phi}{\partial z} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial\phi}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right).$$

We denote by ν the function defined by

$$\nu(z) = (1 \pm |z|^2)\frac{\partial\mu}{\partial z}(z).$$

Note that both μ and ν are smooth, *i.e.*, infinitely differentiable, in the interior of T.

The function ν is essentially $\partial\mu/\partial z$. The factor $(1 \pm |z|^2)$ ensures that ν has certain useful invariance properties that we will make precise momentarily. To solve our main extremal problem, we will want to find the critical points of μ. They are given by the zeros of ν.

To simplify the appearance of the formulas below, we will use the following short-hand notation:

$$D_z\phi = \frac{d}{dz}\log\left(\frac{d\phi}{dz}\right) = \frac{d^2\phi/dz^2}{d\phi/dz}.$$

The following function will play an important role in determining the critical points of μ:

$$H(z) = \mp\frac{D_z f(z)}{zD_z f(z) + 2}.$$

Note that the denominator in the fraction defining H is not identically zero and hence H is a meromorphic in the interior of T. Indeed, if the denominator were identically zero, we would get a differential equation for f. It would then be straightforward to show that f would have to be a Möbius transformation. But, Möbius transformations preserve angle measures, whereas f changes the angle measure at the vertex C of T. This would then be a contradiction.

Remark. The idea to consider this function H comes from the work of Ruscheweyh and Wirths [RW].

Solving for $D_z f$ in the definition of H, we get

$$D_z f(z) = \frac{\mp 2H(z)}{1 \pm zH(z)}. \tag{1}$$

The following proposition describes the invariance behavior of the functions μ and H.

Proposition 2.1. *Let the notation be as above. Consider the function*

$$\tilde{f} = V \circ f \circ U, \quad \tilde{z} \mapsto \tilde{\zeta} = V \circ f \circ U(\tilde{z}),$$

and let $\tilde{\mu}$, $\tilde{\nu}$, and $\widetilde{H}$ be defined as μ, ν, and H, respectively, using $\tilde{f}$ instead of f. Then

(i) $\tilde{\mu} = \mu \circ U$,
(ii) $\tilde{\nu} = \nu \circ U$,
(iii) $U \circ I = I \circ U$,
(iv) $\overline{\widetilde{H}} = U^{-1} \circ \overline{H} \circ U$.

Proof. Equation (i) follows from the invariant definition of μ as the logarithm of (twice) the ratio of the infinitesimal length elements and the fact that U and V are isometries in the appropriate geometries.

Equation (ii) follows from the chain rule and the equation

$$(1 \pm |\tilde{z}|^2)\left|\frac{dU}{d\tilde{z}}(\tilde{z})\right| = 1 \pm |U(\tilde{z})|^2.$$

In the spherical case equation (iii) follows from the fact that rotations map antipodal points to antipodal points. Similarly, suppose two points are mapped to each other by inversion with respect to the unit circle. Then the image points of these points under a Möbius transformation preserving the unit disc have the same property.

To prove (iv), first note that

$$D_{\tilde{z}}\tilde{f} = (D_z f \circ U) \cdot \frac{dU}{d\tilde{z}} + D_{\tilde{z}}U. \tag{2}$$

Moreover, the following identity is true for $a, b \in \overline{\mathbf{C}}$, $a \neq I(b)$,

$$\frac{\mp 2\overline{U(b)}}{1 \pm U(a)\overline{U(b)}} \cdot \frac{dU}{d\tilde{z}}(a) + D_{\tilde{z}}U(a) = \frac{\mp 2\bar{b}}{1 \pm a\bar{b}}. \tag{3}$$

Using (2), (1), and (3) with $a = \tilde{z}$ and $b = U^{-1} \circ \overline{H} \circ U(\tilde{z})$, we obtain

$$D_{\tilde{z}}\tilde{f}(\tilde{z}) = \frac{\mp 2\big(H \circ U(\tilde{z})\big)}{1 \pm U(\tilde{z})\big(H \circ U(\tilde{z})\big)} \cdot \frac{dU}{d\tilde{z}}(\tilde{z}) + D_{\tilde{z}}U(\tilde{z}) = \frac{\mp 2\big(\overline{U^{-1}} \circ \overline{H} \circ U(\tilde{z})\big)}{1 \pm \tilde{z}\big(\overline{U^{-1}} \circ \overline{H} \circ U(\tilde{z})\big)}.$$

Note that in this equation the denominator is not identically zero, since $\overline{H} \not\equiv I$. This equation together with equation (1) where z, f, and H are replaced by $\tilde{z}$, $\tilde{f}$, and $\widetilde{H}$, respectively, implies (iv). □

We now prove that the functions μ, ν, and H have extensions to the boundary of T.

Proposition 2.2. *Let the notation be as discussed above. Then*

(i) *μ extends continuously to T with values in $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$. The function μ can be ∞ or $-\infty$ only at the vertex C, where $\mu(C) = \infty$ in the spherical case and $\mu(C) = -\infty$ in the hyperbolic case.*

(ii) *If one of the vertices A or B of T has internal angle measure strictly less than $\frac{1}{2}\pi$, then this vertex is a local minimum for μ if T is spherical and a local maximum for μ if T is hyperbolic.*

(iii) *The function ν extends continuously to T with values in $\overline{\mathbf{C}}$. The only point where $\nu = \infty$ is the vertex C. We have $\nu = 0$ at the vertices A and B.*

Proof. By Schwarz's reflection principle, at all boundary points of T which are not vertices, f has a unique local extension as a conformal map. It is then clear that μ and ν extend smoothly to the boundary of T, except possibly at the vertices. Thus, we need only check what happens there. Let $\pi\delta$ denote the angle measure of one of the vertices of T (*i.e.* $\delta = \alpha, \beta$, or γ), and let $\pi\tilde{\delta}$ denote the angle measure of the image vertex (*i.e.* $\tilde{\delta} = \alpha, \beta$, or $\tilde{\gamma}$).

We first consider the case that $\delta \neq 0$. In other words, we exclude for the moment the case that T is hyperbolic with the vertex in question lying on the boundary of the unit circle.

By the invariance properties proved in Proposition 2.1, we can move this vertex to $z = 0$ by an isometry. Moreover, we can assume that one of the edges meeting at this vertex lies along the positive real-axis with the interior of the triangle lying to the left and that the other edge meeting at this vertex is a straight line segment. Post-composing f by an euclidean isometry, we can assume that the image vertex is also at the origin, that one of the sides of the euclidean triangle is along the positive real ζ-axis, and that the other side meeting the vertex at $\zeta = 0$ is a line segment in the upper half-plane. In this case, for $z \in T$ near the origin, f is of the form

$$f(z) = [h_1(z^{1/\delta})]^{\tilde{\delta}}.$$

Here and in the following, h_1, h_2, *etc.* will denote functions holomorphic at the origin. Here we have $h_1(0) = 0$, $h_1'(0) \neq 0$, and h_1 positive on the positive real axis. We will always choose branches of the power functions $u \mapsto u^\lambda$, $\lambda \in \mathbf{R}$, that are positive on the positive real axis. Then,

$$f'(z) = (\tilde{\delta}/\delta) h_1(z^{1/\delta})^{\tilde{\delta}-1} h_1'(z^{1/\delta}) z^{1/\delta - 1} = z^{\tilde{\delta}/\delta - 1} h_2(z^{1/\delta}), \quad h_2(0) \neq 0, \tag{4}$$

and

$$D_z f(z) = \frac{\tilde{\delta}/\delta - 1}{z} + \frac{1}{\delta} \cdot \frac{h_2'(z^{1/\delta})}{h_2(z^{1/\delta})} \cdot z^{1/\delta - 1} = \frac{1}{z} h_3(z^{1/\delta}), \quad h_3(0) = \tilde{\delta}/\delta - 1. \tag{5}$$

If $\tilde{\delta} = \delta$, then $\delta < 1$. Hence $D_z f(z) \to 0$ as $z \to 0$, $z \in T$. In this case $D_z f$ continuously extends to the origin with value 0 at this point. On the other hand, if $\tilde{\delta} \neq \delta$, then $D_z f(z) \to \infty$ as $z \to 0$, $z \in T$.

The above expansions for f and its derivatives imply the statements of the proposition. To see this note that if $\tilde{\delta} \neq \delta$, then near 0 we have

$$\mu(z) = (\tilde{\delta}/\delta - 1) \log |z| + O(1).$$

If $\tilde{\delta} = \delta$, we have

$$\mu(z) = \log |h_2(0)| + O(|z|^{\min\{2, 1/\delta\}}).$$

If in addition $\delta < \frac{1}{2}$, then

$$\mu(z) = \log|h_2(0)| \pm |z|^2 + o(|z|^2).$$

Statements (i) and (ii) follow.

Since

$$\nu(z) = \pm\bar{z} + \tfrac{1}{2}(1 \pm |z|^2)D_z f(z), \tag{6}$$

statement (iii) follows from the above remark about $D_z f$.

It remains to treat the case $\delta = 0$. In this case, T is hyperbolic, the vertex in question is C, and it lies on the boundary of the unit circle. Without loss of generality (again cf. Proposition 2.1), we may assume the interior of the triangle is contained in the upper half plane, the vertex C is located at $z = 1$, and one of the sides meeting at C lies along the positive real-axis. We can also translate and rotate the target triangle E so that the image vertex lies at the origin and so that the image of the side of T along the positive real-axis is mapped into the positive real-axis. Note that since T is in the upper half-plane, E will be in the lower half-plane. With this setup, the function f near $z = 1$ can be written as

$$f(z) = h_4\big(e^{\eta(z+1)/(z-1)}\big)^{\tilde{\delta}}$$

with $h_4(0) = 0$, $h_4'(0) \neq 0$, provided we choose η as a positive real number so that the function $z \mapsto \eta(z+1)(z-1)^{-1}$ maps the angle of T at $z = C$ to a strip of width π. Computation yields the following expansions for $z \in T$ near the point 1:

$$f'(z) = \frac{1}{(z-1)^2} e^{\tilde{\delta}\eta(z+1)/(z-1)} h_5\big(e^{\eta(z+1)/(z-1)}\big), \qquad h_5(0) \neq 0,$$

and

$$D_z f(z) = -\frac{2}{z-1} + \frac{1}{(z-1)^2} h_6\big(e^{\eta(z+1)/(z-1)}\big), \qquad h_6(0) = -2\tilde{\delta}\eta \neq 0.$$

Noting that $(1-|z|)/|z-1| \to 1$ as $z \to 1$, $z \in T$, the assertions easily follow from these expressions. We leave the details to the reader. □

Proposition 2.3. *Let the notation be as discussed above. Then, the function H extends continuously to T with values in $\overline{\mathbf{C}}$. Moreover,*

$$\overline{H}(A) = A, \qquad \overline{H}(B) = B, \qquad \text{and} \qquad \overline{H}(C) = I(C). \tag{7}$$

Proof. The Schwarz reflection principle and formula (1) imply that H extends meromorphicly and hence continuously to the boundary points of T different from the vertices. To see what happens at the vertices we can use the invariance property of H and the invariant nature of the equations in (7) to reduce to the same normalizations for T and E as in the proof of Proposition 2.2. Using the same notation as in that proof, we can assume that the vertex is located at the origin if $\delta \neq 0$. If in addition $\tilde{\delta} = \delta$, then $D_z f(z) \to 0$ as $z \to 0$, $z \in T$. This together with the definition of H implies that H has a continuous extension to the origin with $H(0) = 0$.

If $\delta \neq 0$ and $\tilde{\delta} \neq \delta$, then it follows from (5) that H has a continuous extension to the origin if we set $H(0) = \infty = I(0)$.

Finally, if $\delta = 0$, we may assume that the vertex (*i.e.* C) is located at $z = 1$. The expansion for $D_z f$ in the second part of the proof of Proposition 2.2 shows that H extends continuously to C if we put $H(1) = 1 = I(1)$. □

We will be interested in the critical points of μ, and it turns out they occur if and only if $\bar{z} = H(z)$.

Proposition 2.4. *Let the notation be as discussed above. Suppose T is spherical (respectively hyperbolic), and $z_0 \in T$ is a point where μ has a local minimum (respectively maximum). Then $\nu(z_0) = 0$. Moreover, for all $z \in T \setminus \{C\}$, we have $\nu(z) = 0$ if and only if $\bar{z} = H(z)$.*

Proof. By the invariance properties of ν and μ, we may assume $z_0 \neq \infty$. If z_0 is in the interior of T, then z_0 is a critical point of μ, and so $\nu(z_0) = 0$. By Schwarz's reflection principle, this is also true for points on the edges of T which are not vertices. Finally, z_0 cannot be the vertex C, and we already know $\nu(A) = \nu(B) = 0$ by statement (iii) in Proposition 2.2.

To prove that $\nu(z) = 0$ if and only if $\bar{z} = H(z)$, note that for $z \neq C, \infty$, we have $\nu(z) = 0$ if and only if $\partial\mu/\partial z = 0$. By direct computation,

$$\frac{\partial \mu}{\partial z}(z) = \frac{\pm \bar{z}}{1 \pm z\bar{z}} + \frac{1}{2} D_z f(z).$$

Setting this equal to zero and solving for $\bar{z}$, we get

$$\bar{z} = \mp \frac{D_z f(z)}{z D_z f(z) + 2}.$$

The expression on the right is just the definition of H, and so we are done in the case that $z \neq \infty$. The case $z = \infty$ follows from the invariance properties of H and ν. □

Before coming to the main result of this section, we introduce one last idea. Given a distinguished side of the triangle T, which in our case will be the side connecting the vertices A and B, we define the triangle T' *complementary* to T with respect to the distinguished side AB as follows. Each side of T is a circular-arc on the Riemann sphere, and can thus be extended to form a complete circle on the Riemann sphere. If we so extend the two sides of T other than our distinguished side connecting A to B, *i.e.*, the two sides meeting at C, then we obtain two complete circles. They determine a unique subregion of the Riemann sphere which contains T and is bounded by subarcs of these circles. This region is the union of T and another circular-arc triangle. We define this second triangle to be the complementary triangle T'. Note that the distinguished side connecting A and B is a common side of both triangles T and T'. Hence T' shares the vertices A and B with T, and T' has angle measures $\pi(1-\alpha)$ and $\pi(1-\beta)$ at the vertices A and B, respectively. From the fact that T is a geodesic triangle, it follows that the third vertex of T' is $I(C)$, and the angle measure of T' at $I(C)$ is $\pi\gamma$. Note that the relation of T' to be the complementary triangle of T is invariant under Möbius transformations. Figure 2 shows a circular-arc triangle T and its complementary triangle T'. To simplify the drawing, the triangles in this figure are not geodesic.

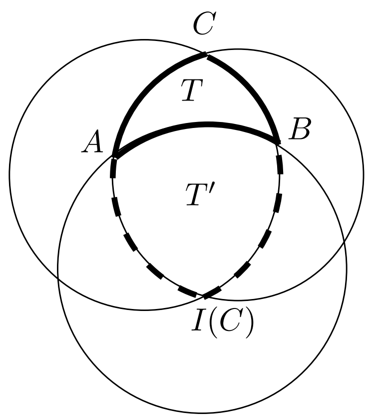

Figure 2.

An *anti-conformal triangle map* of a circular-arc triangle T_1 onto a circular-arc triangle T_2 is a homeomorphism from T_1 onto T_2 which is an anti-conformal map of the interior of T_1 onto the interior of T_2 and which maps the vertices of T_1 onto the vertices of T_2.

We now state the main result of this section, which describes the mapping properties of the meromorphic function H.

Theorem 2.5. *Under the assumptions of Theorem* 1.1, *the function* $\overline{H}$ *is an anti-conformal triangle map of* T *onto the complementary triangle* T' *of* T *with respect to the side of* T *connecting the two vertices whose angle measures are preserved by the given triangle map* f, *i.e. the side connecting* A *and* B. *The correspondence of the vertices of* T *and* T' *is given by* $\overline{H}(A)=A$, $\overline{H}(B)=B$, $\overline{H}(C)=I(C)$.

Note that we have already proven the statement about the vertex correspondence in Proposition 2.3. Before giving the rest of the proof of Theorem 2.5, we outline our strategy. We begin by recalling that we can detect functions which map conformally onto the interior of circular-arc triangles by examining their Schwarzian derivatives. If G is a holomorphic function of a complex variable t, then we define the *Schwarzian derivative* of G with respect to t, denoted $\{G, t\}$, by

$$\{G,t\} = \frac{d}{dt} D_t G - \frac{1}{2}\big(D_t G\big)^2.$$

Schwarzian derivatives are useful because they remain invariant if G is post-composed by an arbitrary Möbius transformation. We recall the following well-known facts, the second of which is an implication of the Schwarz–Christoffel formula; see for example [Ne] or [Ca] for details.

Theorem 2.6. *Let t be a complex variable. A function G meromorphic in the upper half-plane $\operatorname{Im} t > 0$ maps the upper half-plane conformally onto the interior of a circular-arc triangle with internal angle measurements πa, πb, and πc, and maps the real-axis onto the boundary such that $0, 1$ and ∞ map to the vertices of measure $\pi a, \pi b$, and πc respectively if and only if*

$$\{G,t\} = \frac{1-a^2}{2t^2} + \frac{1-b^2}{2(t-1)^2} + \frac{a^2+b^2-c^2-1}{2t(t-1)}.$$

Moreover, if the target is a euclidean triangle, then

$$D_t G(t) = \frac{a-1}{t} + \frac{b-1}{t-1}.$$

The strategy for proving Theorem 2.5 is then to change variables to the upper half-plane, to compute the Schwarzian derivative of H in order to check that it maps to a circular-arc triangle, and then to determine the placement of that triangle.

So, let t be a complex variable, let $\zeta(t)$ be the conformal map from $\operatorname{Im} t > 0$ onto the interior of the euclidean triangle E such that when extended to the boundary $0, 1$, and ∞ map to the vertices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ of E respectively. Similarly, let $z(t)$ be the conformal map from $\operatorname{Im} t > 0$ onto the region T in the z-plane such that when extended to the boundary $0, 1$ and ∞ map to the vertices A, B and C of T respectively. We can use Theorem 2.6 to compute $\{z, t\}$ and $\{\zeta, t\}$, and from this we can compute $\{H \circ z, t\}$. Before we do that, we will first state some useful identities between the Schwarzian derivatives of the various functions we are considering.

Proposition 2.7. *Let*

$$u = \{\zeta, t\} - \{z, t\} \qquad \text{and} \qquad v = \frac{z(dH/dt)}{1 \pm zH}.$$

We then have the following equalities:

$$\{\zeta, z\} = \{f, z\} = \frac{\mp 2dH/dz}{(1 \pm zH)^2}, \tag{8}$$

$$u = \frac{\mp 2(dH/dt)(dz/dt)}{(1 \pm zH)^2}, \tag{9}$$

$$D_t H = \frac{d}{dt}\log u - D_t\zeta \pm 2v. \tag{10}$$

Proof. To show (8), we begin with the definition

$$\{f, z\} = \frac{d}{dz}D_z f - \frac{1}{2}(D_z f)^2.$$

We now use equation (1) to get

$$\{f, z\} = \frac{d}{dz}\left[\frac{\mp 2H}{1 \pm zH}\right] - \frac{1}{2}\left(\frac{\mp 2H}{1 \pm zH}\right)^2.$$

After taking the derivative of the first term, squaring the second term, and cancelling, we are left with the expression in (8).

For (9), note that the "chain rule" for Schwarzian derivatives reads

$$\{\zeta, t\} = \{\zeta, z\}(dz/dt)^2 + \{z, t\}.$$

Thus,

$$u = \{\zeta, t\} - \{z, t\} = \{\zeta, z\}(dz/dt)^2.$$

Combining this with (8), we have

$$u = \frac{\mp 2dH/dz}{(1 \pm zH)^2} \cdot (dz/dt)^2 = \frac{\mp 2(dH/dt)(dz/dt)}{(1 \pm zH)^2},$$

which shows (9).

For (10), we begin by computing the t-derivative of $\log u$, which appears on the right in equation (10). Using (9),

$$\begin{aligned}\frac{d}{dt}\log u &= \frac{d}{dt}\log\left[\frac{\mp 2(dH/dt)(dz/dt)}{(1 \pm zH)^2}\right] \\ &= D_t H + D_t z \mp 2\frac{H(dz/dt) + z(dH/dt)}{1 \pm zH}.\end{aligned} \tag{11}$$

On the other hand,

$$D_t\zeta - D_t z = D_z\zeta \cdot \frac{dz}{dt} = \mp 2\frac{H(dz/dt)}{1 \pm zH}, \tag{12}$$

where the left equality is the chain rule for the operator D and the right equality is equation (1). Solving for $D_t z$ in equation (12), substituting this expression in for $D_t z$ in equation (11), and then cancelling gives us

$$\frac{d}{dt}\log u = D_t H + D_t\zeta \mp \frac{2z(dH/dt)}{1 \pm zH}.$$

Solving this last equation for $D_t H$ results in (10). □

Proof of Theorem 2.5. As we said in our outline, we need to compute $\{H\circ z, t\}$ and verify that it is in the form of a Schwarzian derivative of a triangle function. To compute $\{H,t\}$, we need to know D_tH and its derivative. We have an expression for D_tH by Proposition 2.7. So, we now differentiate equation (10):

$$\begin{aligned}\frac{d}{dt}D_tH &= \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta \pm 2\frac{[(dz/dt)(dH/dt)+z(d^2H/dt^2)](1\pm zH)}{(1\pm zH)^2}\\ &\quad - 2\frac{[H(dz/dt)+z(dH/dt)](z)(dH/dt)}{(1\pm zH)^2}\\ &= \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta \pm 2\frac{(dz/dt)(dH/dt)}{(1\pm zH)^2} \pm 2\frac{z(d^2H/dt^2)}{(1\pm zH)^2}\\ &\quad + 2\frac{z^2H(d^2H/dt^2)}{(1\pm zH)^2} - 2\frac{z^2(dH/dt)^2}{(1\pm zH)^2}.\end{aligned}$$

Using equation (9) to replace the term

$$\pm 2\frac{(dH/dt)(dz/dt)}{(1\pm zH)^2}$$

with $-u$, and using the fact that

$$\frac{d^2H}{dt^2} = D_tH\cdot\frac{dH}{dt},$$

we get

$$\frac{d}{dt}D_tH = \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta - u \pm 2\frac{z(dH/dt)(D_tH)}{(1\pm zH)^2}[1\pm zH] - 2\frac{z^2(dH/dt)^2}{(1\pm zH)^2}.$$

Next, we let v be as in Proposition 2.7, and we have

$$\frac{d}{dt}D_tH = \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta - u \pm 2vD_tH - 2v^2.$$

Replacing the occurrence of D_tH on the right-hand side in the above equation with the right hand side of equation (10), we get

$$\frac{d}{dt}D_tH = \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta - u \pm 2v\left[\frac{d}{dt}\log u - D_t\zeta\right] + 2v^2.$$

Then, we combine this last equation with equation (10) to compute $\{H,t\}$:

$$\begin{aligned}\{H,t\} &= \frac{d}{dt}D_tH - \frac{1}{2}(D_tH)^2\\ &= \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta - u \pm 2v\left[\frac{d}{dt}\log u - D_t\zeta\right] + 2v^2\\ &\quad - \frac{1}{2}\left[\frac{d}{dt}\log u - D_t\zeta \pm 2v\right]^2\\ &= \frac{d^2}{dt^2}\log u - \frac{d}{dt}D_t\zeta - u - \frac{1}{2}\left[\frac{d}{dt}\log u - D_t\zeta\right]^2.\end{aligned}$$

Now, using Theorem 2.6 to compute $\{\zeta,t\}$ and $\{z,t\}$,

$$u(t)=\{\zeta,t\}-\{z,t\}=\frac{\gamma^2-\tilde{\gamma}^2}{2t(t-1)}.$$

Moreover, since the triangle in the ζ-plane is euclidean, the second part of Theorem 2.6 tells us

$$D_t\zeta=-\frac{1-\alpha}{t}-\frac{1-\beta}{t-1}.$$

Thus,

$$\frac{d}{dt}\log u-D_t\zeta=-\frac{\alpha}{t}-\frac{\beta}{t-1}.$$

Differentiating this equation with respect to t, we find

$$\frac{d^2}{dt^2}\log u-\frac{d}{dt}D_t\zeta=\frac{\alpha}{t^2}+\frac{\beta}{(t-1)^2}.$$

Combining these last two equalities with our equation for $\{H,t\}$, we find

$$\begin{aligned}\{H,t\}&=\frac{\alpha}{t^2}+\frac{\beta}{(t-1)^2}-\frac{\gamma^2-\tilde{\gamma}^2}{2t(t-1)}-\frac{1}{2}\left[-\frac{\alpha}{t}-\frac{\beta}{t-1}\right]^2\\&=\frac{2\alpha-\alpha^2}{2t^2}+\frac{2\beta-\beta^2}{2(t-1)^2}+\frac{\tilde{\gamma}^2-\gamma^2-2\alpha\beta}{2t(t-1)}\\&=\frac{1-(1-\alpha)^2}{2t^2}+\frac{1-(1-\beta)^2}{2(t-1)^2}+\frac{\tilde{\gamma}^2-\gamma^2-2\alpha\beta}{2t(t-1)}.\end{aligned}$$

Because E is euclidean, we have $\tilde{\gamma}=1-\alpha-\beta$, and so

$$\tilde{\gamma}^2-\gamma^2-2\alpha\beta=(1-\alpha-\beta)^2-\gamma^2-2\alpha\beta=(1-\alpha)^2+(1-\beta)^2-\gamma^2-1.$$

Thus,

$$\{H,t\}=\frac{1-(1-\alpha)^2}{2t^2}+\frac{1-(1-\beta)^2}{2(t-1)^2}+\frac{(1-\alpha)^2+(1-\beta)^2-\gamma^2-1}{2t(t-1)},$$

and hence by Theorem 2.6, H maps T conformally onto the interior of a circular-arc triangle with internal angle measures: $\pi(1-\alpha)$, $\pi(1-\beta)$, and $\pi\gamma$. Therefore, $\overline{H}$ maps the interior of T anti-conformally to the interior of a circular-arc triangle with these same angle measurements, which are the angle measurements of T'. The Schwarzian derivative computation we have done determines H, and hence $\overline{H}$, up to post-composition by a Möbius transformation. But, any Möbius transformation is determined by the image of any three distinct points, for example the vertices of T. By Proposition 2.3, $\overline{H}$ maps the vertices of T onto the corresponding vertices of T'. Therefore, $\overline{H}$ must map T onto T'. □

The following corollary implies Theorem 1.1 by Proposition 2.4.

Corollary 2.8. *Let $z_0 \in T$ be a point where $\nu(z_0) = 0$. Then z_0 lies on the side of T connecting the two vertices whose angle measures are preserved by f. In particular, if T is spherical (respectively hyperbolic), then any point $z_0 \in T$ where μ attains a global minimum (respectively maximum) over T lies on this side.*

Proof. By Proposition 2.4, $z_0 = \overline{H}(z_0)$. By Theorem 2.5, the image of T under $\overline{H}$ is T'. Therefore, $z_0 \in T \cap T'$. The only common points of T and T' are the points on the edge connecting A and B, except possibly the vertex C if $C = I(C)$. (The possibility $C = I(C)$ can occur only in the hyperbolic case, and then only when $\gamma = 0$, in which case C lies on the unit circle.) Since $z_0 \neq C$ by statement (iii) of Proposition 2.2, the corollary follows. ▫

In case that one of the angles preserved by f has measure $\frac{1}{2}\pi$, we can say even more.

Corollary 2.9. *Under the assumptions of Theorem 1.1, and with the additional assumption that $\alpha = \frac{1}{2}$, i.e. the angle at A has measure $\frac{1}{2}\pi$, we have $\nu(z_0) = 0$ for $z_0 \in T$ if and only if $z_0 = A$ or $z_0 = B$. Moreover, if T is spherical (respectively hyperbolic), then the unique point where μ attains its minimum (respectively maximum) over T is at the vertex B.*

We need the following geometric statement for the proof of the corollary. We denote by R the reflection across the circle which contains the side of T opposite to the vertex C.

Lemma 2.10. *Let the notation be as discussed above. If $\alpha, \beta \leq \frac{1}{2}$, then we have the strict inclusion $R(T) \subset T'$.*

Proof. Let D_1, D_2, D_3, be the unique closed discs in the Riemann sphere that contain T and whose boundary circles contain the sides of T opposite to A, B, and C, respectively. Then R is the reflection across ∂D_3. Moreover,

$$T = D_1 \cap D_2 \cap D_3 \qquad \text{and} \qquad T' = D_1 \cap D_2 \cap R(D_3).$$

The hypothesis $\alpha \leq \frac{1}{2}$ implies $R(D_2 \cap D_3) \subseteq D_2$, and similarly, $\beta \leq \frac{1}{2}$ implies $R(D_1 \cap D_3) \subseteq D_1$. Therefore,

$$R(T) \subseteq R(D_1 \cap D_3) \cap R(D_2 \cap D_3) \cap R(D_3) \subseteq D_1 \cap D_2 \cap R(D_3) = T'.$$

Note that the angle measures of the triangle $R(T)$ at the vertices A and B are $\alpha\pi$ and $\beta\pi$, respectively. The angle measures of the triangle T' at the vertices A and B are $(1-\alpha)\pi$ and $(1-\beta)\pi$, respectively. It follows from this that the above inclusion is strict if $\alpha < \frac{1}{2}$ or $\beta < \frac{1}{2}$. This is always the case under our assumptions, since $\alpha\pi$ and $\beta\pi$ are angle measures of the non degenerate euclidean triangle E which implies $\alpha + \beta < 1$. ▫

Proof of Corollary 2.9. Assume T is normalized so that the vertex A is at the origin, that one side of T lies along the positive real-axis, and that one side of T lies along the positive imaginary-axis. The mapping properties of H and Lemma 2.10 imply that H (not $\overline{H}$!) maps T onto a circular-arc triangle that strictly contains T. This triangle is nothing other than the reflection of T' across the real-axis.

Let H^{-1} denote the inverse mapping of H, which is defined on T. Then H^{-1} maps T onto a proper subset of itself so that the edge of T along the real axis is mapped onto itself and the edge along the imaginary axis is mapped into itself. Since T is a right triangle with right angle at 0, we can reflect T across the real and imaginary axis and through the origin. Denote the union of T with the image triangles of T under these involutions by U. The interior of the set U is a simply connected region and contains the edge AB of T apart from the point B. By the Schwarz reflection principle, the map H^{-1} has a continuous extension to U which is conformal in the interior of U. This mapping maps the side AB of T onto itself and has the fixed points A and B. This map is not the identity map, since the image of U is a proper subset of U. By the Schwarz lemma, a map of a simply connected region into itself that is not the identity can have at most one internal fixed point. Thus, the vertex A, *i.e.*, the origin, is the only fixed point of H^{-1} in the interior of U. Since the edge AB lies along the real-axis, this implies that the only solutions of $\bar{z} = H(z)$ on the edge AB of T are A and B. The first part of the corollary now follows from Proposition 2.4 and Corollary 2.8.

In particular, μ has no critical point in T different from the vertices. Since μ is invariant under reflection across any side of T, the derivative of μ in the direction normal to the side will vanish along each side. Since μ has no critical point except at the vertices, this means that the directional derivative of μ in the direction of a side cannot vanish anywhere along the side, except at the vertices. Thus μ is strictly monotonic along each side.

Suppose T is spherical. As we remarked in proof of the previous lemma, $\alpha + \beta < 1$. Therefore, $\beta < \frac{1}{2}$ and the function μ has a local minimum at B by Proposition 2.2(ii). Since μ is strictly monotonic along the side AB of T, we have $\mu(A) > \mu(B)$. We have seen in the first part of the proof that A and B are the only points where μ can possibly have a global minimum on T. It follows that $z_0 = B$ is the unique point where this global minimum is attained.

If T is hyperbolic, it follows similarly that $z_0 = B$ is the unique point where the global maximum of μ on T is attained. □

3. Applications

We recall that a region $\Omega \subseteq \overline{\mathbf{C}}$ is called *hyperbolic* if and only if there exists a meromorphic universal covering map f from the unit disc onto Ω. This is the case if and only if the complement of Ω in $\overline{\mathbf{C}}$ contains at least three points.

If $\Omega \subseteq \mathbf{C}$ is hyperbolic, the *hyperbolic density* Λ at a point $w \in \Omega$ is defined as the ratio of the hyperbolic length element at any preimage point of w under f to the euclidean length element at w. This definition is independent of the choice of f and of the preimage point w. Explicitly,

$$\Lambda(w) = \frac{2}{(1-|z|^2)|f'(z)|}, \qquad \text{if } w = f(z).$$

As our first application of the results of the previous section, we give a new proof of the following theorem of A. Baernstein II and H. Montgomery.

Theorem 3.1. *Let $\Omega = \mathbf{C} \setminus L_j$, were L_j is one of the following two lattices*:

$$L_1 = \{a + bi : a, b \in \mathbf{Z}\}, \qquad L_2 = \{a + b(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\, i) : a, b \in \mathbf{Z}\}.$$

Then the set of minimum points of the hyperbolic density Λ of Ω consists of the points congruent modulo L_j to $\frac{1}{2}(1+i)$ if $j = 1$, or to either

$$\tfrac{1}{2} + \tfrac{1}{6}\sqrt{3}\, i \qquad \text{or} \qquad 1 + \tfrac{1}{3}\sqrt{3}\, i$$

if $j = 2$.

Remark. L_1, a square lattice, and L_2, a hexagonal lattice, are, up to rotation and stretching, the two lattices that have non-trivial rotational symmetries. In the case of the second lattice L_2, the region Ω is important because of its connection to the conjectural extremal functions for Bloch's and Landau's constants. See [B] for a discussion of this problem, and see [BV] for a proof of the stronger result that in the case of L_2, not only are the specified points z_0 minimum points for Λ_{L_2} for the fixed lattice L_2, but the pairs (L_2, z_0) (z_0 one of the points specified in the statement of the theorem) are local minimum points for $\Lambda_L(z)$ if the lattice L is also allowed to vary (given a certain constraint) together with the point z. The work of H. Montgomery is not phrased in terms of the hyperbolic density. For a discussion of how Montgomery's theorem implies the hexagonal case of Theorem 3.1, see [BV].

Proof. The density Λ is doubly periodic, so we need only minimize it over one period parallelogram. From the further symmetries, we may consider Λ restricted to right euclidean triangles E_j for $j \in \{1, 2\}$ as shown in Figure 3. The right-hand side of Figure 3 shows the square (above) and hexagonal lattice (below), together with lines of symmetry. In each case, the right triangle over which it suffices to minimize Λ is outlined in bold. The left-hand side of Figure 3 shows the unit disc as the universal covering space of Ω, with the hyperbolic triangle outlined in bold as one of the inverse image triangles of the bold triangle on the right. Using the standard notation of the previous section we have $\tilde{\alpha}_1 = \frac{1}{2}$, $\tilde{\beta}_1 = \frac{1}{4}$, $\tilde{\gamma}_1 = \frac{1}{4}$, and $\tilde{\alpha}_2 = \frac{1}{2}$, $\tilde{\beta}_2 = \frac{1}{3}$, $\tilde{\gamma}_2 = \frac{1}{6}$. The universal covering map of the unit disc onto Ω is the analytic continuation of a triangle map $f_j\colon T_j \to E_j$, where $\alpha_j = \tilde{\alpha}_j$, $\beta_j = \tilde{\beta}_j$, and $\gamma_j = 0$ for $j \in \{1,2\}$. Note that $\log \Lambda\bigl(f_j(z)\bigr) = \log 2 - \mu(z)$, where μ is as in the previous section. Therefore, the minimum of Λ occurs at the maximum of μ, which by Corollary 2.9 is at the vertex whose angle measure is preserved by f and less than $\frac{1}{2}\pi$. The location of this vertex, modulo L_j, is located precisely as stated in the theorem. □

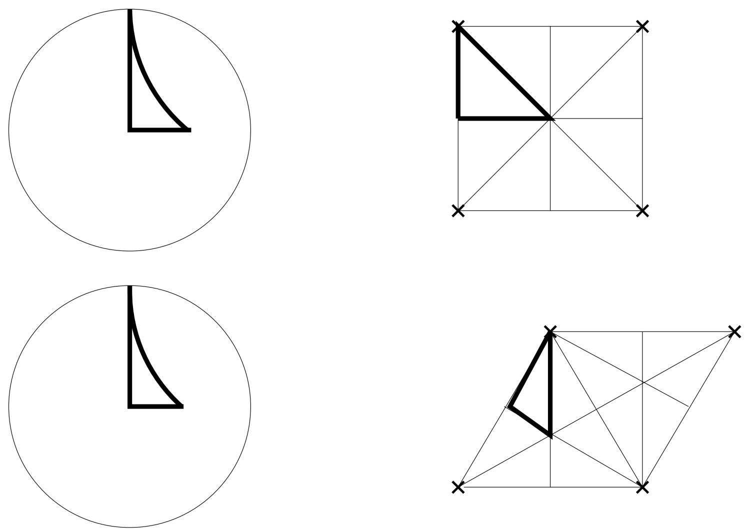

Figure 3.

For our second application, we define the *spherical density* Σ at a point w of a hyperbolic region $\Omega \subseteq \overline{\mathbf{C}}$ as follows. Let f be a meromorphic universal covering map from the unit disc onto Ω. Then $\Sigma(w)$ is the ratio of the hyperbolic length element at any preimage point of w under f to the spherical length element at w. Again this definition is independent of the choice of f and of the preimage point w. Explicitly,

$$\Sigma(w) = \frac{1+|w|^2}{(1-|z|^2)|f'(z)|} \qquad \text{if } w = f(z).$$

Since we have $\Sigma(w) \to \infty$ as $w \to \partial\Omega$, one should expect that the minimum points of Σ are located “as far” from the boundary of Ω as possible. In [BC] we established methods to prove that the minimum points Σ in symmetric situations are located where they should be. The following theorem covers some cases that could not be treated with the methods in [BC].

We need the following well-known fact about tesselations of the Riemann sphere by spherical triangles, cf. [C]. If T is a spherical triangle, and we reflect T through its edges, then we obtain new spherical triangles. If we successively repeat this process with the new triangles, then we obtain a finite tesselation of the Riemann sphere by non-overlapping triangles if and only if T has angle measures at its vertices A, B, C given by π/k, π/l, π/m, respectively, where k, l, m are integers satisfying

$$\frac{1}{k}+\frac{1}{l}+\frac{1}{m}>1.$$

In this case, we say T generates a finite tesselation of the Riemann sphere of type (k,l,m). The reflections along the sides of T generate a finite group of isometries of $\overline{\mathbf{C}}$, the *triangle group* corresponding to T.

Theorem 3.2. *With the above conventions let T be a spherical triangle generating a finite tesselation of the Riemann sphere of type $(2,l,m)$, $l\geq 3$. Suppose Ω is the complement of the orbit of the vertex C of T under the triangle group Γ generated by T. Then the set of minimum points of the spherical density Σ of Ω is equal to the orbit of the vertex B of T under Γ.*

Before proving Theorem 3.2, we state some corollaries and discuss its significance.

Corollary 3.3. *Let Ω be the complement (in the Riemann sphere) of the n-th roots of unity $(n\geq 3)$. Then Σ is minimal at $z=0$ and $z=\infty$.*

Proof. Take for T the circular-arc triangle with vertices at $z=0$, $z=1$, and $z=e^{2\pi i/2n}$. This is the case $(2,l,m)=(2,n,2)$. ◻

Corollary 3.4. *Let Ω be the complement (in the Riemann sphere) of the vertices of a regular polyhedron P inscribed in the Riemann sphere. Then Σ is minimal at the radial projections to the Riemann sphere of the centers of the faces of the inscribed polyhedron. Put another way, the minimum points of Σ occur at the vertices of the inscribed polyhedron dual to P.*

Proof. Choose a vertex V, an edge L, and a face F of P such that L is one of the edges of the face F, and such that L meets V. Take for T the great circle triangle whose vertices are V, the radial projection to the Riemann sphere of the midpoint of L, and the radial projection to the Riemann sphere of the center of the face F. Here we are in one of the cases: $(2,l,m)=(2,3,3)$ (tetrahedron), $(2,l,m)=(2,4,3)$ (cube), $(2,l,m)=(2,3,4)$ (octahedron), $(2,l,m)=(2,5,3)$ (dodecahedron), or $(2,l,m)=(2,3,5)$ (icosahedron).

The triangle T is illustrated for the tetrahedron in Figure 4. ◻

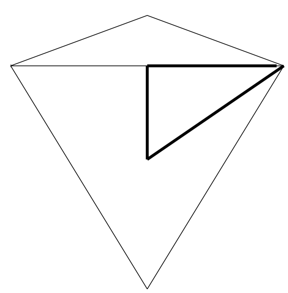

Figure 4.

By a different method, in an earlier paper [BC], we were able to prove Corollary 3.3, and also Corollary 3.4 in the case that the inscribed polyhedron was a tetrahedron, cube, or octahedron. Thus, Theorem 3.2 should perhaps be regarded as providing a new proof for finding the minimum of Σ for some of the regions considered in [BC]. Nonetheless, methods like those used in [BC] do not seem adequate to prove Corollary 3.4 in the case of the dodecahedron or icosahedron, and so for these two regions Theorem 3.2 provides the first proof that the minimum points for Σ are as expected.

We remark, that as discussed in [BC], knowing the minimum value for Σ for a region Ω, allows one to answer the following question.

Question 3.5. *Given a hyperbolic region* Ω *in the Riemann sphere, what is*

$$\sup\{f^\sharp(0) : f\colon \mathbf{D} \to \Omega \text{ a holomorphic map }\}?$$

Here $\mathbf{D}$ denotes the unit disc, and the spherical derivative $f^\sharp(z)$ is defined by

$$f^\sharp(z) = \frac{2|f'(z)|}{1+|f(z)|^2}$$

if $f(z) \neq \infty$, and $f^\sharp(z) = 2|(1/f)'(z)|$ if $f(z) = \infty$. The spherical derivative $f^\sharp(z)$ measures how much f distorts length, locally at z, if the length in the image is measured with respect to the spherical metric, and length in the domain is measured with respect to the ordinary euclidean metric. Thus, combining Corollary 3.4 with the numerical computations in the last section of [BC], we have that if f is a meromorphic function on the unit disc whose image is contained in the complement of the vertices of a regular dodecahedron or icosahedron inscribed in the Riemann sphere, then

$$f^\sharp(0) \leq 1.350058$$

in the case of the icosahedron (12 vertices) and

$$f^\sharp(0) \leq 1.079998$$

in the case of the dodecahedron (20 vertices). (See [BC] for the precise upper bounds expressed in terms of the Γ-function.)

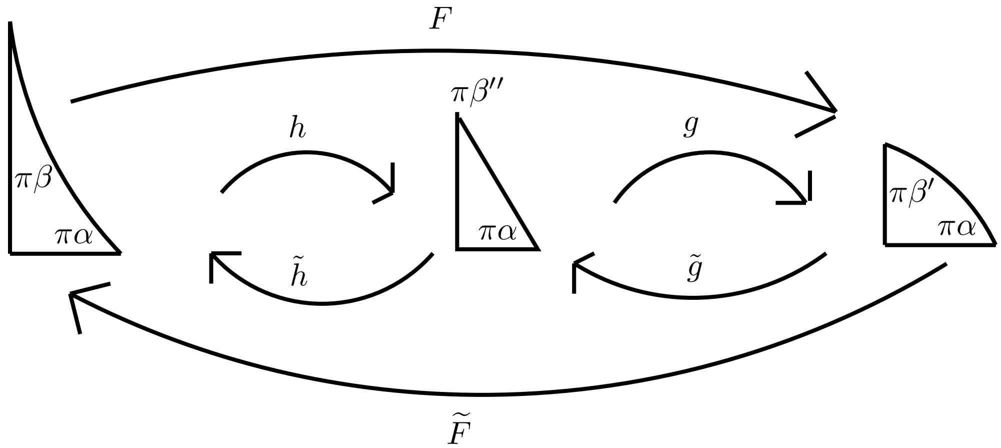

Figure 5.

Proof of Theorem 3.2. Finding the minimum points for Σ is equivalent to finding the minimum points of $\sigma = \log\Sigma$. By symmetry, we only need to minimize σ over the closure of the triangle T. We let F denote the universal covering map of Ω, and we let $\widetilde{F}$ denote its locally defined inverse. We remark that the inverse image under F of the circular-arc triangle T, will be a union of geodesic triangles in the unit disc. We only need to consider F restricted to the closure of one of these triangles. Rather than consider the map F from the hyperbolic triangle to the spherical triangle, we will find it easier to factor F through a euclidean triangle, so we can apply our main theorem in the form of Corollary 2.9. That is, we write $F = g\circ h$, where h conformally maps the interior of the hyperbolic triangle onto the interior of a euclidean triangle, and g conformally maps the interior of that euclidean triangle onto the interior of the spherical triangle T. We choose the intermediate euclidean triangle so that g and h both preserve the two angles preserved by F. We denote by $\tilde{g}$ and $\tilde{h}$ the inverse mappings of g and h, and so we have $\widetilde{F} = \tilde{h}\circ\tilde{g}$. This setup is illustrated in Figure 5.

Notice that the measure of the angle that is not preserved is increased as we move from left to right in Figure 5. Observe further that

$$\begin{aligned}
\sigma(z) &= \log(1+|z|^2) + \log|\widetilde{F}'(z)| - \log\bigl(1-|\widetilde{F}(z)|^2\bigr)\\
&= \log(1+|z|^2) + \log|\tilde{g}'(z)| + \log\bigl|\tilde{h}'\bigl(\tilde{g}(z)\bigr)\bigr| - \log\Bigl(1-\bigl|\tilde{h}\bigl(\tilde{g}(z)\bigr)\bigr|^2\Bigr)\\
&= \log(1+|z|^2) + \log|\tilde{g}'(z)| - \log\Bigl|h'\Bigl(\tilde{h}\bigl(\tilde{g}(z)\bigr)\Bigr)\Bigr| - \log\Bigl(1-\bigl|\tilde{h}\bigl(\tilde{g}(z)\bigr)\bigr|^2\Bigr)\\
&= \mu_+(z) - \mu_-\Bigl(\tilde{h}\bigl(\tilde{g}(z)\bigr)\Bigr),
\end{aligned}$$

where

$$\mu_+(z) = \log(1+|z|^2) + \log|\tilde{g}'(z)|,$$

$$\text{and}\qquad \mu_-(w) = \log(1-|w|^2) + \log|h'(w)|.$$

The point is that μ_+ and μ_- are very similar sorts of functions. In particular, they are both built out of a conformal map from a circular-arc triangle to a euclidean triangle. In other words, they are precisely the kind of function μ we considered in Section 2. By Corollary 2.9, μ_+ attains its minimum at the vertex of T which is interior to Ω and with internal angle measure $< \frac{1}{2}\pi$. Similarly μ_- attains its maximum at the vertex of the hyperbolic triangle which is inside the unit disc and which has internal angle measure $< \frac{1}{2}\pi$. Thus, $-\mu_-\bigl(\tilde{h}\bigl(\tilde{g}(z)\bigr)\bigr)$ attains its minimum at exactly the same place where $\mu_+(z)$ attains its minimum. Since σ is the sum of these two functions, it must be minimal at exactly the same location. □

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