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GENERALIZED HECKE GROUPS AND HECKE POLYGONS

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Abstract. In this paper, we study certain Fuchsian groups $\mathscr{H}(p_1,\dots,p_n)$, called generalized Hecke groups. These groups are isomorphic to $\prod^{*\,n}_{j=1} Z_{p_j}$. Let Γ be a subgroup of finite index in $\mathscr{H}(p_1,\dots,p_n)$. By Kurosh's theorem, Γ is isomorphic to $F_r * \prod^{*\,k}_{i=1} Z_{m_i}$, where F_r is a free group of rank r, and each m_i divides some p_j. Moreover, $\mathbf{H}^2/\Gamma$ is Riemann surface. The numbers $m_1,\dots,m_k$ are branching numbers of the branch points on $\mathbf{H}^2/\Gamma$. The signature of Γ is $(g; m_1,\dots,m_k; t)$, where g and t are the genus and the number of cusps of $\mathbf{H}^2/\Gamma$, respectively.

A purpose of this paper is to consider two problems. First, determine the necessary and sufficient conditions for the existence of a subgroup of finite index of a given type in $\mathscr{H}(p_1,\dots,p_n)$. We also extend this work to extended generalized Hecke groups $\mathscr{H}^*(p_1,\dots,p_n)$ which are isomorphic to $\mathbf{D}_{p_1} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{p_n}$ (amalgamated over Z_2's generated by reflections), where each $\mathbf{D}_{p_j}$ is a dihedral group of order $2p_j$.

The second problem is the realizability problem for the existence of a subgroup with a given signature in $\mathscr{H}(p_1,\dots,p_n)$. This is a special case of the Hurwitz problem about the realizability of branched covers. Special cases of this work were also studied by Millington, Singerman, Hoare, Edmonds, Ewing and Kulkarni. Our approach is based on constructing special Poincaré polygons which are the same as fundamental domains for $\mathscr{H}(p_1,\dots,p_n)$, $\mathscr{H}^*(p_1,\dots,p_n)$ and their subgroups.

1. Introduction

Suppose that integers $p_1, p_2, \dots, p_n$ are given, where each $p_j \ge 2$. The purpose of this paper is to study the geometry and topology of a Fuchsian group $\mathscr{H}(p_1,\dots,p_n)$, called a *generalized Hecke group*, and its certain extension $\mathscr{H}^*(p_1,\dots,p_n)$, called an *extended generalized Hecke group.* As an abstract group, $\mathscr{H}(p_1,\dots,p_n)$ is isomorphic to $\prod^{*\,n}_{i=1} Z_{p_i}$, and $\mathscr{H}^*(p_1,\dots,p_n)$ is isomorphic to $\mathbf{D}_{p_1} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{p_n}$ (amalgamated over Z_2's generated by reflections), where throughout the paper $\prod^*$ denotes a free product of groups, each Z_{p_j} is a finite cyclic group of order p_j, and each $\mathbf{D}_{p_j}$ is a dihedral group of order $2p_j$; cf. Section 2.

Let Γ be a subgroup of finite index in $\mathscr{H}(p_1,\dots,p_n)$. Then $\mathbf{H}^2/\Gamma$ is a Riemann surface. Let g and t be the genus and the number of cusps of $\mathbf{H}^2/\Gamma$ respectively, and let $m_1,\dots,m_k$ be the branching numbers of the branch points on $\mathbf{H}^2/\Gamma$. The signature of Γ is $(g; m_1,\dots,m_k; t)$.

It follows from Kurosh's theorem that a subgroup of a generalized Hecke group $\prod^{*\,n}_{i=1} Z_{p_i}$ is isomorphic to $F * \bigl(\prod^{*\,k}_{i=1} Z_{m_i}\bigr)$, where F is a free group, and each

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m_j divides some p_i, for $j = 1, \dots, k$. A group $\prod^{*}{}_{i=1}^{n} Z_{p_i}$ may not always contain a subgroup of a given type. For instance, $Z_2 * Z_2 * Z_2 * Z_2$ does not embed in $Z_3 * Z_6$ as a subgroup of index 2. Indeed, it is easy to see that there is a unique normal subgroup of index 2 in $Z_3 * Z_6$, and it is isomorphic to $Z_3 * Z_3 * Z_3$.

Millington [11] investigated the existence of subgroups with given signatures in the modular group which is isomorphic to $Z_2 * Z_3$. We state Millington's theorem as follows.

Theorem 1.1. *Let d, k_1, k_2, g, t be nonnegative integers, and $t, d \geq 1$. If the Riemann–Hurwitz relation*

$$d = 3k_1 + 4k_2 + 12g + 6t - 12$$

holds, the modular group contains a subgroup of index d and with a signature $(g; \underbrace{2, \dots, 2}_{k_1}, \underbrace{3, \dots, 3}_{k_2}; t)$.

This result was partially extended. A group Γ can be embedded as a subgroup of index d in $Z_{p_1} * Z_{p_2}$, where p_1, p_2 are distinct primes if and only if the Euler characteristic condition is satisfied, i.e. $\chi(\Gamma) = d\chi(Z_{p_1} * Z_{p_2})$ [6, Theorem 5.1], where χ is the Euler characteristic of a group in the sense of Wall; cf. [15]. Notice that this result is partial since we do not know whether the group can be realized as a *Fuchsian* group with a prescribed signature, subject to Euler characteristic (that is the same as Riemann–Hurwitz) condition. However when $p_1, \dots, p_n$ are not distinct primes, the Riemann–Hurwitz condition is not sufficient to embed a group as a subgroup of finite index in $\prod^{*}{}_{i=1}^{n} Z_{p_i}$.

In [6], Kulkarni derived a further necessary condition, a diophantine condition, and showed that this condition together with the Riemann–Hurwitz condition is also sufficient to embed a group $F_r * \prod^{*}{}_{m} Z_m$ in $\prod^{*}{}_{i=1}^{n} Z_{p_i}$ as a subgroup of finite index, where henceforth F_r denotes a free group of rank r. We describe this theorem as follows.

Theorem 1.2. *Let k, r be nonnegative integers. Let $\Gamma = \prod^{*}{}_{i=1}^{n} Z_{p_i}$, and $\Phi = F_r * \prod^{*}{}_{i=1}^{k} Z_{m_i}$, where each m_i divides some p_j. Then Φ can be realized as a subgroup of Γ of index d if and only if the following conditions are satisfied*:

(i) (*The Riemann–Hurwitz condition*)

$$\sum_{i=1}^{k} \frac{1}{m_i} - (k + r) + 1 = d\left(\sum_{i=1}^{n} \frac{1}{p_i} - n + 1\right).$$

(ii) (*The diophantine condition*) *Let $m_0 = 1$, and let $m_1, \dots, m_s$ be the maximal set of distinct m_i, where each m_j, $1 \leq j \leq s$, occurs b_j times. Set*

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if } m_j \nmid p_i, \\ 1 & \text{if } m_j \mid p_i, \end{cases} \qquad \delta_{ij} = \frac{p_i}{m_j}\varepsilon_{ij}.$$

Then the system

$$\sum_{i=1}^{n} \varepsilon_{ij} x_{ij} = b_j, \qquad j = 1, \ldots, s,$$
$$\sum_{j=0}^{s} \delta_{ij} x_{ij} = d, \qquad i = 1, \ldots, n$$

has a solution for x_{ij} in nonnegative integers.

Moreover Kulkarni [7] extended Millington's theorem to $Z_{p_1} * Z_{p_2}$.

Theorem 1.3. *Let k, g, t, r be nonnegative integers, where $t \geq 1$, $r = 2g + t - 1$. Let $\Gamma = Z_{p_1} * Z_{p_2}$, and $\Phi = F_r * \prod^{*}{}_{i=1}^{k} Z_{m_i}$, where each m_i divides p_1 or p_2. Then Φ can be realized as a subgroup of Γ of index d and with a signature $(g; m_1, \ldots, m_k; t)$ if and only if the following conditions are satisfied*:

(i) (*The Riemann–Hurwitz condition*)

$$\sum_{i=1}^{k} \frac{1}{m_i} - (k + r) + 1 = d\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right).$$

(ii) (*The diophantine condition*) *Let $m_0 = 1$, and let $m_1, \ldots, m_s$ be the maximal set of distinct m_i, where each m_j, $1 \leq j \leq s$, occurs b_j times. Set*

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if } m_j \nmid p_i, \\ 1 & \text{if } m_j \mid p_i, \end{cases} \qquad \delta_{ij} = \frac{p_i}{m_j} \varepsilon_{ij}.$$

Then the system

$$\varepsilon_{1j} x_{1j} + \varepsilon_{2j} x_{2j} = b_j, \qquad j = 1, \ldots, s,$$
$$\sum_{j=0}^{s} \delta_{ij} x_{ij} = d, \qquad i = 1, 2,$$

has a solution for x_{ij} in nonnegative integers.

A motivation of this paper was to study realizability of signatures by subgroups of finite index in $\mathscr{H}(p_1, \ldots, p_n)$ considered as a Fuchsian group.

A noncocompact Fuchsian group Γ is a free product of cyclic groups. A system of generators for Γ is said to be *independent* if the group is a free product of cyclic subgroups generated by elements in the generating system. This notion is due to Rademacher; cf. [12]. A fundamental domain P for Γ is called an *admissible* fundamental domain for Γ if the side pairings of P is an independent system of generators for Γ; cf. [7]. A fundamental domain is in general *not* admissible. Indeed, the usual fundamental domain for the modular group and the well-known fundamental domain constructed by Fricke for congruence subgroups are not admissible. In Section 2, we introduce a special kind of Poincaré polygon, called a *Hecke polygon*, which is an admissible fundamental domain for the group generated by the side pairings of it.

There is a correspondence between Hecke polygons and subgroups of finite index in $\mathscr{H}(p_1,\dots,p_n)$. Each subgroup of finite index in $\mathscr{H}(p_1,\dots,p_n)$ admits an admissible fundamental domain which is a Hecke polygon. From this result, we give new proofs of Theorems 1.2 and 1.3 by constructing a Hecke polygon. Meanwhile the diophantine condition (which is the same as the integrality condition of Theorem 1.4) is interpreted geometrically as the relationship between the index of a subgroup and the number of Ω_j-polygons of a Hecke polygon; cf. Section 3. In our set-up Theorem 1.2 is restated as follows.

Theorem 1.4. *Let* $k_0=0,k_1,\dots,k_n,r$ *be nonnegative integers, where* $k_i\le k_{i+1}$, *for* $i=0,\dots,n-1$. *Let* $\Gamma=F_r*\prod^{*}{}_{j=1}^{n}\prod^{*}{}_{i=k_{j-1}+1}^{k_j}Z_{p_j/m_i}$, *where* $m_i\mid p_j$, $i=k_{j-1}+1,\dots,k_j$, $j=1,\dots,n$. *Then* Γ *can be embedded in* $\mathscr{H}(p_1,\dots,p_n)$ *as a subgroup of index* d *if and only if the following conditions hold*:

(i) (*The Riemann–Hurwitz condition*)

$$\sum_{j=1}^{n}\sum_{i=k_{j-1}+1}^{k_j}\frac{m_i}{p_j}-(k_n+r)+1=d\left(\sum_{j=1}^{n}\frac{1}{p_j}-n+1\right).$$

(ii) (*The integrality condition*) *The numbers* $s_1,\dots,s_n$ *satisfying*

$$s_jp_j+\sum_{i=k_{j-1}+1}^{k_j}m_i=d,\qquad j=1,\dots,n,$$

are nonnegative integers.

In particular, if $p_1,\dots,p_n$ are distinct primes, the integrality condition reduces to $d\ge k_j$, $j=1,\dots,n$, where k_j is the number of copies of Z_{p_j}'s in Γ (see Corollary 5.2).

In Section 4, we study a special kind of a NEC (non-euclidean crystallographic) group $\mathscr{H}^*(p_1,\dots,\ p_n)$ in which $\mathscr{H}(p_1,\dots,\ p_n)$ is a subgroup of index 2.

The algebraic structure of a NEC group with noncompact quotient space was determined by Macbeath and Hoare [9]. It follows that each subgroup of finite index in $\mathbf{D}_{p_1}*_{Z_2}\cdots*_{Z_2}\mathbf{D}_{p_n}$ is isomorphic to $F_r*\prod^{*}{}_{m}Z_m*\prod^{*}{}_{i}(\mathbf{D}_{x_{i1}}*_{Z_2}\cdots*_{Z_2}\mathbf{D}_{x_{ik_i}})*\prod^{*}{}_{j}E_j$, where each m divides some p_j, each x_{ij} divides some p_l, and each E_j has a presentation

$$\langle y_j,a_{j1},\dots,a_{js_j}\mid a_{j1}y_ja_{js_j}y_j^{-1}=a_{jl}^2=a_{jl+1}^2=(a_{jl}a_{jl+1})^{u_{jl}}=1,\ l=2,\dots,s_j-1\rangle.$$

We extend Theorem 1.4 in the case of subgroups of finite index in $\mathscr{H}^*(p_1,\dots,\ p_n)$. In this case, the necessary and sufficient conditions are still called the Riemann–Hurwitz and diophantine conditions (see Theorem 4.2). When $p_1,\dots,p_n$ are distinct primes, the diophantine condition can be stated in a more concise way (see Theorem 5.1).

Singerman [14] gave a permutation-theoretic approach to the realizability problem for signatures of subgroups of finitely generated Fuchsian groups. A generalization to NEC groups was done by Hoare [4]. Singerman's theorem is as follows.

Theorem 1.5. *Suppose that Γ has a presentation*

$$\Big\langle a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_r, f_1, \dots, f_t \mid x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{t} f_k = 1 \Big\rangle$$

with a signature $(g; m_1, \dots, m_r; t)$. Then Γ contains a subgroup Φ of index d with a signature $(h; n_{11}, n_{12}, \dots, n_{1\rho_1}, \dots, n_{r1}, n_{r2}, \dots, n_{r\rho_r}; s)$ if and only if there exists a finite permutation group G transitive on d points, and an epimorphism $\theta\colon \Gamma \to G$ satisfying the following conditions:

(i) *The permutation $\theta(x_j)$ has precisely ρ_j disjoint cycles of lengths m_j/n_{j1}, $\dots, m_j/n_{j\rho_j}$.*

(ii) *If $\delta(f)$ denotes the number of cycles in the permutation $\theta(f)$, then $s = \sum_{i=1}^{t} \delta(f_i)$.*

In Section 6, we show how to associate a system of permutations to a Hecke polygon such that the signature of the group generated by the side pairings of this polygon can be determined from the action of those permutations. The permutations which we construct (in the special case of generalized Hecke groups) are different from the ones in Singerman's theorem. In particular, we use permutations to construct the appropriate Hecke polygon, and in fact get an explicit geometric realization of the corresponding surface.

It is of interest to note that the Riemann–Hurwitz and diophantine conditions are not sufficient for the existence of a subgroup with a prescribed signature in $\mathscr{H}(p_1, \dots, p_n)$ if $n \geq 3$. An obvious additional necessary end-condition for the existence of a subgroup Γ of index d in a group is that the number t of cusps of the quotient space $\mathbf{H}^2/\Gamma$ is at most d. This condition does not follow from the Riemann–Hurwitz or diophantine condition; cf. the example in Section 7. The realizability problem for the existence of a subgroup of $\mathscr{H}(p_1, \dots, p_n)$ with a given signature for any possible $t \leq d$ is still open. Indeed even for torsion free subgroups, this problem appears to be difficult. Curiously, in the cocompact case for the torsion free subgroups, only Riemann–Hurwitz condition is sufficient; cf. [2]. In our case, the result in [2] implies that if $n \geq 3$ and $t \mid d$, there exists a torsion free subgroup of index d whose corresponding surface has t cusps; cf. Theorem 7.2. Here we use a different approach and consider the realizability of torsion free subgroups with $t \leq d$. Special cases are dissussed in Section 7. Some further cases for groups with torsions in the cocompact case are dealt with in [3].

There is a close relation between the Hurwitz problem on realizability of a branched covering of a sphere and the problem of the existence of a subgroup of finite index in $\mathscr{H}(p_1,\dots,p_n)$ [5]. Given a subgroup Γ of index d in $\mathscr{H}(p_1,\dots,p_n)$, let π_Γ: $\mathbf{H}^2 \to \mathbf{H}^2/\Gamma$ be the natural projection. Then π_Γ and $\pi_{\mathscr{H}(p_1,\dots,p_n)}$ induce a branched covering ϕ: $\mathbf{H}^2/\Gamma \to \mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$ of degree d of a punctured sphere $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$. In particular, in any of the cases (i) $p_1 = p_n = 2$, $p_2 = \cdots = p_{n-1} = p$ (ii) $n = 4$, $p_j \geq 4$, $(j = 1,\dots,4)$ (iii) $n = 5$, $p_j \geq 3$, $(j = 1,\dots,5)$ (iv) $n \geq 6$, $p_j \geq 2$, $(j = 1,\dots,n)$, the covering space $\mathbf{H}^2/\Gamma$ of genus g with t cusps of a once-punctured sphere $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$, branched at $\{x_1,\dots,x_k\}$ to order $\{2,\dots,2,p\dots,\ p\}$ for case (i) and to order $\{p_1,\dots,p_1,\dots,p_n,\dots,p_n\}$ for case (ii), (iii), (iv), can be realized for any $t \leq d$; cf. Corollary 7.4 and Corollary 7.8.

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2. Hecke polygons

Let $p_1, p_2, \dots, p_n$, be integers, where each $p_j \geq 2$. For each $j = 1,\dots,n-1$, let C_j be a circle $|z - a_j| = \delta_j$, where $a_j \in \mathbf{R}$, $a_j < a_{j+1}$, and $\delta_j^2 + \delta_{j+1}^2 \leq (a_j - a_{j+1})^2 < (\delta_j + \delta_{j+1})^2$. Then C_j intersects only C_{j-1} and C_{j+1}, for $j = 2,\dots,n-2$. Suppose that C_{j-1} and C_j intersect at a point $b_j \in \mathbf{H}^2$ with an angle π/p_j, for $j = 2,\dots,n-2$. Let $b_1 = a_1 - \delta_1 e^{-\pi i/p_1}$ and $b_n = a_{n-1} + \delta_{n-1} e^{\pi i/p_n}$. Let $\mathscr{D}^*$ be the hyperbolic polygon with vertices at $b_1,\dots,b_n$, and ∞. An *extended generalized Hecke group* $\mathscr{H}^*(p_1,\dots,p_n)$ is a discrete group generated by the reflections in the edges of $\mathscr{D}^*$. The stabilizers of each vertex b_j and each edge of $\mathscr{D}^*$ are $\mathbf{D}_{p_j}$ and Z_2 respectively, where Z_2's are reflections of the dihedral groups $\mathbf{D}_{p_j}$, i.e., the elements in the nonidentity coset of the rotation group Z_{p_j}. Therefore $\mathscr{H}^*(p_1,\dots,p_n)$ is isomorphic to $\mathbf{D}_{p_1} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{p_n}$.

Let $\mathscr{H}(p_1,\dots,p_n)$ be the subgroup of $\mathscr{H}^*(p_1,\dots,p_n)$, called a *generalized Hecke group*, which consists of all orientation-preserving transformations in $\mathscr{H}^*(p_1,\dots,p_n)$. Then $\mathscr{H}(p_1,\dots,p_n)$ is isomorphic to ${\prod^*}_{j=1}^{n} Z_{p_j}$.

We will need the following definitions. The elements of the $\mathscr{H}^*(p_1,p_2,\dots,p_n)$-orbits of b_j and ∞ are called the b_j*-vertices* and the *cusps*, respectively, $j = 1,\dots,n$. Suppose that C_j and the hyperbolic line through a_j and ∞ intersect at a point c_j, for $j = 1,\dots,n-1$ (see Figure 1). The elements of the $\mathscr{H}^*(p_1,\dots,p_n)$-orbits of c_j's are called the c_j*-vertices*. The elements of the $\mathscr{H}^*(p_1,\dots,p_n)$-orbits of the edges joining c_j to ∞ are called the c_j*-edges*. The elements of the $\mathscr{H}^*(p_1,\dots,p_n)$-orbits of the edges joining b_j to ∞ are called b_j*-edges*. The elements of $\mathscr{H}^*(p_1,\dots,p_n)$-orbits of the edges joining b_j to c_j and c_j to b_{j+1} respectively, for $j = 1,\dots,n$, are called e_j-edges and f_j*-edges* respectively. Each of the e_j- and f_j-edges has finite length, and each of the b_j-edges has infinite length. The hyperbolic line joining a_j to ∞ consists of two c_j-edges, for $j = 1,\dots,n-1$.

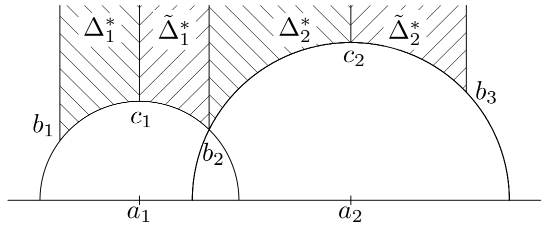

Figure 1. A fundamental polygon $\mathscr{D}^*$ for $\mathscr{H}^*(p_1, p_2, p_3)$.

Its $\mathscr{H}^*(p_1, \dots, p_n)$ translates are called the c_j-*lines*.

The $\mathscr{H}^*(p_1, \dots, p_n)$ translates of the polygon with vertices at $\{b_1, c_1, \infty\}$, $\{c_1, b_2, c_2, \infty\}$, $\dots$, $\{c_{n-2}, b_{n-1}, c_{n-1}, \infty\}$ and $\{c_{n-1}, b_n, \infty\}$, respectively, are called the Ω_j^*-*polygons*. The $\mathscr{H}(p_1, \dots, p_n)$ translates of the polygon with vertices at $\{b_1, a_1, \infty\}$, $\{a_1, b_2, a_2, \infty\}$, $\dots$, $\{a_{n-2}, b_{n-1}, a_{n-1}, \infty\}$ and $\{a_{n-1}, b_n, \infty\}$, respectively, are called the Ω_j-*polygons*. If each p_j is greater than 2, then Ω_1- and Ω_n-polygons are triangles, and the rest of Ω_j-polygons are quadrilaterals.

Let Δ_j^* and $\tilde{\Delta}_j^*$ be triangles with vertices at $\{b_j, c_j, \infty\}$ and $\{c_j, b_{j+1}, \infty\}$, where $j = 1, \dots, n-1$. For $j = 1, \dots, n-1$, let $\Delta_j = \Delta_j^* \cup \sigma_j(\Delta_j^*)$ and $\tilde{\Delta}_j = \tilde{\Delta}_j^* \cup \sigma_j(\tilde{\Delta}_j^*)$, where σ_j is a reflection in the circle C_j.

A usual construction of a fundamental domain for $\mathscr{H}(p_1, \dots, p_n)$ would be $\mathscr{D}^* \cup \sigma(\mathscr{D}^*)$, where σ is a reflection in an edge of $\mathscr{D}^*$. But we find it more convenient to take $\mathscr{D} = \bigcup_{j=1}^{n-1}(\Delta_j \cup \tilde{\Delta}_j)$ as a fundamental domain (see Figure 2).

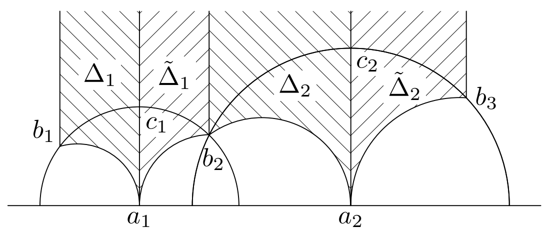

Figure 2. A fundamental polygon for $\mathscr{H}(p_1, p_2, p_3)$.

A *Hecke polygon* is defined to be a convex polygon P whose boundary is a finite union of c_j-lines and b_j-edges satisfying the following conditions:

S$_1$. Each c_j-line in ∂P is paired to another c_j-line in ∂P such that one of them is a side of an Ω_j-polygon in P, and the other is a side of an Ω_{j+1}-polygon in P.

S$_2$. The b_j-edges in ∂P come in pairs. The edges of each pair meet at a b_j-vertex with an interior angle $2k\pi/p_j$, where $k \mid p_j$, and are identified.

S$_3$. $a_1, \dots, a_{n-1}$, and ∞ are among the vertices of P.

The main point about Hecke polygons is the following theorem.

Theorem 2.1. *Let P be a Hecke polygon, and let Γ_P be the subgroup of $\mathscr{H}(p_1, \dots, p_n)$ generated by the side pairing transformations of P. Then P is* *an admissible fundamental domain for Γ_P. Conversely, every subgroup of finite index in $\mathscr{H}(p_1,\dots,p_n)$ admits an admissible fundamental domain which is a Hecke polygon.*

Proof. The argument is similar to the one in Theorem 3.3 [7]. Suppose that P is a Hecke polygon and that Γ_P is the subgroup of $\mathscr{H}(p_1,\dots,p_n)$ generated by the side pairing transformations of P. It follows from the Poincaré polygon theorem [10, Section IV.H] that the set S of the side pairing transformations is an independent set of generators of Γ_P, that is, $\Gamma_P = \prod^*{}_{f\in S}\langle f\rangle$. So the fundamental polygon P is an admissible fundamental domain for Γ_P.

Conversely, suppose that Γ is a subgroup of finite index in $\mathscr{H}(p_1,\dots,p_n)$. Let $\mathscr{T}^*$ be the tessellation of $\mathbf{H}^2$ whose tiles are $\mathscr{H}^*(p_1,\dots,p_n)$ translates of $\mathscr{D}^*$. Let $\varphi\colon \mathbf{H}^2 \to \mathbf{H}^2/\Gamma$ be the canonical projection. Since Γ preserves $\mathscr{T}^*$, we have an induced tessellation $\mathscr{T}^*_\Gamma$ of $\mathbf{H}^2/\Gamma$. The φ-images of c_j-vertices, b_j-vertices, c_j-edges, b_j-edges, e_j- and f_j-edges will again be called c_j-vertices, b_j-vertices, c_j-edges, b_j-edges, e_j- and f_j-edges, respectively, in $\mathbf{H}^2/\Gamma$. Let $\mathscr{E}$ be the union of the e_j- and f_j-edges in $\mathbf{H}^2/\Gamma$. Consider $\mathscr{E}$ as a graph whose vertices are the c_j-vertices and b_j-vertices in $\mathbf{H}^2/\Gamma$, and whose edges are the e_j- and f_j-edges in $\mathbf{H}^2/\Gamma$. Note that each c_j-vertex is of valence 2, and each b_j-vertex is of valence 1 or k (respectively 2 or $2k$), where $k \mid p_j$, if $j=1,n$, (respectively $j=2,\dots,n-1$).

Since the union of the e_j- and f_j-edges in $\mathbf{H}^2$ is connected, so is $\mathscr{E}$. Let T be a maximal tree in $\mathscr{E}$. Let A be the union of all the c_j-edges in $\mathbf{H}^2/\Gamma$ at the c_j-vertices of valence 1 and all the b_j-edges at the b_j-vertices of valence k and $2k$, where $k \mid p_j$, $k \neq p_j$, in T. Make $\mathbf{H}^2/\Gamma$ into a polygon P in $\mathbf{H}^2$ by cutting A such that $a_1,\dots,a_{n-1}$, and ∞ are among the vertices of P. For each c_j-vertex u and each b_j-vertex v in A, there is a pair of c_j-lines and a pair of b_j-edges adjacent to u and v, respectively. Correspondingly, we obtain a pair of c_j-lines (respectively b_j-edges) on ∂P which are paired. Hence P is a Hecke polygon which is a fundamental domain for Γ. ▫

3. A new proof of an extension of Kurosh's theorem

We now give a new proof of an extension of Kurosh's theorem to the groups $\mathscr{H}(p_1,\dots,p_n)$. We mean by a $(k+2)$-gon (respectively $2k+2$-gon) centered at a b_j-vertex a $(k+2)$-gon (respectively $(2k+2)$-gon) consisting of k Ω_j-polygons with a common b_j-vertex which attach to each other along the b_j-edges, where $k \mid p_j$, $j=1,n$ (respectively $j=2,\dots,n-1$), provided that $p_1,p_n \neq 2$ (see Figure 3).

Proof of Theorem 1.4. Let Γ be a subgroup of index d in $\mathscr{H}(p_1,\dots,p_n)$. Let P be the Hecke polygon for Γ, and let s_j be the number of ideal p_j-gons or $2p_j$-gons centered at b_j-vertices in P. Then $s_jp_j+\sum_{i=k_{j-1}+1}^{k_j} m_i$ is the total number of Ω_j-polygons in P, for $j=1,\dots,n$. Hence the conditions (i) and (ii)

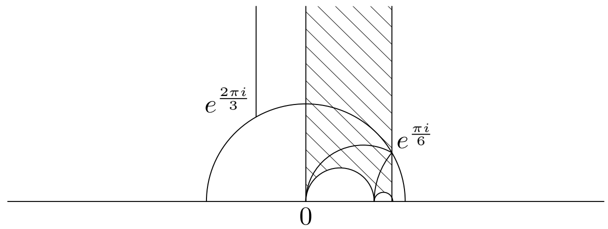

Figure 3. A 5-gon centered at $e^{\pi i/6}$ for a group $\mathscr{H}^*(6,3)$.

follow directly from the Gauss–Bonnet theorem and the geometric interpretation of the Hecke polygon P.

Conversely, suppose that conditions (i) and (ii) hold. Substituting $s_j + \sum_{i=k_{j-1}+1}^{k_j} m_i/p_j$ for d/p_j, $j=1,\dots,n$ in condition (i), we have

$$r = (n-1)d - k_n - \sum_{j=1}^{n} s_j + 1.$$

Without loss of generality, we may assume that $s_1 \ge s_n$ and $s_j \ge 2s_n - 1$, for all $j \ne 1, n$. We shall find a Hecke polygon P for Γ which consists of s_j ideal p_j-gons or $2p_j$-gons (an *ideal* polygon in $\mathbf{H}^2$ is a hyperbolic polygon with vertices at the circle at infinity $\mathbf{R}\cup\{\infty\}$), and the (m_i+2)-gon or $(2m_i+2)$-gon centered at a b_j-vertex, for $i = k_{j-1}+1,\dots,k_j$, and $j=1,\dots,n$, such that a subgroup of $\mathscr{H}(p_1,\dots,p_n)$ generated by the side pairing transformations of P is Γ.

Start with an ideal p_1-gon Q_1 centered at b_1. Then attach an ideal $2p_2$-gon to Q_1 along the c_1-line through ∞ and obtain a new polygon Q_2. Next attach an ideal $2p_3$-gon to Q_2 along the c_2-line through ∞. Continuing in this way, after $2(n-1)s_n - n + 2$ steps we obtain a polygon P_0 whose boundary consists of c_j-lines and which contains s_n Ω_1-polygons, s_n Ω_n-polygons, and $(2s_n-1)$ Ω_j-polygons, for $j=2,\dots,n-1$.

Now there are $s_1 - s_n$ ideal p_1-gons, $s_j - 2s_n + 1$ ideal $2p_j$-gons, for $j = 2,\dots,n-1$, and the (m_i+2)-gon or $(2m_i+2)$-gon centered at a b_j-vertex, for $i = k_{j-1}+1,\dots,k_j$, $j=1,\dots,n$, to be attached. For each $j=1,\dots,n-1$, the number of c_j-lines on the boundary of those polygons and P_0 that are sides of Ω_j-polygons or Ω_{j+1}-polygons is

$$\begin{cases} s_n(p_j-2)+1+(s_j-s_n)p_1+\sum_{i=k_{j-1}+1}^{k_j} m_i, & j=1,n,\\ (2s_n-1)(p_j-1)+(s_j-2s_n+1)p_j+\sum_{i=k_{j-1}+1}^{k_j} m_i, & j\ne 1,n,\end{cases}$$

$$= \begin{cases} d-2s_n+1, & j=1,n,\\ d-2s_n+1, & j\ne 1,n.\end{cases}$$

Hence, after attaching those

$$s_1 - s_n + \sum_{j=2}^{n-1}(s_j - 2s_n + 1) + \sum_{j=1}^{n}(k_j - k_{j-1}) = k_n + \sum_{j=1}^{n} s_j - 2(n-1)s_n + n - 2$$

polygons to P_0, we have

$$(n-1)(d-2s_n+1)-\left[k_n+\sum_{j=1}^{n}s_j-2(n-1)s_n+n-2\right]=(n-1)d-k_n-\sum_{j=1}^{n}s_j+1=r$$

pairs of c_j-lines, and each pair consists of a side of an Ω_j-polygon and a side of an Ω_{j+1}-polygon on the boundary. Therefore we obtain a convex polygon P whose boundary is the union of k_j-k_{j-1} pairs of b_j-edges making an interior angle $2m_i\pi/p_j$, where $i=k_{j-1}+1,\dots,k_j$, $j=1,\dots,n$, and r pairs of c_j-lines.

Each pair of b_j-edges of an interior angle $2m_i\pi/p_j$ are identified. Each pair of $2r$ c_j-lines are identified. Now P becomes a Hecke polygon. Then a subgroup of $\mathscr{H}(p_1,\dots,p_n)$ generated by the side pairing transformations of P is isomorphic to Γ. $\square$

For the case $n=2$ in Theorem 1.4, one can pair the r pairs of c_j-lines on ∂P as in the proof with the desired patterns. We state this result as a corollary of Theorem 1.4.

Theorem 3.1. *Let $k_0=0$, k_1, k_2, g, t, r be nonnegative integers, where $k_1\le k_2$, $t\ge 1$, and $r=2g+t-1$. Let $\Gamma=F_r*\prod^{*\,2}_{j=1}\prod^{*\,k_j}_{i=k_{j-1}+1}Z_{p_j/m_i}$, where $m_i\mid p_j$, $i=k_{j-1}+1,\dots,k_j$, $j=1,2$. Then Γ can be embedded in $\mathscr{H}(p_1,p_2)$ as a subgroup of index d and with a signature*

$$\left(g;\frac{p_1}{m_1},\dots,\frac{p_1}{m_{k_1}},\frac{p_2}{m_{k_1+1}},\dots,\frac{p_2}{m_{k_2}};t\right)$$

if and only if the following conditions hold:

(i) (*The Riemann–Hurwitz condition*)

$$\sum_{j=1}^{2}\sum_{i=k_{j-1}+1}^{k_j}\frac{m_i}{p_j}-(k_2+r)+1=d\left(\frac{1}{p_1}+\frac{1}{p_2}-1\right).$$

(ii) (*The integrality condition*) *The numbers s_1, s_2 satisfying*

$$s_jp_j+\sum_{i=k_{j-1}+1}^{k_j}m_i=d,\qquad j=1,2,$$

are nonnegative integers.

4. Subgroups of finite index in $\mathscr{H}^*(p_1, \ldots, p_n)$

In this section we determine the necessary and sufficient conditions for the existence of a subgroup of finite index of a given type in $\mathscr{H}^*(p_1, \ldots, p_n)$.

Suppose that Γ^* is a subgroup of finite index in $\mathscr{H}^*(p_1, \ldots, p_n)$ containing a reflection. Then $S_{\Gamma^*} = \mathbf{H}^2/\Gamma^*$ is a (possibly nonorientable) surface with boundary (see Figure 4). The boundary ∂S_{Γ^*} is formed by the projection of the fixed lines of reflections in Γ^*. Also, ∂S_{Γ^*} contains a corner when the fixed lines of two reflections in Γ^* intersect. Each component C of ∂S_{Γ^*} is the projection of a simple curve $\widetilde{C}$ in $\mathbf{H}^2$ which is either a finite union of e_j- and f_j-edges or the union of two of the b_j-edges and a finite number of the e_j- and f_j-edges, where any two consecutive edges intersect at a b_j-vertex v, and make an angle $k\pi/p_j$, where $k \mid p_j$. If v is a center of a rotation which is the product of two reflections in Γ^*, the stabilizer of v is isomorphic to a dihedral group $\mathbf{D}_{p_j/k}$.

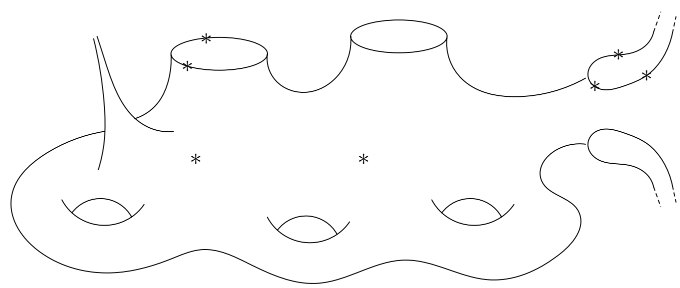

Figure 4. The marked point is an elliptic fixed point if it is in the interior, and a center of a rotation which is a product of two reflections if it is on the boundary.

We generalize the construction of Hecke polygons to extended Hecke polygons.

Definition. An *extended Hecke polygon* is a convex hyperbolic polygon P^* of finite area containing a_1 and ∞ as vertices such that each component of ∂P^* is of one of the following forms:

(i) a c_j-line;

(ii) a pair of b_j-edges making an interior angle $2k\pi/p_j$, where $k \mid p_j$;

(iii) a simple curve which is the union of two of the b_j-edges and a finite number of the e_j- and f_j-edges,

satisfying the following conditions:

$\mathbf{S}_1^*$. Each c_j-line which is a side of an Ω_j-polygon in P^* is paired to another c_j-line which is a side of an Ω_j-polygon in P^* by an orientation-preserving or reversing transformation in $\mathscr{H}^*(p_1, \ldots, p_n)$.

$\mathbf{S}_2^*$. The b_j-edges of each pair as in (ii) are paired by a transformation in $\mathscr{H}(p_1, \ldots, p_n)$.

$\mathbf{S}_3^*$. Each of the e_j- and f_j-edges as in (iii) is paired to itself by a reflection in $\mathscr{H}^*(p_1, \ldots, p_n)$.

$\mathbf{S}_4^*$. Each of the b_j-edges as in (iii) is paired to itself by a reflection or to the other b_j-edge on the same component of ∂P by an orientation-preserving transformation in $\mathscr{H}^*(p_1, \ldots, p_n)$.

Note that the group $\mathscr{H}^*(p_1, \ldots, p_n)$ may contain a subgroup with a fundamental domain whose boundary has only one component, and contains only one cusp ∞ as a vertex. Such a fundamental domain is not an extended Hecke polygon. In this case, this subgroup is isomorphic to $\mathbf{D}_{m_1} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{m_k}$, where each m_i divides some p_j and Z_2's are generated by reflections.

Theorem 4.1. *Let P^* be an extended Hecke polygon, and let Γ_{P^*} be the subgroup of $\mathscr{H}^*(p_1, \ldots, p_n)$ generated by the side pairing transformations of P^*. Then P^* is a fundamental domain for Γ_{P^*}, and Γ_{P^*} is a subgroup of finite index in $\mathscr{H}^*(p_1, \ldots, p_n)$ which is isomorphic to a free product of the groups $\mathbf{Z}$, Z_r, and $\mathbf{D}_{m_1} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{m_k}$, where r, and each m_i divide some p_j. Conversely, every subgroup of finite index in $\mathscr{H}^*(p_1, \ldots, p_n)$ but $\neq \mathbf{D}_{m_1} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{m_k}$, where each m_i divides some p_j, admits an extended Hecke polygon.*

Proof. The proof is similar to that of Theorem 2.1. The first assertion follows from the Poincaré polygon theorem.

Suppose that Γ is a subgroup of finite index in $\mathscr{H}^*(p_1, \ldots, p_n)$. Let $\mathscr{E}^*$ be the union of e_j- and f_j-edges in $\mathbf{H}^2/\Gamma$. Let T^* be the maximal tree in $\mathscr{E}^*$. Let A^* be the union of all the c_j-edges in $\mathbf{H}^2/\Gamma$ at the c_j-vertices of valence 1 and all the b_j-edges at the b_j-vertices of valence k and $2k$ in T^*, where $k \mid p_j$, $k \neq p_j$. Now as in the argument of Theorem 2.1, cut $\mathbf{H}^2/\Gamma$ open along the edges in A^* into a set which is isometric to a simply connected convex hyperbolic polygon P and then obtain an extended Hecke polygon which is a fundamental domain for Γ. □

We take a *positive orientation* on $\mathbf{H}^2$ to be the usual counterclockwise orientation on $\mathbf{H}^2$. Suppose that P^* is an extended Hecke polygon for Γ^*. Let C be a boundary component of $S_{\Gamma^*} = \mathbf{H}^2/\Gamma^*$ which is the projection of a simple curve $\widetilde{C}$ on ∂P^*. Suppose that $\{w_1, w_2, \ldots, w_k\}$ is a set of the b_j-vertices on $\widetilde{C}$ in positive order on ∂P^* such that w_1 and w_k are on the infinite edges. Note that for each j, no two b_j-vertices are adjacent along $\widetilde{C}$. Let π: $\mathbf{H}^2 \to \mathbf{H}^2$ be the projection map. Suppose that the corresponding stabilizer of w_j is $\mathbf{D}_{m_j}$. If $\pi(w_1) \neq \pi(w_k)$, the ordered set $(m_1, m_2, \ldots, m_k)$ is called a *boundary cycle* on $\widetilde{C}$ for P^*. If $\pi(w_1) = \pi(w_k)$, the ordered set $(m_1, m_2, \ldots, m_k)$ is called a *closed boundary cycle* on $\widetilde{C}$ for P^*. Each m_j is called a *branching number on the boundary.* If w_i is a b_j-vertex, the integer p_j/m_i is the number of Ω_j^*-polygons in P^* with a vertex at w_i.

Suppose that $(y_1/x_1, y_2/x_2, \ldots, y_k/x_k)$ is a boundary cycle of Γ^*, where $x_i \mid y_i$ and $y_i \in \{p_1, \ldots, p_n\}$. Let $y_i = p_j$, for some j. Then from the property of a Hecke polygon for Γ^* we have the following results.

(i) $y_1, y_k \in \{p_1, p_n\}$.

(ii) If $k=1$, then x_1 is an even number, and if $k>1$, then x_1 and x_k are odd numbers.

(iii) If x_i is an odd number, then $y_{i-1}=p_{j-1}$, $y_{i+1}=p_{j+1}$, or $y_{i-1}=p_{j+1}$, $y_{i+1}=p_{j-1}$.

(iv) If x_i is an even number, then $y_{i-1}=y_{i+1}=p_{j-1}$, or $y_{i-1}=y_{i+1}=p_{j+1}$.

The above results (i)–(iv) are also true for a close boundary cycle $(y_1/x_1, y_2/x_2,\dots,y_k/x_k)$ except for (i) which now becomes $y_1=y_k\in\{p_1,p_n\}$.

Suppose that

$$\Gamma = F_r * \prod\nolimits^{*n}_{\alpha=1}\prod\nolimits^{*k_\alpha}_{i=k_{\alpha-1}} Z_{p_\alpha/m_i} \\ * \prod\nolimits^{*h_0}_{i=1}\prod\nolimits^{*u_i}_{j=1}(\mathbf{D}_{y_{ij1}/x_{ij1}} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{y_{ija_{ij}}/x_{ija_{ij}}}) * \prod\nolimits^{*h}_{i=h_0+1} E_i$$

is a subgroup of index d in $\mathscr{H}^*(p_1,\dots,p_n)$, where E_i has a presentation

$$\langle f_i, v_{i1},\dots,v_{ia_{i1}} \,|(v_{i1}f_iv_{ia_{i1}}f_i^{-1})^{y_{i1a_{i1}}/x_{i11}+x_{i1a_{i1}}} = v_{il}^2 \\ = v_{il+1}^2 = (v_{il}v_{il+1})^{y_{i1l}/x_{i1l}},\ l=2,\dots,a_{i1}-1\rangle,$$

$x_{ijl}\mid y_{ijl}$, $y_{ijl}\in\{p_1,\dots,p_n\}$, for all i,j,l, and $(x_{i11}+x_{i1a_{i1}})\mid y_{i11}$, for $i=h_0+1,\dots,h$.

Suppose that P is a fundamental polygon for Γ which is an extended Hecke polygon. Let $B_{ij}=(y_{ij1}/x_{ij1}, y_{ij2}/x_{ij2},\dots,y_{ija_{ij}}/x_{ija_{ij}})$, for $i=1,\dots,h$, and $j=1,\dots,u_i$. For each i,j, we will construct a polygon R_{ij} whose boundary contains a corresponding boundary component for B_{ij}. For instance, assume that $B_{ij}=(p_1/x_{ij1}, p_2/x_{ij2},\dots,p_n/x_{ijn})$. Start with an $(x_{ij1}+2)$-gon Q_1 centered at a b_1-vertex $\overline{w}_1$ such that ∂Q_1 has precisely one component C_1 consisting of two b_1-edges (respectively one b_1-edge, one e_1-edge and one c_1-edge) if $n=1$ (respectively $n>1$). If $n>1$, attach a $(2x_{ij}+2)$-gon Q_2 centered at a b_2-vertex $\overline{w}_2$ to Q_1 along a c_1-edge on C_1 such that the vertices $\overline{w}_1$ and $\overline{w}_2$ are on $\partial(Q_1\cup Q_2)$ in positive order. Continuing in this way, we obtain a polygon R_{ij} such that the vertices $w_{ij},\overline{w}_1,\overline{w}_2,\dots,\overline{w}_n,w'_{ij}$ are on ∂R_{ij} in positive order, where w_{ij} and w'_{ij} are the cusps on the b_1-edge through $\overline{w}_1$ and the b_n-edge through $\overline{w}_n$, respectively. Note that ∂R_{ij} contains c_j-lines through w_{ij} and w'_{ij}. Hence we can think of P as a polygon $P-\bigcup_{i=1}^h\bigcup_{j=1}^{u_i}R_{ij}$ attached to R_{ij} along the c_j-line through w_{ij} or w'_{ij}, $i=1,\dots,h$, $j=1,\dots,u_i$.

Let $k_0=0$. Apply the Gauss–Bonnet theorem to P. It follows that

$$\frac{d}{2}\left(\sum_{\alpha=1}^{n}\frac{1}{p_\alpha}-n+1\right)=\sum_{\alpha=1}^{n}\sum_{i=k_{\alpha-1}+1}^{k_\alpha}\frac{m_i}{p_\alpha}+\sum_{i,j,l}\frac{x_{ijl}}{2y_{ijl}} \\ -\left(k_n+\frac{1}{2}\sum_{i=1}^{h}\sum_{j=1}^{u_i}a_{ij}+\frac{1}{2}\sum_{i=1}^{h}u_i+r\right)+h-h_0+1.$$

On the other hand, for each $\alpha = 1, \dots, n$, since the number of Ω^*_α-polygons is equal to d, there exists a nonnegative integer s_α such that

$$2s_\alpha p_\alpha + \sum_{i=k_{\alpha-1}+1}^{k_\alpha} 2m_i + \sum_{i=1}^{h}\sum_{j=1}^{u_i}\sum_{l\in\Phi_{ij}(\alpha)} x_{ijl} = d$$

where $\Phi_{ij}(\alpha) = \{l \mid y_{ijl} = p_\alpha,\ 1 \le l \le a_{ij}\}$.

Conversely, all the above equalities are also sufficient for a subgroup to exist in $\mathscr{H}^*(p_1, \dots, p_n)$. We will use the previous notations to describe and prove this result.

Theorem 4.2. *Let $k_0 = 0, k_1, \dots, k_n, h_0, h, r$ be nonnegative integers, where $k_i \le k_{i+1}$, for $i = 1, \dots, n$, and $h_0 \le h$. Suppose that*

$$\begin{aligned}\Gamma = F_r * {\prod}^{*n}_{\alpha=1}{\prod}^{*k_\alpha}_{i=k_{\alpha-1}} Z_{p_\alpha/m_i} \\ * {\prod}^{*h_0}_{i=1}{\prod}^{*u_i}_{j=1}(\mathbf{D}_{y_{ij1}/x_{ij1}} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{y_{ija_{ij}}/x_{ija_{ij}}}) * {\prod}^{*h}_{i=h_0+1} E_i,\end{aligned}$$

where E_i has a presentation

$$\begin{aligned}\langle f_i, v_{i1}, \dots, v_{ia_{i1}} \mid (v_{i1} f_i v_{ia_{i1}} f_i^{-1})^{y_{i1a_{i1}}/x_{i11}+x_{i1a_{i1}}} = v_{il}^2 \\ = v_{il+1}^2 = (v_{il}v_{il+1})^{y_{i1l}/x_{i1l}},\ l = 2, \dots, a_{i1}-1\rangle,\end{aligned}$$

$x_{ijl} \mid y_{ijl}$, $y_{ijl} \in \{p_1, \dots, p_n\}$, for all i, j, l, and $(x_{i11} + x_{i1a_{i1}}) \mid y_{i11}$, for $i = h_0+1, \dots, h$. Then Γ can be embedded in $\mathscr{H}^*(p_1, \dots, p_n)$ as a subgroup of index d if and only if the following conditions are satisfied:

(i) (*The Riemann–Hurwitz condition*)

$$\begin{aligned}\frac{d}{2}\left(\sum_{\alpha=1}^{n}\frac{1}{p_\alpha} - n + 1\right) = \sum_{\alpha=1}^{n}\sum_{i=k_{\alpha-1}+1}^{k_\alpha}\frac{m_i}{p_\alpha} + \sum_{i=1}^{h}\sum_{j=1}^{u_i}\sum_{l=1}^{a_{ij}}\frac{x_{ijl}}{2y_{ijl}} \\ - \left(k_n + \frac{1}{2}\sum_{i=1}^{h}\sum_{j=1}^{u_i} a_{ij} + \frac{1}{2}\sum_{i=1}^{h} u_i + r\right) + h - h_0 + 1.\end{aligned}$$

(ii) (*The integrality condition*) *The numbers $s_1, \dots, s_n$ satisfying*

$$2s_\alpha p_\alpha + \sum_{i=k_{\alpha-1}+1}^{k_\alpha} 2m_i + \sum_{i=1}^{h}\sum_{j=1}^{u_i}\sum_{l\in\Phi_{ij}(\alpha)} x_{ijl} = d, \qquad \alpha = 1, \dots, n,$$

are nonnegative integers.

Proof. The proof of the necessity is in the above argument. Conversely, if (i) and (ii) are satisfied, it follows that

$$r=(n-1)\frac{d}{2}-k_n-\frac{1}{2}\sum_{i=1}^{h}\sum_{j=1}^{u_i}a_{ij}-\frac{1}{2}\sum_{i=1}^{h}u_i-\sum_{\alpha=1}^{n}s_\alpha+h-h_0+1. \tag{1}$$

We will construct an extended Hecke polygon P which consists of s_α ideal p_α- or $2p_\alpha$-polygons, the $(m_\alpha+2)$- or $(2m_\alpha+2)$-gons, for $i=1,\dots,k_n$, $\alpha=1,\dots,n$, and the polygons R_{ij}, for $i=1,\dots,h$, $j=1,\dots,u_i$, such that a subgroup generated by the side pairing transformations of P is isomorphic to Γ.

Let P_0 be a polygon in $\mathbf{H}^2$ whose boundary consists of c_j-lines as we constructed in the proof of Theorem 1.4. For each $\alpha=1,\dots,n-2$, let μ_α and μ'_α be the numbers of polygons among those s_1-s_n ideal p_1-gons and (m_i+2)-gons if $\alpha=1$, and the $s_\alpha-2s_n+1$ ideal $2p_\alpha$-gons and $(2m_i+2)$-gons if $\alpha\neq 1$, which are attached to P_0 or any other polygon along the c_α-lines and the $c_{\alpha+1}$-lines, respectively, where $i=k_\alpha+1,\dots,k_{\alpha+1}$. Call this new polygon P_1. We will prove that after attaching the polygons R_{ij} to P_1 to obtain a polygon P, the numbers of c_α-lines on ∂P which are sides of Ω_α-polygons and $\Omega_{\alpha+1}$-polygons, respectively, are the same, where $\alpha=1,\dots,n-1$, and there are r pairs of such c_j-lines on ∂P.

Suppose that for $i=1,\dots,h$, $j=1,\dots,u_i$, there exist integers $\xi_{ij\alpha}$, $\sigma_{ij\alpha\beta_1}$, $\eta_{ij\alpha}$, $\tau_{ij\alpha\beta_2}$, $\zeta_{ij\alpha}$, and $\rho_{ij\alpha\beta_3}$, where $\alpha=1,\dots,n-1$, $\beta_1=1,\dots,\xi_{ij\alpha}$, $\beta_2=1,\dots,\xi_{ij\eta_{ij\alpha}}$, and $\beta_3=1,\dots,\zeta_{ij\alpha}$, such that in each boundary cycle B_{ij}, there are $\xi_{ij\alpha}$ collections of branching numbers on the boundary of type (I_α):

$$\frac{p_\alpha}{l_1},\ \frac{p_{\alpha+1}}{l_2},\ \dots,\ \frac{p_\alpha}{l_{2\sigma_{ij\alpha\beta}-1}},\ \frac{p_{\alpha+1}}{l_{2\sigma_{ij\alpha\beta}}}\quad\text{or}\quad\frac{p_{\alpha+1}}{l_1},\ \frac{p_\alpha}{l_2},\ \dots,\ \frac{p_{\alpha+1}}{l_{2\sigma_{ij\alpha\beta}-1}},\ \frac{p_\alpha}{l_{2\sigma_{ij\alpha\beta}}},$$

$\eta_{ij\alpha}$ collections of branching numbers on the boundary of type (II_α):

$$\frac{p_\alpha}{l_1},\ \frac{p_{\alpha+1}}{l_2},\ \dots,\ \frac{p_{\alpha+1}}{l_{2\tau_{ij\alpha\beta}-1}},\ \frac{p_\alpha}{l_{2\tau_{ij\alpha\beta}}},$$

and $\zeta_{ij\alpha}$ collections of branching numbers on the boundary of type (III_α):

$$\frac{p_{\alpha+1}}{l_1},\ \frac{p_\alpha}{l_2},\ \dots,\ \frac{p_\alpha}{l_{2\rho_{ij\alpha\beta}-1}},\ \frac{p_{\alpha+1}}{l_{2\rho_{ij\alpha\beta}}}.$$

Since each boundary cycle except for the types (II_1) and (III_{n-1}) starts and ends up with branching numbers on the boundary of types (I_1) or (I_{n-1}), we have the following equation:

$$u_i=\sum_{j=1}^{u_i}\left[\tfrac{1}{2}(\xi_{ij1}+\xi_{ijn-1})+\eta_{ij1}+\zeta_{ijn-1}\right],\quad i=1,\dots,h. \tag{2}$$

Note that for each $\alpha = 1, \ldots, n-1$, $l_1, l_{2\sigma ij\alpha\beta}, l_{2\tau ij\alpha\beta}$, and $l_{2\rho ij\alpha\beta}$ are odd numbers, and any other $l_2, \ldots, l_{2\sigma ij\alpha\beta-1}$ (or $l_{2\tau ij\alpha\beta-1}$, or $l_{2\rho ij\alpha\beta-1}$) are even numbers. Also, the number of the branching numbers $p_{\alpha+1}/l_{2\sigma ij\alpha\beta}$ or $p_{\alpha+1}/l_1$ as in type (I_α) and $p_{\alpha+1}/l_1$, $p_{\alpha+1}/l_{2\rho ij\alpha\beta}$ as in type (II_α) is equal to the number of the same branching numbers as in type ($\mathrm{I}_{\alpha+1}$) and ($\mathrm{II}_{\alpha+1}$). Then we have

$$\xi_{ij\alpha} + 2\zeta_{ij\alpha} = \xi_{ij\alpha+1} + 2\eta_{ij\alpha+1}, \quad \alpha = 1, \ldots, n-2, \tag{3}$$

and

$$\begin{aligned} a_{ij} &= \sum_{\alpha=1}^{n-1}\left[\sum_{\beta=1}^{\xi_{ij\alpha}} 2\sigma_{ij\alpha\beta} + \sum_{\beta=1}^{\eta_{ij\alpha}}(2\tau_{ij\alpha\beta}-1) + \sum_{\beta=1}^{\zeta_{ij\alpha}}(2\rho_{ij\alpha\beta}-1)\right] \\ &\quad - \sum_{\alpha=1}^{n-2}(\xi_{ij\alpha} + 2\zeta_{ij\alpha}) \\ &= 2\sum_{\alpha=1}^{n-1}\left(\sum_\beta \sigma_{ij\alpha\beta} + \sum_\beta \tau_{ij\alpha\beta} + \sum_\beta \rho_{ij\alpha\beta}\right) \\ &\quad - \sum_{\alpha=1}^{n-1}(\xi_{ij\alpha} + \eta_{ij\alpha} + 3\zeta_{ij\alpha}) \\ &\quad + \xi_{ijn-1} + 2\zeta_{ijn-1}, \end{aligned} \tag{4}$$

where $i = 1, \ldots, h$, $j = 1, \ldots, u_i$.

To compute the numbers of c_j-lines, let $\varepsilon(\alpha)$ and $\delta(\alpha)$ be the numbers of c_α-lines on ∂P and ∂R_{ij}, for $i = 1, \ldots, h$, $j = 1, \ldots, u_i$, which are sides of Ω_α-polygons and $\Omega_{\alpha+1}$-polygons, respectively, where $\alpha = 1, \ldots, n-1$. First, we have

$$\begin{aligned} \varepsilon(1) = {} & \left(s_1p_1 + \sum_{i=1}^{k_1} m_i - 2s_n + 1\right) - (s_1 - s_n) - k_1 - \mu_1 - \mu_1' + \sum_{i,j}\sum_{l\in\Phi_{ij}(1)} \tfrac{1}{2}x_{ijl} \\ & - \sum_{i,j}\sum_{l\in\Phi_{ij}(1)} \#(x_{ijl} \text{ is even}) - \frac{1}{2}\sum_{i,j}\sum_{l\in\Phi_{ij}(1)} \#(x_{ijl} \text{ is odd}) \\ = {} & \left(s_1p_1 + \sum_{i=1}^{k_1} m_i - 2s_n + 1\right) - (s_1 - s_n) - k_1 - \mu_1 - \mu_1' + \sum_{i,j}\sum_{l\in\Phi_{ij}(1)} \tfrac{1}{2}x_{ijl} \\ & - \sum_{i,j}\left[\sum_{\beta=1}^{\xi_{ij1}}(\sigma_{ij1\beta} - 1) + \sum_{\beta=1}^{\eta_{ij1}}(\tau_{ij1\beta} - 2) + \sum_{\beta=1}^{\zeta_{ij1}}(\rho_{ij1\beta} - 1)\right] \\ & - \frac{1}{2}\sum_{i,j}(\xi_{ij1} + 2\eta_{ij1}) \end{aligned}$$

$$= \tfrac{1}{2}d - 2s_n + 1 - (s_1 - s_n) - k_1 - \mu_1 - \mu_1'$$
$$- \sum_{i,j}\Bigl(\sum_{\beta}\sigma_{ij1\beta} + \sum_{\beta}\tau_{ij1\beta} + \sum_{\beta}\rho_{ij1\beta}\Bigr) + \sum_{i,j}\bigl(\tfrac{1}{2}\xi_{ij1} + \eta_{ij1} + \zeta_{ij1}\bigr),$$

and

$$\begin{aligned}
\delta(1) &= \Bigl(s_2p_2 + \sum_{i=k_1+1}^{k_2} m_i - 2s_n + 1\Bigr) - (s_1 - s_n) \\
&\quad - k_1 - \mu_1 - \mu_1' + \sum_{i,j}\sum_{l\in\Phi_{ij}(2)} \tfrac{1}{2}x_{ijl} \\
&\quad - \sum_{i,j}\sum_{l\in\Phi_{ij}(2)} \#(x_{ijl} \text{ is even}) - \frac{1}{2}\sum_{i,j}\sum_{l\in\Phi_{ij}(2)} \#(x_{ijl} \text{ is odd}) \\
&= \Bigl(s_2p_2 + \sum_{i=k_1+1}^{k_2} m_i - 2s_n + 1\Bigr) - (s_1 - s_n) \\
&\quad - k_1 - \mu_1 - \mu_1' + \sum_{i,j}\sum_{l\in\Phi_{ij}(2)} \tfrac{1}{2}x_{ijl} \\
&\quad - \sum_{i,j}\Bigl[\sum_{\beta=1}^{\xi_{ij1}}(\sigma_{ij1\beta} - 1) + \sum_{\beta=1}^{\eta_{ij1}}(\tau_{ij1\beta} - 1) + \sum_{\beta=1}^{\zeta_{ij1}}(\rho_{ij1\beta} - 2)\Bigr] \\
&\quad - \frac{1}{2}\sum_{i,j}(\xi_{ij1} + 2\zeta_{ij1})
\end{aligned}$$

$$\begin{aligned}
&= \tfrac{1}{2}d - 2s_n + 1 - (s_1 - s_n) - k_1 - \mu_1 - \mu_1^{'} \\
&\quad - \sum_{i,j}\Bigl(\sum_{\beta}\sigma_{ij1\beta} + \sum_{\beta}\tau_{ij1\beta} + \sum_{\beta}\rho_{ij1\beta}\Bigr) \\
&\quad + \sum_{i,j}\bigl(\tfrac{1}{2}\xi_{ij1} + \eta_{ij1} + \zeta_{ij1}\bigr) = \varepsilon(1).
\end{aligned}$$

Similarly, for $\alpha = 2, \dots, n-2$,

$$\begin{aligned}
\varepsilon(\alpha) &= \tfrac{1}{2}d - 2s_n + 1 - (s_\alpha - 2s_n + 1 - \mu_{\alpha-1}) - (k_\alpha - k_{\alpha-1} - \mu_{\alpha-1}^{'}) - \mu_\alpha - \mu_\alpha' \\
&\quad - \sum_{i,j}\Bigl(\sum_{\beta}\sigma_{ij\alpha\beta} + \sum_{\beta}\tau_{ij\alpha\beta} + \sum_{\beta}\rho_{ij\alpha\beta}\Bigr) \\
&\quad + \sum_{i,j}\bigl(\tfrac{1}{2}\xi_{ij\alpha} + \eta_{ij\alpha} + \zeta_{ij\alpha}\bigr) = \delta(\alpha),
\end{aligned}$$

and

$$\begin{aligned}\varepsilon(n-1) = \tfrac{1}{2}d - 2s_n + 1 - (s_{n-1} - 2s_n + 1 - \mu_{n-1}) - (k_{n-1} - k_{n-2} - \mu'_{n-1}) \\ - (k_n - k_{n-1}) - \sum_{i,j}\Bigl(\sum_\beta \sigma_{ijn-1\beta} + \sum_\beta \tau_{ijn-1\beta} + \sum_\beta \rho_{ijn-1\beta}\Bigr) \\ + \sum_{i,j}\bigl(\tfrac{1}{2}\xi_{ijn-1} + \eta_{ijn-1} + \zeta_{ijn-1}\bigr) = \delta(n-1).\end{aligned}$$

This implies that after attaching those polygons R_{ij}, for $i = 1,\dots,h$, $j = 1,\dots,u_i$, to P_1, which is called a polygon P, the numbers of c_α-lines on ∂P which are sides of Ω_α-polygons and sides of $\Omega_{\alpha+1}$-polygons, respectively, where $\alpha = 1,\dots,n-1$, are the same.

On the other hand, from equations (1), (2), (3), and (4), it follows that

$$\sum_{\alpha=1}^{n-1}\varepsilon(\alpha) = \sum_{\alpha=1}^{n-1}\delta(\alpha) = r + \sum_{i=1}^{h_0} u_i.$$

This proves that on ∂P, there are $r-(h-h_0)$ pairs of c_j-lines, and each pair of them are sides of an Ω_α-polygon and an $\Omega_{\alpha+1}$-polygon. Therefore we obtain a convex polygon P whose boundary is the union of $k_j - k_{j-1}$ pairs of b_j-edges making an interior angle $2m_i\pi/p_j$, where $i = k_{j-1}+1,\dots,k_j$, $j = 1,\dots,n$, r pairs of c_j-lines, and the e_j- and f_j-edges on R_{ij} corresponding to B_{ij}.

Let each pair of those c_j-lines be identified, and each pair of b_j-edges in an interior angle $2m_i\pi/p_j$ be identified. For $i = 1,\dots,h$, let each of an e_j-edge and an f_j-edge on $R_{ij}\cap\partial P$ be identified with itself by a reflection. Let each b_j-edge on $R_{ij}\cap\partial P$ be identified with itself by a reflection if $i = 1,\dots,h_0$, and be identified with the other b_j-edge on $R_{ij}\cap\partial P$ by an orientation-preserving transformation if $i = h_0+1,\dots,h$. Now P becomes an extended Hecke polygon. Hence a subgroup generated by the side pairings of P is isomorphic to Γ. ▫

5. Special cases

Theorem 5.1. *Suppose that $p_1,\dots,p_n$ are distinct primes. Let*

$$\begin{aligned}\Gamma = F_r * \textstyle\prod^{*n}_{\alpha=1} \underbrace{(Z_{p_\alpha} * \cdots * Z_{p_\alpha})}_{k_\alpha} \\ * \textstyle\prod^{*h_0}_{i=1}\prod^{*u_i}_{j=1}(\mathbf{D}_{y_{ij1}/x_{ij1}} *_{Z_2} \cdots *_{Z_2} \mathbf{D}_{y_{ija_{ij}}/x_{ija_{ij}}}) * \prod^{*h}_{i=h_0+1} E_i,\end{aligned}$$

where each E_i has a presentation as in Theorem 4.2. Then Γ can be embedded in $\mathscr{H}^*(p_1,\dots,p_n)$ as a subgroup of finite index d if and only if the Riemann–Hurwitz *condition holds, i.e.*

(5)
$$\frac{d}{2}\left(\sum_{\alpha=1}^{n}\frac{1}{p_\alpha}-n+1\right)=\sum_{\alpha=1}^{n}\frac{k_\alpha}{p_\alpha}+\sum_{i=1}^{h}\sum_{j=1}^{u_i}\sum_{l=1}^{a_{ij}}\frac{x_{ijl}}{2y_{ijl}}$$
$$-\left(\sum_{\alpha=1}^{n}k_\alpha+\frac{1}{2}\sum_{i=1}^{h}\sum_{j=1}^{u_i}a_{ij}+\frac{1}{2}\sum_{i=1}^{h}u_i+r\right)+h-h_0+1,$$

and for each $\alpha=1,\dots,n$, $d-2k_\alpha-\sum_{i=1}^{h}\sum_{j=1}^{u_i}\sum_{l\in\Phi_{ij}(\alpha)}x_{ijl}$ *is a nonnegative even integer, where* $\Phi_{ij}(\alpha)=\{l \mid y_{ijl}=p_\alpha,\ 1\le l\le a_{ij}\}$.

Proof. It is sufficient to prove that the integrality condition follows from the two conditions in the theorem. Let $\beta\in\{1,\dots,n\}$. Multiplying $\prod_{\alpha=1}^{n}p_\alpha$ to equation (5), we have

$$\left(\prod_{\alpha\neq\beta}p_\alpha\right)\left(d-2k_\beta-\sum_{i,j}\sum_{l\in\Phi_{ij}(\beta)}x_{ijl}\right)$$
$$=\left(\prod_{\alpha=1}^{n}p_\alpha\right)\left[(n-1)d-d\sum_{\alpha\neq\beta}\frac{1}{p_\alpha}+2\sum_{\alpha\neq\beta}\frac{k_\alpha}{p_\alpha}+\sum_{i,j}\sum_{l\notin\Phi_{ij}(\beta)}\frac{x_{ijl}}{y_{ijl}}\right.$$
$$\left.-\left(2\sum_{\alpha}k_\alpha+\sum_{i,j}a_{ij}+\sum_{i}u_i+2r\right)+2(h-h_0)+2\right].$$

Note that the right-hand side of this equation is a nonnegative even integer divisible by p_β. Hence there is a nonnegative integer s_β such that

$$2s_\beta p_\beta+2k_\beta+\sum_{i,j}\sum_{\Phi_{ij}(\beta)}x_{ijl}=d. \ \square$$

In particular, if Γ as in Theorem 5.1 contains only orientation-preserving transformations, then $h=0$ and the index d is an even number. Therefore we have the following corollary.

Corollary 5.2. *Let* $p_1,\dots,p_n$ *be distinct primes, and*

$$\Gamma=F_r*{\prod^*}_{j=1}^{n}\Big(\underbrace{Z_{p_j}*\cdots*Z_{p_j}}_{k_j}\Big).$$

Then Γ *can be embedded in* $\mathscr{H}(p_1,\dots,p_n)$ *as a subgroup of finite index* d *if and only if the Riemann–Hurwitz condition*

$$d\left(\sum_{j=1}^{n}\frac{1}{p_j}-n+1\right)=\sum_{j=1}^{n}\frac{k_j}{p_j}-\left(\sum_{j=1}^{n}k_j+r\right)+1$$

holds and $d\ge k_j$, *for* $j=1,\dots,n$.

6. Hecke polygons with associated permutations

We will show how to associate a collection of permutations to a Hecke polygon.

Suppose that Γ is a subgroup of index d in $\mathscr{H}(p_1, \ldots, p_n)$. By Theorem 2.1, Γ has a fundamental domain P which is a Hecke polygon. Suppose that P consists of s_j ideal p_j-gons or $2p_j$-gons $Q_{j1}, \ldots, Q_{js_j}$, which are a union of p_j Ω_j-polygons, and the m_i+2-gon or $2m_i+2$-gon Q_{ji} centered at a b_j-vertex, which is a union of m_i Ω_j-polygons, for $i = s_j+1, \ldots, s_j+k_j-k_{j-1}$, $j = 1, \ldots, n$.

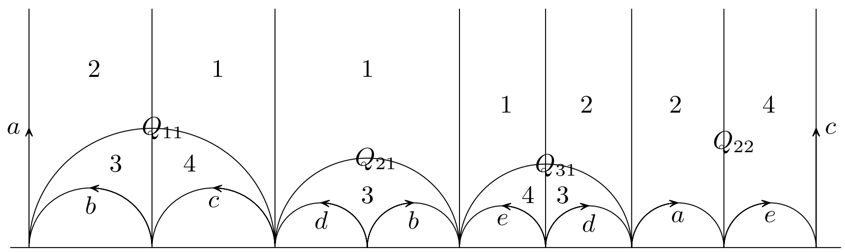

Figure 5. For the case of $\mathscr{H}(4,2,4)$, a Hecke polygon $P = Q_{11} \cup Q_{21} \cup Q_{22} \cup Q_{31}$ with the indicated side pairings, where $A_1(Q_{11}) = (1\ 2\ 3\ 4)$, $A_2(Q_{21}) = (1\ 3)$, $A_2(Q_{22}) = (2\ 4)$, $A_3(Q_{31}) = (1\ 4\ 3\ 2)$.

We assign an element in $\{1, \ldots, d\}$ to each of Ω_j-polygons in P for each j as follows (see also Figure 5). For $j = 1, \ldots, n$, let A_j be a function of a collection $\mathscr{M}_j$ of Ω_j-polygons onto $\{1, \ldots, d\}$ such that

(1) $A_j(R_1) \neq A_j(R_2)$, for any two elements $R_1, R_2 \in \mathscr{M}_j$;

(2) $A_j(R_1) = A_{j+1}(R_2)$, if $R_1 \in \mathscr{M}_j$, $R_2 \in \mathscr{M}_{j+1}$, and they have an identified c_j-line.

Write all the elements of $A_j(Q_{ji})$ in counterclockwise order, say $\{l_1, \ldots, l_r\}$. An element $(l_1, \ldots, l_r)$ of a symmetric group S_d is called a *permutation associated to* Q_{ji}. Let α_j be a product of the permutations associated to Q_{ji}, $i = 1, \ldots, s_j + k_j - k_{j-1}$, $j = 1, \ldots, n$. Note that α_j is a product of disjoint s_j p_j-cycles and m_i-cycles, $i = k_{j-1}+1, \ldots, k_j$. Then $\{\alpha_1, \ldots, \alpha_n\}$ is called a *system of permutations associated to* P or Γ with respect to p_j's and m_i's.

In fact, the group $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$. For, if two elements a, b of $\{1, \ldots, d\}$ are in disjoint cycles of α_1, say $a \in A_1(Q_{11})$, $b \in A_1(Q_{12})$, by the connectivity of the set $\mathscr{E}$ of the union of the e_j- and f_j-edges in P there is a path of e_j- and f_j-edges in $\mathscr{E}$, for some $j = j_1, \ldots, j_r$, which connects the b_1-vertex in Q_{11} to the b_1-vertex in Q_{12}. Note that any of the e_j- and f_j-edges in the same Q_{ji} can be mapped to an e_j- or f_j-edge under some power of α_j. Then a gets mapped to b through some powers of $\alpha_1, \alpha_{j_1}, \ldots, \alpha_{j_r}, \alpha_1$, respectively. Hence any Hecke polygon gives us a group of permutations in S_d acting transitively on $\{1, \ldots, d\}$.

On the other hand, we will show that the number of cusps on P/Γ is the number of disjoint cycles of $\sigma = \alpha_n \cdots \alpha_1$. If x is a cusp on P/Γ, then there is a sequence of c_j-lines and b_j-edges around x as follows. Start with a c_1-line or a b_1-edge L_1 through x on an Ω_1-polygon R_1 in P such that R_1 remains on the left when we walk along L_1 toward x. Suppose that $A_1(R_1) = 1$. Let R_2 be an Ω_1-polygon, possibly $R_1 = R_2$, with $A_1(R_2) = \alpha_1(1)$. Let L_2 be a c_1-line of R_2 which contains a cusp equivalent to x. Then there is a c_1-line M_2 on an Ω_2-polygon R_3 with $A_2(R_3) = \alpha_1(1)$. Again there is a c_2-line L_3 on an Ω_2-polygon R_4 which contains a cusp equivalent to x with $A_2(R_4) = \alpha_2\alpha_1(1)$. Continuing this way, we generate a sequence of edges $\{L_1, L_2, M_2, \ldots, L_n, M_n\}$ each of which contains a cusp equivalent to x, and a sequence of Ω_j-polygons $\{R_1, R_2, \ldots, R_{2n-1}, R_{2n}\}$ such that $A_j(R_{2j-1}) = \alpha_{j-1}\cdots\alpha_1(1)$, $A_j(R_{2j}) = \alpha_j\cdots\alpha_1(1)$, where $j = 1, \ldots, n$, and $\alpha_0 =$ identity.

Next there is an Ω_{n-1}-polygon R_{2n+1}, an Ω_{n-2}-polygon R_{2n+2}, $\ldots$, and an Ω_2-polygon R_{3n-2} attached to M_n cyclically, where $A_2(R_{3n-2}) = \cdots = A_{n-1}(R_{2n+1}) = A_n(R_{2n})$. If $A_n(R_{2n}) \neq 1$, then repeat the same argument for an edge on an Ω_1-polygon R_{3n-1} with $A_1(R_{3n-1}) = A_n(R_{2n})$. This will stop at the l-th step when $A_n(R_{2ln}) = 1$ (see Figure 6).

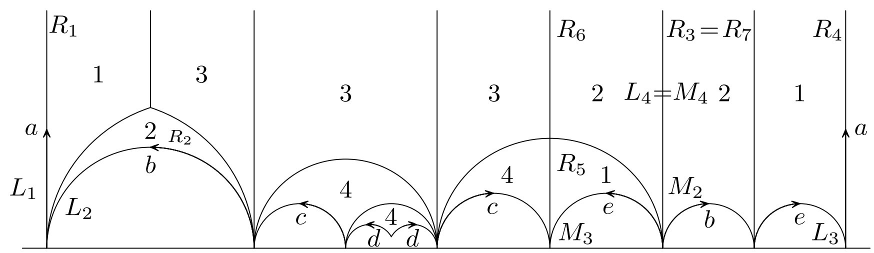

Figure 6. The subgroup generated by the side pairings of this Hecke polygon is of index 4 in $\mathscr{H}(3, 2, 4)$.

From the above observation we see that the number of cusps on P/Γ is exactly the number of disjoint cycles of σ.

Conversely, given p_j's and m_i's satisfying the conditions (i) and (ii) in Theorem 1.4, let α_j be a permutation of disjoint s_j p_j-cycles and m_i-cycles, $i = k_{j-1}+1, \ldots, k_j$, $j = 1, \ldots, n$. Suppose that $\langle\alpha_1, \ldots, \alpha_n\rangle$ is acting transitively on $\{1, \ldots, d\}$, and $\sigma = \alpha_n\cdots\alpha_1$ has t cycles. We will construct a Hecke polygon P such that a group generated by the side pairings of P has a signature $(g; p_1/m_1, \ldots, p_n/m_{k_n}; t)$.

For each j, let Q_{ji}, be an ideal p_j-gon or an $2p_j$-gon, for $i = 1, \ldots, s_j$, and an m_i+2-gon or a $2m_i+2$-gon, for $i = s_j+1, \ldots, s_j+k_j-k_{j-1}$. If $(i_1, \ldots, i_r)$ is a cycle of α_j, assign those elements $i_1, \ldots, i_r$ cyclically in counterclockwise order to Ω_j-polygons of some Q_{ji} in which the number of Ω_j-polygons is r. Then this Q_{ji} with those assigned numbers is called a *polygon associated to* $(i_1, \ldots, i_r)$.

Let a, b be any two elements of $\{1, \ldots, d\}$. Since $\langle\alpha_1, \ldots, \alpha_n\rangle$ is acting transitively on $\{1, \ldots, d\}$, there are some permutations, say, $\alpha_1, \ldots, \alpha_q$ such that

$$\alpha_q^{l_q}\cdots\alpha_1^{l_1}(a) = b,$$

for some powers $l_1, \dots, l_q$. Then there is a cycle β_j in α_j, for each $j = 1, \dots, q$, such that $\beta_q^{l_q} \cdots \beta_1^{l_1}(a) = b$. Let $z_j = \beta_j^{l_j} \cdots \beta_1^{l_1}(a)$, $j = 1, \dots, q$. Then $z_1 = \beta_1^{l_1}(a) \in \beta_1 \cap \beta_2$, $z_2 \in \beta_2 \cap \beta_3$, $\dots$, $z_{q-1} \in \beta_{q-1} \cap \beta_q$, $b = z_q \in \beta_q$.

Suppose that Q_{j1} is a polygon associated to β_j, $j = 1, \dots, q$. Then there is an Ω_j-polygon $R_j \subset Q_{j1}$ and an Ω_{j+1}-polygon $R'_j \subset Q_{j+11}$ with $A_j(R_j) = A_{j+1}(R'_j) = z_j$, for $j = 1, \dots, q-1$. Hence Q_{j+11} can be attached to Q_{j1} along the c_j-lines which are sides of R_j and R'_j, $j = 1, \dots, q-1$. Call this polygon P_0. Suppose that $Q_{q+11}, \dots, Q_{n1}$ are the polygons whose associated permutations contain b. Let W_j be an Ω_j-polygon contained in Q_{j1} with $A_j(W_j) = b$, for $j = q+1, \dots, n$. Now attach $Q_{q+11}, \dots, Q_{n1}$ to P_0 along the c_j-lines which are sides of $W_{q+1}, \dots, W_n$. Call this polygon P_1. Similarly, the Q_{ji}'s whose associated permutations contain a can be attached to P_1. Call this polygon P_2. Hence all the polygons Q_{ji}'s whose associated permutations contain a and b are attached together.

Since any element in $\{1, \dots, d\}$ is mapped to a under some permutation in $\langle \alpha_1, \dots, \alpha_n \rangle$, all the rest of the Q_{ji}'s can be attached to P_2 in the same way as above. Call this polygon P. The boundary of P consists of c_j-lines and pairs of b_j-edges making an angle $2m_i\pi/p_j$.

Before the Q_{ji}'s are attached to each other, there are d c_j-lines on Q_{ji}'s and $Q'_{j+1i}s$, respectively. When any two of the Q_{ji}'s are attached along a c_j-line, we lose one c_j-line from each of Q_{ji}'s and Q_{j+1i}'s. Hence the number of c_j-lines on ∂P which are sides of Ω_j-polygons and Ω_{j+1}-polygons, respectively, is the same. Therefore, to find the number of c_j-lines on ∂P, it is sufficient to count the number of c_j-lines on ∂P which are sides of Ω_j-polygons, where $j = 1, \dots, n-1$. There are $(n-1)d$ c_j-lines on Q_{ji}'s, $j = 1, \dots, n-1$, altogether. Then after attaching those $\sum_{j=1}^n s_j + \sum_{j=1}^n (k_j - k_{j-1})$ polygons Q_{ji}'s, there are

$$r = (n-1)d - \sum_{j=1}^{n} s_j - k_n + 1$$

c_j-lines on ∂P which are sides of Ω_j-polygons, $j = 1, \dots, n-1$.

Now pair a c_j-line of an Ω_j-polygon U_j on ∂P to a c_j-line of an Ω_{j+1}-polygon U_{j+1} on ∂P with $A_j(U_j) = A_{j+1}(U_{j+1})$. Any two b_j-edges on ∂P making an interior angle $2m_i\pi/p_j$ are identified. Then P together with those side pairings becomes a Hecke polygon.

Moreover, if Γ is a subgroup of $\mathscr{H}(p_1, \dots, p_n)$ generated by the side pairings of P, we see by using the previous argument that the number of cusps of the surface $\mathbf{H}^2/\Gamma$ is t, and Γ has a signature $(g; p_1/m_1, \dots, p_n/m_{k_n}; t)$, where $r = 2g + t - 1$. Therefore we proved the following theorem.

Theorem 6.1. *Let $k_0 = 0, k_1, \dots, k_n, g, t, r$ be nonnegative integers, where $k_i \le k_{i+1}$, for $i = 1, \dots, n-1$, $t \ge 1$, and $r = 2g + t - 1$. Let m_i be positive integers, where $m_i \mid p_j$, $i = k_{j-1}+1, \dots, k_j$, $j = 1, \dots, n$. Then $\mathscr{H}(p_1, \dots, p_n)$*

contains a subgroup of index d *with a signature* $(g; p_1/m_1, \dots, p_n/m_{k_n}; t)$ *if and only if*

(i) *The numbers* r, p_j*'s* m_i*'s and* s_j*'s satisfy the Riemann–Hurwitz and integrality conditions as in Theorem* 1.4.

(ii) *For* $j = 1, \dots, n$, *there exists a permutation* α_j *in* S_d *such that*

(a) α_j *is a product of disjoint* p_j*-cycles (in all* s_j *of them) and* m_i*-cycles,* $i = k_{j-1} + 1, \dots, k_j$.

(b) *The group* $\langle \alpha_1, \dots, \alpha_n \rangle$ *acts transitively on* $\{1, \dots, d\}$.

(c) *The permutation* $\sigma = \alpha_n \cdots \alpha_1$ *has* t *disjoint cycles.*

7. Branched coverings of punctured spheres

In this section we will construct branched coverings of a punctured sphere $\mathbf{H}^2/\mathscr{H}(p_1, \dots, p_n)$ by applying Theorem 6.1 or using a Hecke polygon.

First note that the Riemann–Hurwitz and integrality conditions are not sufficient for the existence of a subgroup with a given signature if $n \geq 3$ (see the following example).

Example. Consider a torsion free subgroup isomorphic to F_7 of index 6 with a signature $(0; 8)$ in $\mathscr{H}(2,3,6)$. Here, $\chi(F_7) = -6$ and $\chi\big(\mathscr{H}(2,3,6)\big) = -1$. Integrality conditions are also satisfied. However, by Proposition 7.1 below F_7 cannot be regarded as a subgroup of $\mathscr{H}(2,3,6)$ with a signature $(0; 8)$.

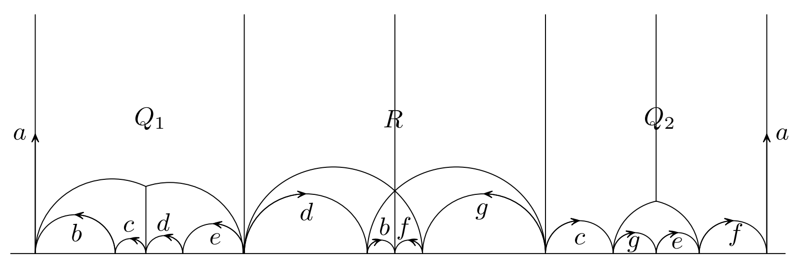

Figure 7. The c_1- and c_2-lines marked by the same letters are identified.

Note that F_7 can be regarded as a subgroup in $\mathscr{H}(2,3,6)$ with a signature $(1; 6)$, or $(2; 4)$, or $(3; 2)$. Indeed, take two ideal hexagons Q_1, Q_2 which both consist of six Ω_2-polygons and one ideal hexagon R which consists of six Ω_3-polygons. Glue those hexagons together along the c_2-lines through ∞ to get a polygon P as in Figure 7. Let the c_1-lines through ∞ on ∂P be identified. Then the side pairings of P

$$abcded^{-1}b^{-1}fgc^{-1}g^{-1}e^{-1}f^{-1}a^{-1}$$
$$abcdeb^{-1}d^{-1}fgc^{-1}g^{-1}e^{-1}f^{-1}a^{-1}$$
$$abcdeb^{-1}d^{-1}fgc^{-1}f^{-1}e^{-1}g^{-1}a^{-1}$$

correspond to subgroups with signatures $(1; 6)$ (see Figure 7), $(2; 4)$ and $(3; 2)$, respectively.

Proposition 7.1. *Suppose that* Γ *is a subgroup of index* d *in* $\mathscr{H}(p_1,\dots,p_n)$ *whose number of cusps is* t. *Then we have a partition* $d=\sum_{i=1}^{t} d_i$. *In particular* $t\le d$.

Proof. The surface $\mathbf{H}^2/\Gamma$ is a branched cover of degree d of the once-punctured sphere $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$. Let $\{\tilde{x}_1,\dots,\tilde{x}_t\}$ and x be the cusps in $\mathbf{H}^2/\Gamma$ and $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$. Compactify $\mathbf{H}^2/\Gamma$ and $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$ by filling with cusps. The original branched covering is extended to the one of degree d between the compactified surfaces. Then there are t points $\{\tilde{x}_1,\dots,\tilde{x}_t\}$ in the fiber of x. Let $\tilde{\gamma}_i$ be a simple closed curve around $\tilde{x}_i$. It projects in $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$ as a (not necessarily simple) closed curve γ_i around x. Let d_i be the winding number of γ_i around x. Then $d=\sum_{i=1}^{t} d_i$. In particular $t\le d$. ◻

The result in [2] implies that in our case, if $n\ge 3$ and $t\mid d$, then t can be realized as the number of cusps of a subgroup of index d.

Theorem 7.2. *Suppose that* $n\ge 3$, $lcm(p_1,\dots,p_n)\mid d$ *and* $t\mid d$. *Then there exists a torsion free subgroup of index* d *with a signature* $(g;t)$ *in* $\mathscr{H}(p_1,\dots,p_n)$ *if* $2g+t-2=d\bigl(n-1-\sum_{j=1}^{n} 1/p_j\bigr)$.

Proof. Let $d=kt$, and let $E,V_1,\dots,V_{n+1}$ be integers satisfying $E=(n+1)d$, $p_jV_j=d$, where $j=1,\dots,n+1$, and $p_{n+1}=k$. Then $\sum_{j=1}^{n+1} V_j-E+2d=2-2g$. From Theorem 1.3 in [2], there is a tessellation of a surface M of genus g into $2d$ $(n+1)$-gons with E edges and $\sum_{j=1}^{n+1} V_j$ vertices, V_j of valence $2p_j$, $j=1,\dots,n+1$, such that each face has vertices of valence $2p_1,\dots,2p_{n+1}$, up to cyclic order. Remove the vertices of valence $2k$ from M to obtain a *topological* surface X of genus g with t cusps. Then X is a branched cover of a once-punctured sphere S with n branch points $P_1,\dots,P_n$ with branching numbers $p_1,\dots,p_n$.

Let $\pi\colon X\to S$ be the corresponding projection map. If $\pi_S\colon \mathbf{H}^2\to S$ is the universal branched covering with branching numbers $p_1,\dots,p_n$, the covering group of $\pi_S\colon \mathbf{H}^2\to S$ is isomorphic to $\mathscr{H}(p_1,\dots,p_n)$. Then π_S factors through $\mathbf{H}^2\xrightarrow{\pi_X}X\xrightarrow{\pi}S$. The set $\pi^{-1}(P_j)$ is precisely the V_j vertices of the tessellation of X lying over P_j. The condition $p_jV_j=d$ ensures that $\mathbf{H}^2\xrightarrow{\pi_X}X$ is an *unbranched* covering. This corresponds to a torsion free subgroup Γ of the covering group of $\pi_S\colon \mathbf{H}^2\to S$. To realize Γ as a subgroup of the *Fuchsian* group $\mathscr{H}(p_1,\dots,p_n)$, we proceed as follows.

First add a vertex to each compact edge of X as a "midpoint". Next on each face of X, add all noncompact edges through a cusp and any other vertices. Now each face has $n+1$ triangles. Let T be one of these triangles. Replace T by one of the triangles Δ_j^*'s and $\tilde{\Delta}_j^*$'s, called $\widetilde{T}$, as in Section 2. On the interior of $\widetilde{T}$, we have a well-defined hyperbolic metric. These $\widetilde{T}$'s can be glued along edges by uniquely defined isometries. So at the end, we get a *complete* Riemannian metric of constant curvature -1 on X. This extends uniquely to a universal covering

$\mathbf{H}^2 \to X$ which is an isometry on each component of the inverse image on each face. Therefore X is homeomorphic to $\mathbf{H}^2/\Gamma$, for some torsion free subgroup Γ of index d in $\mathscr{H}(p_1, \ldots, p_n)$. ▫

We now discuss some special cases for the realizability of signatures by subgroups of finite index in $\mathscr{H}(p_1, \ldots, p_n)$. We suppose that n and p_j's satisfy any of the following conditions: (i) $p_1 = p_n = 2$, $p_2 = \cdots = p_{n-1} = p$; (ii) $n = 4$, $p_j \geq 4$, $(j = 1, \ldots, 4)$; (iii) $n = 5$, $p_j \geq 3$, $(j = 1, \ldots, 5)$; (iv) $n \geq 6$, $p_j \geq 2$, $(j = 1, \ldots, n)$.

Theorem 7.3. *Let g and $t \geq 1$ be nonnegative integers. Suppose that p and d are positive integers, where $p \geq 2$ and d is divisible by $lcm(2,p)$. Then $\mathscr{H}(2, \underbrace{p, \ldots, p}_{n}, 2)$ contains a torsion free subgroup of index d with a signature $(g;t)$ if and only if $2g + t = (1 - 1/p)nd + 2$ and $t \leq d$.*

Proof. The necessity of the conditions follow from the Riemann–Hurwitz condition and Proposition 7.1. We will prove the sufficiency by constructing a Hecke polygon.

First, take d/p ideal $2p$-gons centered at b_j-vertices, for each $j = 1, \ldots, n$. Glue these ideal polygons together along the c_j-lines through ∞ to obtain a polygon P. Then we can identify the c_j-lines on ∂P with the desired pattern to have a surface of genus g with t cusps. ▫

Corollary 7.4. *Let Γ be a torsion free subgroup of index d in $\mathscr{H}(2, \underbrace{p, \ldots, p}_{n}, 2)$, where $p \geq 2$ and d is divisible by $lcm(2,p)$. Then the surface $\mathbf{H}^2/\Gamma$ of genus g with t cusps is a branched cover of degree d of the once-punctured sphere $\mathbf{H}^2/\mathscr{H}(2, p, \ldots, p, 2)$ branched at all b_j-vertices to order $\{\underbrace{2, \ldots, 2}_{d}, \underbrace{p, \ldots, p}_{nd/p}\}$ if and only if $2g + t = (1 - 1/p)nd + 2$ and $t \leq d$.*

Theorem 7.5. *Suppose that $n \geq 3$, $p_j \geq 2$, $j = 1, \ldots, n$, and d is divisible by $lcm(p_1, \ldots, p_n)$. Let $g, t \geq 1$ be nonnegative integers, and let $d = k_j p_j$, for $j = 1, \ldots, n$. If $2g + t = (n-1)d - \sum_{j=1}^{n} k_j + 2$ and $t \leq \min\{d, (n-2)d - \sum_{j=1}^{n} k_j + 3\}$, then $\mathscr{H}(p_1, \ldots, p_n)$ contains a torsion free subgroup of index d with a signature $(g;t)$.*

Proof. To find a subgroup with a signature $(g;t)$ amounts to choosing an appropriate α_j in S_d, $j = 1, \ldots, n$, such that $\alpha_n \cdots \alpha_1$ has disjoint t cycles.

There will be two cases. First, suppose that $d = 4$. Then each p_j is either 2 or 4. If each p_j equals 2, then the result follows directly from Theorem 7.3. Suppose that there are n_1 of those p_j's equal to 2 and n_2 of them equal to 4. Then $t = 2n + n_2 - 2 - 2g$.

If n_2 is an odd number, t is also an odd number, namely 1 or 3. Consider two collections

$$\mathscr{E}_1 = \{\underbrace{[2,2],\ldots,[2,2]}_{n_1}, \underbrace{[4],\ldots,[4]}_{n_2}, [4]\}$$

and

$$\mathscr{E}_2 = \{\underbrace{[2,2],\ldots,[2,2]}_{n_1}, \underbrace{[4],\ldots,[4]}_{n_2}, [1,1,2]\}$$

of partitions of d. Then the total branchings [3] $v(\mathscr{E}_1) = 2n_1 + 3n_2 + 3$ and $v(\mathscr{E}_2) = 2n_1 + 3n_2 + 1$ are even numbers.

If n_2 is an even number, t is also an even number, namely 2 or 4. Consider two collections

$$\mathscr{E}_3 = \{\underbrace{[2,2],\ldots,[2,2]}_{n_1}, \underbrace{[4],\ldots,[4]}_{n_2}, [2,2]\}$$

and

$$\mathscr{E}_4 = \{\underbrace{[2,2],\ldots,[2,2]}_{n_1}, \underbrace{[4],\ldots,[4]}_{n_2}, [1,1,1,1]\}$$

of partitions of d. Then the total branchings $v(\mathscr{E}_3) = 2n_1 + 3n_2 + 2$ and $v(\mathscr{E}_4) = 2n_1 + 3n_2$ are also even numbers. Moreover, for $j = 1,2,3,4$, $v(\mathscr{E}_j) \geq 2n_1 + 3n_2 = 2n + n_2 \geq 2d - 2 = 6$, because $n \geq 3$. Therefore $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$ and $\mathscr{E}_4$ are realizable by Complement 5.6 in [3].

Secondly, suppose that $d \neq 4$. Let $\mathscr{F} = \{A_1,\ldots,A_{n+1}\}$ be a collection of partitions of d, where $A_j = [p_j,\ldots,p_j]$, for $j = 1,\ldots,n$, and $A_{n+1} = [m_1,\ldots,m_t]$ with $\sum_{i=1}^{t} m_i = d$. The total branching is

$$v(\mathscr{F}) = \sum_{j=1}^{n}(p_j - 1)k_j + \sum_{i=1}^{t}(m_i - 1) = (n+1)d - \sum_{j=1}^{n} k_j - t.$$

Since $t \leq (n-2)d - \sum_{j=1}^{n} k_j + 3$, it follows that $v(\mathscr{F}) \geq 3(d-1)$. By Theorem 5.4 in [3], $\mathscr{F}$ is realized as the branch data of a connected branched covering of a closed sphere S^2. Hence, by Lemma 2.1 in [3], for each $j = 1,\ldots,n$, there exists α_j in S_d which is a product of k_j disjoint p_j-cycles such that $\alpha_1 \cdots \alpha_n$ is a product of disjoint m_i-cycles, $i = 1,\ldots,t$, and $\langle \alpha_1,\ldots,\alpha_n \rangle$ acts transitively on $\{1,\ldots,d\}$. □

Corollary 7.6. *Suppose that $n \geq 3$, $p_j \geq 2$, $j = 1,\ldots,n$, and d is divisible by $lcm(p_1,\ldots,p_n)$. Let $d = k_j p_j$, for $j = 1,\ldots,n$. If $2g + t = (n-1)d - \sum_{j=1}^{n} k_j + 2$ and $t \leq \min\{d, (n-2)d - \sum_{j=1}^{n} k_j + 3\}$, then $\mathscr{H}(p_1,\ldots,p_n)$ contains a torsion free subgroup Γ of index d such that $\mathbf{H}^2/\Gamma$ is a surface of genus g with t cusps which is a branched cover of degree d of the once-punctured sphere $\mathbf{H}^2/\mathscr{H}(p_1,\ldots,p_n)$, branched at all b_j-vertices to order $\{\underbrace{p_1,\ldots,p_1}_{k_1},\ldots,\underbrace{p_n,\ldots,p_n}_{k_n}\}$.*

We would like to know whether or not the Riemann–Hurwitz condition and the condition on t in Theorem 7.5 are also necessary for the existence of a subgroup of $\mathscr{H}(p_1,\dots,p_n)$ with a prescribed signature. In fact, in each of the cases: (i) $n=4$, $p_j\ge 4$, $(j=1,\dots,4)$; (ii) $n=5$, $p_j\ge 3$, $(j=1,\dots,5)$; (iii) $n\ge 6$, $p_j\ge 2$, $(j=1,\dots,n)$, it follows that $(n-2)d-\sum_{j=1}^n k_j+3\ge d$. Then the sufficient condition on t in Theorem 7.5 (which is now reduced to $t\le d$) is the same as the necessary end-condition in Proposition 7.1. Those consequences are stated as the following theorem.

Theorem 7.7. *Suppose that n and p_j, $j=1,\dots,n$, satisfy any of the following conditions*:

(i) $n=4$, $p_j\ge 4$, $j=1,\dots,4$; (ii) $n=5$, $p_j\ge 3$, $j=1,\dots,5$; (iii) $n\ge 6$, $p_j\ge 2$, $j=1,\dots,n$, *and that d is divisible by $lcm(p_1,\dots,p_n)$. Let $g,t\ge 1$ be nonnegative integers, and let $d=k_jp_j$, for $j=1,\dots,n$. Then $\mathscr{H}(p_1,\dots,p_n)$ contains a torsion free subgroup of index d with a signature $(g;t)$ if and only if $2g+t=(n-1)d-\sum_{j=1}^n k_j+2$ and $t\le d$.*

Corollary 7.8. *Suppose that n and p_j, $j=1,\dots,n$, satisfy any of the following conditions*: (i) $n=4$, $p_j\ge 4$, $j=1,\dots,4$; (ii) $n=5$, $p_j\ge 3$, $j=1,\dots,5$; (iii) $n\ge 6$, $p_j\ge 2$, $j=1,\dots,n$, *and that d is divisible by $lcm(p_1,\dots,p_n)$. Let Γ be a torsion free subgroup of index d in $\mathscr{H}(p_1,\dots,p_n)$. Then the surface $\mathbf{H}^2/\Gamma$ of genus g with t cusps is a branched cover of degree d of the once-punctured sphere $\mathbf{H}^2/\mathscr{H}(p_1,\dots,p_n)$, branched at all b_j-vertices to order* $\{\underbrace{p_1,\dots,p_1}_{k_1},\dots,\underbrace{p_n,\dots,p_n}_{k_n}\}$ *if and only if $2g+t=(1-1/p)nd+2$ and $t\le d$.*

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