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Affine-regular hexagons of extreme areas inscribed in a centrally symmetric convex body

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Abstract. Let M be a planar centrally symmetric convex body. If H is an affine regular hexagon of the smallest (the largest) possible area inscribed in M, then M contains (respectively, the interior of M does not contain) an additional pair of symmetric vertices of the affine-regular 12-gon T_H whose every second vertex is a vertex of H. Moreover, we can inscribe in M an octagon whose three pairs of opposite vertices are vertices of an affine-regular hexagon H and the remaining pair is a pair of opposite vertices of T_H. A corollary concerns packing M with its three homothetical copies. Another corollary is that the unit disk of any Minkowski plane contains three points in distances at least $1 + \sqrt{3}/3$.

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Every non-degenerate affine image of the regular n-gon is called an *affine-regular n-gon*. It is well known that in every convex body we can inscribe at least one affine-regular hexagon. This was proved by Besicovitch [1]. In the case of a centrally symmetric convex body M, for every direction there is an affine regular hexagon *abcdef* inscribed in M such that the side *ab* is of this direction (this was established in many papers and the priority seems to belong to Gołąb [4]). What is more, the vertices of M vary continuously along the boundary of M as the prescribed direction of the side *ac* rotates (see [6]).

Our basic aim is to consider the positions of H in M with the smallest possible area and with the greatest possible area. We show that in the first case M contains an additional pair of symmetric vertices of the affine-regular 12-gon T_H whose every second vertex is a vertex of H, and that in the second case the interior of M does not contain a pair of symmetric vertices of T_H.

Applying continuity arguments, we conclude that there is an affine-regular hexagon H inscribed in M such that at least one additional pair of symmetric vertices of T_H is in the boundary of M. In other words, in M we can inscribe a centrally symmetric octagon with four pairs of vertices at some vertices of an affine-regular 12-gon and such that the convex hull of three of those pairs is an affine-regular hexagon. Recall here the result of Grünbaum [5] who proved that in every centrally symmetric planar convex body we can inscribe an affine-regular octagon.

Theorems on affine-regular polygons inscribed in convex bodies have many applications. Also our results lead to some corollaries. A corollary asserts that every centrally symmetric convex body M can be packed with three homothetical copies of M of ratio $\frac{4+\sqrt{3}}{13} \approx 0.441$. This ratio is not far from the value $\frac{5+2\sqrt{2}}{17} \approx 0.4605$ conjectured by Doyle, Lagarias and Randall [3], and also, in an equivalent form, by the author [7]. Another corollary says that the boundary of the unit disk of any Minkowski plane contains three points in equal distances at least $1 + \frac{1}{3}\sqrt{3} \approx 1.577$. This improves the estimate $1.546\ldots$ of Bezdek, Fodor and Talata [2].

1 Inscribed affine-regular hexagons of extreme areas

Theorem 1. *Consider the family $\mathscr{H}$ of all affine-regular hexagons inscribed in a planar centrally symmetric convex body M. For each $H \in \mathscr{H}$ denote by T_H the affine-regular* 12*-gon whose every second vertex is a vertex of H. If $H \in \mathscr{H}$ has the smallest possible area, then M contains at least one additional pair of symmetric vertices of T_H. If $H \in \mathscr{H}$ has the greatest possible area, then the interior of M contains at most two additional pairs of symmetric vertices of T_H.*

Proof. We present the proof of the first statement. The proof of the second is analogous; basically it is sufficient to exchange every symbol $\leqslant$ into the symbol $\geqslant$.

Denote by o the center of M and H. Let $H = h_1h_2h_3h_4h_5h_6$ (see Figure 1). By the

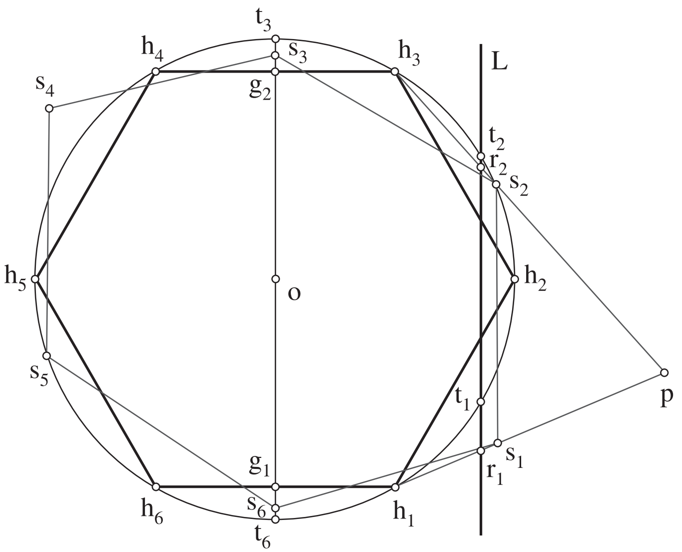

Figure 1

definition of the affine-regular hexagon, we do not make our considerations narrower assuming that H is a regular hexagon of sides of length 1. Denote by g_1 the midpoint of h_6h_1 and by g_2 the midpoint of h_3h_4.

Since M is centrally symmetric, we can inscribe in M an affine-regular hexagon with a side of any given direction ([4]). Let $S = s_1s_2s_3s_4s_5s_6$ be such an affine-regular hexagon inscribed in M with a side parallel to g_1g_2. We choose the notation such that on the boundary of M we successively find $h_1, s_1, h_2, s_2, \ldots$. There are points $t_1, \ldots, t_6$ such that $T_H = h_1t_1h_2t_2h_3t_3h_4t_4h_5t_5h_6t_6$. Since the hexagon H is regular, also the 12-gon T_H is regular. Observe that the points $o, g_1, g_2, s_3, s_6, t_3, t_6$ are collinear.

If $|os_3| \geqslant 1$, that is, if $|s_6s_3| \geqslant 2$, then $t_3 \in M$ and also $t_6 \in M$, and thus the first statement of Theorem 1 holds true.

Now we consider the opposite case when $|os_3| < 1$. Denote by L the straight line through t_1 and t_2. Since the area of S is not smaller than the area of H and since $|os_3| < 1 = |ot_3|$, the line L intersects the segments h_1s_1 and s_2h_3. Let r_1 and r_2 be the corresponding points of intersection.

Denote by p the intersection of the straight line containing h_1, r_1, s_1 with the straight line containing h_3, r_2, s_2.

Put $\chi = |h_1h_3|$, $\rho = |r_1r_2|$ and $\sigma = |s_1s_2|$. On the other hand, denote by γ', χ', ρ' and σ' the distances of p from the straight lines containing the segments g_1g_2, h_1h_3, r_1r_2 and s_1s_2, respectively. Of course, we have

$$\frac{\chi}{\chi'} = \frac{\rho}{\rho'} = \frac{\sigma}{\sigma'}.$$

Since $t_1t_2t_3t_4t_5t_6$ is a regular hexagon of side of length 1, we have $\gamma' - \rho' = \frac{1}{2}\sqrt{3}$. Since H is a regular hexagon, $\gamma' - \chi' = \frac{1}{2}$. Thus we obtain $\rho' = \chi' - \frac{1}{2}\sqrt{3} + \frac{1}{2}$. This together with $\frac{\chi}{\chi'} = \frac{\rho}{\rho'}$ and $\chi = \sqrt{3}$ gives

$$\rho = \sqrt{3} - \frac{1}{2\chi'}(3 - \sqrt{3}). \tag{1}$$

The hexagon S has area not smaller than the area of the regular hexagon H of side 1 and thus not smaller than the area of the regular hexagon $t_1t_2t_3t_4t_5t_6$ of side 1. This and $|s_1s_2| = |os_3| < 1$ lead to the conclusion that the area $\frac{1}{2}(\gamma' - \sigma')\sigma$ of the triangle os_1s_2 is not smaller than the area $\frac{1}{4}\sqrt{3}$ of the triangle ot_1t_2. Consequently, $\gamma' - \sigma' \geqslant \frac{1}{2\sigma}\sqrt{3}$. Thus $\gamma' - \chi' = \frac{1}{2}$ implies $\sigma' \leqslant \chi' - \frac{1}{2\sigma}\sqrt{3} + \frac{1}{2}$. This inequality, $\frac{\chi}{\chi'} = \frac{\sigma}{\sigma'}$ and $\chi = \sqrt{3}$ lead to

$$\chi' \geqslant \frac{\sqrt{3}}{2\sigma}. \tag{2}$$

From (1) and (2) we conclude that

$$\rho \geqslant \sqrt{3} - (\sqrt{3} - 1)\sigma. \tag{3}$$

Of course, $\sigma = |s_1s_2| = |os_3| < 1$. Consequently, from (3) we obtain that $\rho > 1$. Thus $|r_1r_2| > 1$.

From $|r_1r_2| > 1$ and from $|t_1t_2| = 1$ we see that $|r_1r_2| \geqslant |t_1t_2|$. Thus, since r_1, r_2 are points of M and since M is convex, we conclude that at least one of the points t_1, t_2 belongs to M. Consequently, from the central symmetry of M we obtain the first thesis of Theorem 1. □

It is easy to see that the largest discrepancy between the areas of affine-regular hexagons H inscribed in M is when M is an affine regular hexagon. Then the maximum area of H is equal to $\mathrm{Area}(M)$ and the minimum area is $\frac{1}{3}\sqrt{3}\cdot\mathrm{Area}(M)$. Every affine-regular hexagon inscribed in M has area at least $\frac{3}{4}\cdot\mathrm{Area}(M)$ as shown in [8]. This estimate cannot be improved when M is any parallelogram. So we see that the area of the largest affine-regular hexagon inscribed in M is always between $\frac{3}{4}\cdot\mathrm{Area}(M)$ and $\mathrm{Area}(M)$. The author conjectures that the area of each affine-regular hexagon of the smallest possible area inscribed in M is always at most $\frac{3\sqrt{3}}{2\pi}\cdot\mathrm{Area}(M)$ and that is attained only for ellipses. Sometimes the areas of all affine-regular hexagons inscribed in a centrally symmetric convex body M are equal. This happens when M is an ellipse or a parallelogram (are there other bodies M with this property?). Thus for every affine-regular hexagon inscribed in any ellipse or a parallelogram both statements of Theorem 1 hold true.

2 Inscribed octagons with vertices at vertices of an affine-regular 12-gon

Theorem 2. *In every planar centrally symmetric convex body we can inscribe a centrally symmetric octagon whose three pairs of opposite vertices are vertices of an affine-regular hexagon H and whose remaining two opposite vertices are vertices of the affine-regular 12-gon T_H whose every second vertex is a vertex of H.*

Proof. Consider a planar centrally symmetric convex body M. From [6] we know that for every direction l there exists exactly one affine-regular hexagon $H(l)$, with successive vertices $a(l), b(l), c(l), d(l), e(l), f(l)$ in the positive order, inscribed in M such that the vector $\overrightarrow{a(l)c(l)}$ has direction l. Moreover, as l rotates, then the vertices of $H(l)$ vary continuously along the boundary of M (see [6]).

For every hexagon $H(l)$ denote by $p(l)$ the vertex of the related affine-regular 12-gon such that the neighboring vertices of it are $a(l)$ and $b(l)$. Of course, as l rotates, then $p(l)$ and the symmetric vertex $r(l)$ vary continuously.

For a position l_1 of l we obtain an affine-regular hexagon $H(l_1)$ of the smallest possible area from amongst all the hexagons $H(l)$. What is more, we may assume that $p(l_1)$ and $r(l_1)$ are the two additional vertices of T_H (promised in the first statement of Theorem 1) which also belong to M. Analogously, for a position l_2 of l we obtain an affine-regular hexagon $H(l_2)$ of the greatest possible area. We may assume that $p(l_2)$, $r(l_2)$ are those two additional vertices of T_H as in the second statement of Theorem 1, which are not in the interior of M.

Since $p(l_1), r(l_1) \in M$ and since $p(l_2), r(l_2)$ do not belong to the interior of M, the continuity mentioned earlier implies that there is a direction l_0 for which $p(l_0)$ and $r(l_0)$ are in the boundary of M. Consequently, $H(l_0)$ is the affine-regular hexagon promised in the formulation of Theorem. □

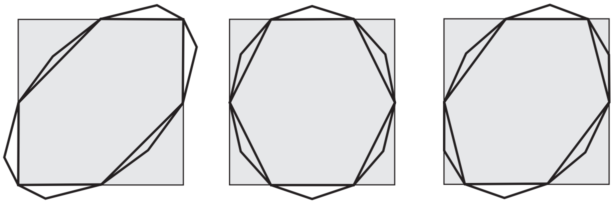

Figure 2

Usually, we cannot inscribe an affine-regular hexagon H such that more than one additional pair of vertices of the 12-gon T_H is in the boundary of M. A simple example is a square. The square always contains one or two additional pairs of vertices of the 12-gon T_H which depends on the direction l. Figure 2 shows such possibilities. The last of those figures shows the position of an affine-regular hexagon with one additional pair of vertices of T_H on the boundary of the square. So we see here the inscribed centrally symmetric octagon and the inscribed affine-regular hexagon as in Theorem 2.

We may distinguish three kinds of centrally symmetric octagons with vertices at the vertices of the regular 12-gon $v_1 \dots v_{12}$. The first kind is when some vertices v_i, v_{i+1} are not vertices of the octagon, the second kind is when some vertices v_i, v_{i+2} are not vertices of the octagon, and the third kind is when some vertices v_i, v_{i+3} are not vertices of the octagon. Theorem 2 says that in every centrally symmetric convex body we can inscribe an octagon which is affinely equivalent to the octagon of the second kind, as mentioned above. A question appears if analogous properties about an inscribed octagon hold true when we exchange here the word "second" into "first" and into "third".

In connection with Theorems 1 and 2 the following question appears. Does every centrally symmetric convex body M permit to inscribe an affine-regular hexagon H such that two additional pairs of vertices of T_H belong to M? A dual question is about the existence of an affine-regular hexagon H with two additional pairs of vertices of T_H outside of the interior of M. The example of the square in the part of M shows that the claim, as in Theorem 1, that "each affine-regular hexagon H of the minimum (respectively, maximum) area inscribed in M has this property" is false.

3 Some applications

Let C be a convex body. By the *C-distance* of points a and b we mean the ratio of $|ab|$ to $\frac{1}{2}|a'b'|$, where $a'b'$ is a longest chord of C parallel to the segment ab (see [7]). If there is no doubt about the body C, we also use the term *the relative distance of* a and b. If we consider a centrally symmetric body M, then the M-distance is nothing else but the distance in the Minkowski space whose unit ball is M.

Conjectures say that in the boundary of every planar convex body there are three points in relative distances at least $\frac{1}{2}(\sqrt{5}+1)$, and that in the boundary of every centrally symmetric convex body there are three points in pairwise relative distances at least $1+\frac{1}{2}\sqrt{2} \approx 1.707$ (i.e. that the boundary of the unit disk of the Minkowski plane contains three points in pairwise distances at least $1+\frac{1}{2}\sqrt{2}$), see [7]. Theorem 2.3 of

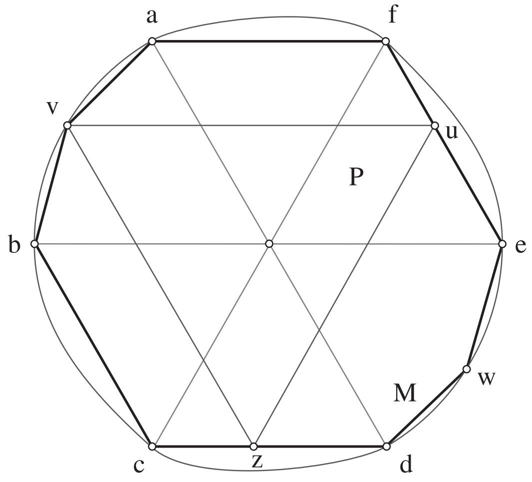

Figure 3

[2] guarantees that there are three such points in relative distances at least $1.546\ldots$. The following Corollary 1 improves this estimate.

Corollary 1. *The boundary of any planar centrally symmetric convex body contains three points in equal pairwise relative distances at least* $1 + \frac{1}{3}\sqrt{3} \approx 1.577$.

Proof. Consider a centrally symmetric convex body M. By Theorem 2 there is an affine-regular hexagon $H = abcdef$ inscribed in M such that an additional pair of opposite vertices v, w of the related affine-regular 12-gon T_H is also in the boundary of M. To fix notation, let a, v, b be successive vertices of the 12-gon (see Figure 3).

Denote by P the triangle with sides of the directions of the main diagonals of the hexagon H and with one vertex at v and the other two vertices u, z on the segments ef and cd.

Since the relative distance does not change under affine transformations, we may assume for simplicity that H is a regular hexagon of sides of length 1. Since M is centrally symmetric, the longest chord in any given direction passes through the center of M. Hence the longest chords of M in the directions of the main diagonals of H are just the main diagonals of H. Consequently, they are of the Euclidean length 2. We omit a simple calculation which shows that the sides of P are of the Euclidean length $1 + \frac{1}{3}\sqrt{3}$.

We have shown that an arbitrary planar centrally symmetric convex body M contains a triangle whose vertices are in pairwise M-distances $1 + \frac{1}{3}\sqrt{3}$. Consider the largest positive homothetical copy of this triangle which is still contained in M. Since M is centrally symmetric, this copy is a triangle inscribed in M. Its vertices are the three points promised in the formulation of Corollary 1. □

In other words, Corollary 1 says that *in the unit disk M of each Minkowski* *plane we can inscribe an equilateral triangle whose vertices are in distances at least* $1 + \frac{1}{3}\sqrt{3} \approx 1.577$.

Corollary 2. *An arbitrary planar centrally symmetric convex body M can be packed with three of its homothetical copies of ratio* $\frac{4+\sqrt{3}}{13} \approx 0.441$.

Proof. Let us treat M as the unit disk of a Minkowski plane. Corollary 1 guarantees that on the boundary of M there are three points in pairwise equal Minkowski distances at least $\delta = 1 + \frac{1}{3}\sqrt{3}$. From Theorem 3.1 of [3] we know that if an m-gon with sides of Minkowski length δ can be inscribed in the unit disk M, then M can also be packed with m translates of M of radius $1 + \frac{2}{\delta}$. In our proof we wish to pack smaller homothetic copies of M into M, so the above described picture, where $m = 3$, is reduced by a factor of $1 + \frac{2}{\delta}$ times. Consequently, a packing of three homothetic copies of M into M is possible provided the homothety ratio is $1/(1 + 2/\delta) = \delta/(\delta + 2) = \frac{4+\sqrt{3}}{13}$. □

In connection with Corollary 2 remember that Doyle, Lagarias and Randall say that the regular octagon is the worst centrally-symmetric body, for packing itself with three homothetic copies, that they have found (see [3]). In this case the homothety ratio of those three copies is $\frac{5+2\sqrt{2}}{17} \approx 0.4605$. Also observe that by Theorem 3.1 of [3] their conjecture is equivalent to the conjecture from [7] mentioned just before Corollary 1.

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