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Note on span-symmetric generalized quadrangles

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Abstract. We determine all span-symmetric generalized quadrangles of order (s,t) for which $t < s^2$.

In a generalized quadrangle $\mathscr{Q}$ of order (s,t) with $s,t > 1$, a line L is called an *axis of symmetry* if the group $T(L)$ of all automorphisms ("symmetries") that fix every line meeting L has the maximal possible order s. Moreover, $\mathscr{Q}$ is called *span-symmetric* if there are two disjoint axes of symmetry. These notions were introduced in [6] and [10] in view of the known examples ($\mathrm{Q}(4,q)$ and $\mathrm{Q}(5,q)$, arising respectively from quadrics in 4- and 5-dimensional projective spaces). In this note we will prove the following

Theorem. *Any span-symmetric generalized quadrangle for which* $t \neq s^2$ *is isomorphic to* $\mathrm{Q}(4,s)$.

The hypotheses provide two disjoint axes of symmetry, L and M, and hence also the group $G = \langle T(L), T(M) \rangle$ of automorphisms they generate. The proof of the theorem is an elementary combination of the classification of 2-transitive permutation groups in which the stabilizer of a point has a normal subgroup regular on the remaining points ([4], [9]), standard results about central extensions of such groups ([8], [1], [3]), and the fact that $|G| = (s+1)s(t-1)$ ([10, IV.2], [7, 10.7.3]) (proved combinatorially using eigenvalue techniques!).

The case $s < t$ of the theorem was announced long ago [5] and mentioned in ([10, p. 88], [7, p. 225]); the simplification and variation of that proof in the special case $s = t$ were noticed one week after [5]. The proof given below is straightforward, and hence was never published. Publication at this point stems from the need for the impossibility of $s < t < s^2$ in lovely new results of K. Thas on span-symmetric generalized quadrangles [12]. The case $s = t$ of the theorem was also obtained by him [11], independently, using the exact same results ([4], [9], [8], [1], [3]), but handling the sharply 2-transitive possibility differently (employing geometric results in place of elementary group theory).

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Proof of the Theorem. By [7, pp. 224–225], $s \leqslant t$, the orbit $L^G = \{L_0, \dots, L_s\}$ has size $s+1$, and $G = \langle T(L), T(M) \rangle$ acts on this set as a 2-transitive permutation group of just the sort we noted above was classified in [4] and [9] (here $T(L)$ is the required normal subgroup of the stabilizer G_L of the line L). Thus, if K is the kernel of this action, then G/K is one of the following: (i) $\mathrm{PSL}(2,q)$, $s = q$; (ii) $\mathrm{PSU}(3,q)$, $s = q^3$; (iii) a Suzuki group $\mathrm{Sz}(q)$, $s = q^2$; (iv) a Ree group $\mathrm{R}(q)$, $s = q^3$; or (v) a sharply 2-transitive group. We will view the cases $q \leqslant 3$ of (i) and $q = 2$ of (ii) and (iii) as lying in case (v), so that G/K is a simple group in (i)–(iv) unless $G/K \cong R(3) \cong \mathrm{P\Gamma L}(2,8)$.

The groups K and $T(L)$ normalize one another, and hence $[K, T(L)] \leqslant K \cap T(L)$, where $K \cap T(L) = 1$ by ([10, p. 85], [7, p. 225]), here $[K, T(L)] := \langle k^{-1}u^{-1}ku \mid k \in K, u \in T(L) \rangle$. Thus, K commutes with each $T(L_i)$ and hence is contained in the center $Z(G)$ of G. Consequently, $K = Z(G)$ since $Z(G/K) = 1$. Note that $|G_{LM}/K|\,|K| = |G_{LM}| = |G|/(s+1)s = t-1$.

In (i)–(iv) we claim that G equals its derived group G' if we (temporarily) exclude the case $G/K \cong R(3)$. For, $(G/K)' = G/K$, so that $G = G'K$. If $t = s$ then $(|T(L)|, |K|)$ divides $(s, t-1) = 1$, so that $T(L) \leqslant G'$ and hence $G = G'$. For general t we use the structure of G_L/K in order to show that $T(L) \leqslant G'$: in each of the groups we are considering in (i–iv), $T(L)K/K \cong T(L)$ lies in $(G_L/K)'$. Since the actions of G_L on $T(L)$ and $T(L)K/K$ are equivalent, it follows that $T(L) \leqslant G'$ and hence that $G \leqslant G'$ for any t, as claimed.

Consequently, G is a group such that $G = G'$ and $G/Z(G)$ is one of the groups in (i)–(iv). The references ([8], [1], [3]) obtain the unique (up to isomorphism) *largest* group $H = H'$ such that $H/Z(H)$ is isomorphic to one of the groups in (i)–(iv), so that $G \cong H/H_0$ for some $H_0 \leqslant Z(H)$.

With this preparation, we can now consider the individual cases (i)–(v).

(i) Here $G \cong \mathrm{PSL}(2,q)$ or $\mathrm{SL}(2,q)$, unless q is 4 or 9 and $G \cong 2.\mathrm{PSL}(2,4)$, $3.\mathrm{PSL}(2,9)$ or $6.\mathrm{PSL}(2,9)$ [8, p. 119]. Since $q = s \leqslant t$ and $|G| = (q+1)q(t-1)$ it follows that $G \cong \mathrm{SL}(2,q)$ and $s = t$: the possibilities $s = 4$ and $t-1 = 2\cdot 3$, as well as $s = 9$ and $t-1 = 3\cdot 4$ or $6\cdot 4$, are all eliminated by the standard divisibility condition $(s+t) \mid st(s+1)(t+1)$ [PT2, 1.2.2]. The subgroups $T(L_i)$ are uniquely determined as the Sylow subgroups of G for the prime dividing q. Since $\mathcal{Q}$ is uniquely reconstructible from G and the $T(L_i)$ ([6, p. 235], [7, p. 227]), $\mathcal{Q}$ is as stated in the theorem.

(ii) Here $G \cong \mathrm{PSU}(3,q)$ or $\mathrm{SU}(3,q)$ [3] and $s = q^3$, so that $t-1 = |G|/(s+1)s$ is $(q^2-1)/3$ or $q^2-1 < s-1$, a contradiction.

(iii) Here $G \cong \mathrm{Sz}(q)$, $2.\mathrm{Sz}(8)$ or $2^2.\mathrm{Sz}(8)$ [1] and $s = q^2$, which produce the contradiction $t-1 = |G|/(s+1)s \leqslant 4(q-1) < s-1$.

(iv) If $q \neq 3$ then $G \cong \mathrm{R}(q)$ [1] and $s = q^3$ produce the contradiction $t-1 = |G|/(s+1)s = q-1 < s-1$.

Suppose that $q = 3$. Then G/K has a normal subgroup $S/K \cong \mathrm{PSL}(2,8)$ of index 3, and $|T(L) \cap S| = |(T(L) \cap S)K/K| = 9$. We can apply an earlier argument to the subgroup H generated by the G—conjugates of $T(L) \cap S$: we have $HK/K = S/K$, $T(L) \cap H = T(L) \cap S$, $T(L) \cap H \cong (T(L) \cap H)K/K \leqslant (H_L K/K)'$ and hence $T(L) \cap H \leqslant H'$. Then $H = H'$ and $H/Z(H) \cong \mathrm{PSL}(2,8)$, so that $Z(H) = 1$ and $H \cong \mathrm{PSL}(2,8)$ by [Sch]. Since H is transitive on the G—conjugates of $T(L) \cap H$ it is transitive on the conjugates of $T(L)$, so that $HT(L)$ contains all such conjugates and hence is G. Now $|G| = |H| \cdot |T(L) : T(L) \cap H| = |R(3)|$ produces the same contradiction as before.

(v) This and (iv) with $q = 3$ are the only cases requiring some effort. Here $s + 1 = p^e$ for some prime p, and there is an elementary abelian normal subgroup N/K of order p^e. Since $K \leqslant Z(N)$, N is nilpotent and hence has a unique Sylow p-subgroup P. Since P is transitive on $L^G = \{L_0, \dots, L_s\}$, the group $\langle P, T(L) \rangle = P \cdot T(L)$ contains all of the groups $T(L_i)$ and hence is just G. Thus, $K \leqslant P$.

Since $|G/K| = (s+1)s$ we have $|K| = t - 1$. We may assume that $t > 3$ [7, Ch. 6]. If $s = t$ then $s - 1$ and $s + 1$ are both powers of p, so that $s - 1 \leqslant 2$, which is not the case.

This concludes the proof when $s = t$. It remains to derive a contradiction when $s < t$. Clearly, $P' \leqslant K$. Maschke's Theorem [2, pp. 66, 177] implies that $P/P' = (K/P') \times (B/P')$ for some subgroup B normalized by $T(L)$ and hence also by $P \cdot T(L) = G$. As above, it follows that $G = \langle B, T(L) \rangle = B \cdot T(L)$ and hence that $K/P' = 1$, so that $P' = K = Z(G)$.

Let $x \in P - K$. For any $y, z \in P$ we have $[x, yz] = [x, y][x, z]$ (cf. [2, p. 18]). Thus, $A := \{[x, y] \mid y \in P\}$ is a subgroup of $[P, P] = P' = K$. Here, A depends only on the coset xK of x, while $[x, yK] = [x, y]$ for any y and $[x, K] = 1$, so that $|A| \leqslant |P/K| - 1 = s$. The 2-transitivity of G/K implies that G_L acts transitively (by conjugation) on the set of nontrivial cosets xK of K in P, while centralizing K and hence A. Thus, A is the same for each such coset xK, and hence $A = [P, P] = K$.

Now $|K| = |A| \leqslant s < t = |K| + 1$, and hence $t = s + 1$, whereas $s + t$ must divide $st(s+1)(t+1)$. □

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