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Divisors on real curves

Jean-Philippe Monnier*

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Abstract. Let X be a smooth projective curve over $\mathbb{R}$. In the first part, we study effective divisors on X with totally real or totally complex support. We give some numerical conditions for a linear system to contain such a divisor. In the second part, we describe the special linear systems on a real hyperelliptic curve and prove a new Clifford inequality for such curves. Finally, we study the existence of complete linear systems of small degrees and dimension r on a real curve.

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Introduction

In this note, a real algebraic curve X is a smooth proper geometrically integral scheme over $\mathbb{R}$ of dimension 1. A closed point P of X will be called a real point if the residue field at P is $\mathbb{R}$, and a non-real point if the residue field at P is $\mathbb{C}$. The set of real points, $X(\mathbb{R})$, will always be assumed to be non-empty. It decomposes into finitely many connected components, whose number will be denoted by s. By Harnack's theorem we know that $1 \leqslant s \leqslant g+1$, where g is the genus of X. A curve with $g+1-k$ real connected components is called an $(M-k)$-curve.

The group $\mathrm{Div}(X)$ of divisors on X is the free abelian group generated by the closed points of X. Let $D \in \mathrm{Div}(X)$ be an effective divisor. We may write $D = D_r + D_c$, in a unique way, such that D_r and D_c are effective and with real, respectively non-real, support. We call D_r (resp. D_c) the real (resp. non-real) part of D. In the sequel, we will say that D is totally real (resp. non-real), if $D = D_r$ (resp. $D = D_c$).

By $\mathbb{R}(X)$, we denote the function field of X. Let $\mathrm{Pic}(X)$ denote the Picard group of X, which is the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors, i.e. divisors of elements in $\mathbb{R}(X)$. Since a principal divisor has an even degree on each connected component of $X(\mathbb{R})$ ([4] Lemma 4.1), we may introduce the following invariants of X:

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(i) $N(X)$, the smallest integer $n \geqslant 1$ such that any divisor of degree n is linearly equivalent to a totally real effective divisor (by [11] Theorem 2.7, we know that $N(X)$ is finite),

(ii) $M(X)$, the smallest integer $m \geqslant 1$ such that any divisor D of degree $2m$ such that the degree of D on each connected component of $X(\mathbb{R})$ is even, is linearly equivalent to a totally non-real effective divisor. If such an integer does not exist, then $M(X) = +\infty$.

The principal goal of the paper is to bound the previous invariants in terms of g and s. The problem for $N(X)$ was raised by Scheiderer in [11].

We briefly describe the structure of the paper. In Section 2, we show that

$$g \leqslant M(X) \leqslant 2g.$$

Moreover, if X is a real rational curve or a real elliptic curve, then $M(X) = 1$. Using this, we also prove that if $X \subseteq \mathbb{P}^n_{\mathbb{R}}$, $n \geqslant 2$, is a non-degenerate linearly normal curve of degree d with no pseudo-line in $X(\mathbb{R})$ (see the Section 2 for the corresponding definitions), and if X satisfies one of the two following conditions

(i) X is rational or elliptic,

(ii) $g \geqslant 2$ and $d \geqslant 4g$,

then $X(\mathbb{R})$ is affine in $\mathbb{P}^n_{\mathbb{R}}$, i.e. there exists a real hyperplane H such that $H(\mathbb{R}) \cap X(\mathbb{R}) = \emptyset$.

In Section 3, we extend a result proved in [6] for M-curves, to $(M-1)$-curve:

$$N(X) \leqslant 2g - 1.$$

Under the assumption of a conjecture of Huisman [9] on unramified curves, we further extend this result to $(M-2)$-curves, the bound being slightly different.

In the last section of the paper, we give a large family of curves for which the invariant N is explicitely calculated. For these computations, we use the results established in Sections 4 and 5.

In Section 4, we prove a stronger version of the Clifford inequality for real hyperelliptic curves, which sharpen Huisman's general result for real curves [8]: if X is a real hyperelliptic curve such that $s \neq 2$ and $D \in \mathrm{Div}(X)$ is an effective and special divisor of degree d, then

$$\dim|D| \leqslant \frac{1}{2}(d - \delta(D)),$$

with $\delta(D)$ the number of connected components C of $X(\mathbb{R})$ such that the degree of the restriction of D to C is odd.

Section 5 deals with the existence of complete linear systems of degree d and dimension $r \geqslant 1$ on X.

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1 Preliminaries

We recall here some classical concepts and more notations that we will be using throughout this paper.

Let X be a real curve. We will denote by $X_{\mathbb{C}}$ the base extension of X to $\mathbb{C}$. The group $\mathrm{Div}(X_{\mathbb{C}})$ of divisors on $X_{\mathbb{C}}$ is the free abelian group on the closed points of $X_{\mathbb{C}}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the complex variety $X_{\mathbb{C}}$ and also on $\mathrm{Div}(X_{\mathbb{C}})$. We will always indicate this action by a bar. If P is a non-real point of X, identifying $\mathrm{Div}(X)$ and $\mathrm{Div}(X_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$, then $P = Q + \bar{Q}$ with Q a closed point of $X_{\mathbb{C}}$.

If D is a divisor on X or $X_{\mathbb{C}}$, we will denote by $[D]$ its class in the Picard group, and by $\mathcal{O}(D)$ its associated invertible sheaf. The dimension of the space of global sections of this sheaf will be denoted by $\ell(D)$ for D on X, and by $\ell_{\mathbb{C}}(D)$ for D on $X_{\mathbb{C}}$.

We will always denote by $C_1, \ldots, C_s$ the connected components of $X(\mathbb{R})$. Let $D \in \mathrm{Div}(X)$, and denote by $\deg_{C_i}(D)$ the degree of the restriction of D to C_i. Following [4], we will denote by c the surjective morphism

$$\mathrm{Pic}(X) \to (\mathbb{Z}/2)^s,$$
$$[D] \mapsto (\ldots, \deg_{C_i}(D) \bmod 2, \ldots),$$

and we will write $\delta(D)$ for the number of connected components C of $X(\mathbb{R})$ such that $\deg_C(D)$ is odd. The connected components of $\mathrm{Pic}^d(X)$, the subgroup of divisor classes of $\mathrm{Pic}(X)$ of degree d, correspond to the fibres of the restriction of c to $\mathrm{Pic}^d(X)$. Let $u = (u_1, \ldots, u_s) \in (\mathbb{Z}/2)^s$, we will denote by $U(d; u_1, \ldots, u_s) = U(d, u)$ the connected component of $\mathrm{Pic}^d(X)$ that corresponds to $c^{-1}(u)$. Obviously, $U(d; u_1, \ldots, u_s) \neq \emptyset$ if and only if $\sum_{i=1}^{s} u_i \equiv d \bmod 2$. We will also denote the coordinates of $u = (u_1, \ldots, u_s) \in (\mathbb{Z}/2)^s$ by $c_i(u) = u_i$.

Let J be the Jacobian of X. It is well known that $\mathrm{Pic}^0(X)$ can be identified with $J(\mathbb{R})$ since $X(\mathbb{R}) \neq \emptyset$. We will denote by $J(\mathbb{R})_0$ the connected component of the identity of $J(\mathbb{R})$. Then $J(\mathbb{R})_0 = U(0; 0, \ldots, 0)$ ([11] Lemma 2.6).

We now reformulate the definition of the invariants N and M.

Definition 1.1. (i) $N(X)$ is the smallest integer $n \geqslant 1$ such that for any real point P, and for any $\alpha \in J(\mathbb{R})$, there exist $P_1, \ldots, P_n \in X(\mathbb{R})$, such that $\alpha = \sum_{i=1}^{n} [P_i - P]$, and

(ii) $M(X)$ is the smallest integer $m \geqslant 1$, such that for any real closed point P, and for any $\alpha \in J(\mathbb{R})_0$, there exist non-real points $Q_1, \ldots, Q_m$ such that $\alpha = \sum_{i=1}^{m} [Q_i - 2P]$. If such an integer does not exist, then $M(X) = +\infty$.

2 Divisors with a complex support

In this section, we bound the invariant $M(X)$ from above and from below, and give a geometric consequence.

The following proposition justifies the definition of the invariant M.

Proposition 2.1. *Let P be a real point of X and $\alpha \in J(\mathbb{R})_0$. There is an integer $m \geqslant 1$ and non-real points $Q_1, \ldots, Q_m$ such that $\alpha = \sum_{i=1}^{m} [Q_i - 2P]$.*

Proof. Let P be a real closed point of X and $\alpha \in J(\mathbb{R})_0$. Since $J(\mathbb{R})_0$ is a divisible group, there is $\beta \in J(\mathbb{R})_0$ such that $2\beta = \alpha$. By Riemann–Roch, the map

$$\varphi_d : (S^d X)(\mathbb{R}) \to \mathrm{Pic}^d(X)$$

is surjective for $d \geqslant g$, where $S^d X$ denotes the symmetric d-fold product of X over $\mathbb{R}$. Hence there exists D an effective divisor of degree g such that $\beta + [gP] = [D]$. By Riemann–Roch, there is an integer k such that the divisor kP is very ample as a complex divisor, and also as a real divisor, since $kP \in \mathrm{Div}(X)$. Hence $D + kP$ is also very ample.

Let ψ denote the embedding of X in $\mathbb{P}^k_{\mathbb{R}}$ associated to the linear system $|D + kP|$. Let S be the quadric hypersurface of $\mathbb{P}^k_{\mathbb{R}}$ with equation $x_0^2 + \cdots + x_k^2 = 0$. Thus $2D + 2kP$ is linearly equivalent to the effective divisor D' of degree $2(g + k)$ obtained by intersecting S and X. Since $S(\mathbb{R}) = \varnothing$, D' is totally non-real. Hence $\alpha = [D'] - [2(g + k)P]$, and the result follows. □

The method of the previous proof allows us to give an upper bound for $M(X)$ in terms of g. The following theorem gives a better result.

Theorem 2.2. *Let X be a curve of positive genus. We have $M(X) \leqslant 2g$.*

Proof. Let P be a real point of X and $V = X(\mathbb{C}) \backslash X(\mathbb{R})$, where $X(\mathbb{C})$ denote the set of closed points of $X_{\mathbb{C}}$. $X(\mathbb{R})$ is seen as a subset of $X(\mathbb{C})$. By Riemann–Roch, the map $X(\mathbb{C})^g \to \mathrm{Pic}^g(X_{\mathbb{C}})$ is surjective. Moreover, the map $S^g X \to J$ is well known to be a birational morphism of complete varieties. The image U of the map

$$V^g \to J(\mathbb{C}), (Q_1, \ldots, Q_g) \mapsto \sum_{i=1}^{g} [Q_i - P],$$

contains therefore an open dense subset of $J(\mathbb{C})$. Thus $U + U = J(\mathbb{C})$. The image of the norm map $N : J(\mathbb{C}) \to J(\mathbb{R}), \alpha \mapsto \alpha + \bar{\alpha}$, is $J(\mathbb{R})_0$ (see [11]). So $N(U) + N(U) = J(\mathbb{R})_0$, and $M(X) \leqslant 2g$. □

Since any two divisors with the same degree on a rational real curve are linearly equivalent, we trivially get the following proposition:

Proposition 2.3. *Let X be a real rational curve, then $M(X) = 1$.*

For real elliptic curves, the result of Theorem 2.2 can be improved.

Theorem 2.4. *Let X be a real elliptic curve, then $M(X) = 1$.*

Proof. Let P be a real point of X and $\alpha \in J(\mathbb{R})_0$. Arguing as in the proof of Proposition 2.1, there is $\beta \in J(\mathbb{R})_0$ such that $2\beta = \alpha$ and $\beta + [P] = [P_0]$, with P_0 a real point. Then

$$\alpha = [2P_0] - [2P].$$

The linear system $|3P_0|$ gives a closed immersion $X \subseteq \mathbb{P}^2_{\mathbb{R}}$. Using Riemann–Roch and after linear changes of coordinates, we obtain a closed immersion $\varphi : X \to \mathbb{P}^2_{\mathbb{R}}$ such that the image is the curve

$$y^2 = (x - a)R(x),$$

with $a \in \mathbb{R}$ and $R(x) \in \mathbb{R}[x]$ a monic and separable polynomial of degree 2. The point P_0 goes to the point at infinity $(0 : 1 : 0)$ on the y-axis (see [5] Proposition 4.6, p. 319). If we project from P_0 onto the x-axis, we get a finite morphism $f : X \to \mathbb{P}^1_{\mathbb{R}}$ of degree 2, sending P_0 to ∞, and being ramified at least at one more real point of $\mathbb{P}^1_{\mathbb{R}}$, besides ∞. In fact, f may be defined using the linear system $|2P_0|$. Since f is ramified with order 2 at ∞, then locally on one side of ∞ the fiber over $\mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ is totally real and on the other side the fiber is totally non-real. In particular, there exists $\lambda \in \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ such that $f^{-1}(\lambda) = \{Q\}$, with Q a non-real point of X. Then $[2P_0] = [Q]$ and $\alpha = [Q] - [2P]$. □

For a given complete linear system of degree sufficiently big, an upper bound exists for the least degree of the real part of divisors in the linear system.

Corollary 2.5. *For any complete linear system $|D|$ with $\deg(D) \geqslant 4g + \delta(D)$ if $g \geqslant 2$, $\deg(D) \geqslant 2 + \delta(D)$ if $g \in \{0, 1\}$, there exists $D' \in |D|$ such that the real part of D' has degree $\delta(D)$.*

Proof. We give the proof only for the case $g \geqslant 2$. Let $P_1, \ldots, P_{\delta(D)}$ be some real points belonging to the connected components of $X(\mathbb{R})$ where the degree of D is odd, and such that no two of them belong to the same connected component of $X(\mathbb{R})$. We set $d = \deg(D)$. We remark that $d - \delta$ is necessarily even. By Theorem 2.2, $D - \sum_{i=1}^{\delta(D)} P_i$ is linearly equivalent to a totally non-real effective divisor and the proof is done. □

We give a lower bound for the invariant $M(X)$.

Proposition 2.6. *Assume $g \geqslant 2$. Then $M(X) \geqslant g$.*

Proof. Let $P \in X(\mathbb{R})$ and consider the divisor $D' = K - P$, where K denotes the canonical divisor. Choose $P' \neq P \in X(\mathbb{R})$ belonging to the same connected component of $X(\mathbb{R})$ as P. Since X is not rational, using the fact that $\ell(P - P') = 0$, it follows that

$$\ell(D' + P') = g - 1 = \ell(D').$$

Hence P' is a base point of the linear system $|D' + P'|$. Since $D' + P'$ has degree $2g - 2$ and has an even degree on each connected component of $X(\mathbb{R})$ (see [9] Proposition 2.1), we easily see that $M(X) > g - 1$. □

We give now a geometric consequence of the previous results. Let $X \subseteq \mathbb{P}^n_{\mathbb{R}}$ be a non-degenerate real curve, i.e. X is not contained in a real hyperplane of $\mathbb{P}^n_{\mathbb{R}}$. We will say that $X(\mathbb{R})$ is affine in $\mathbb{P}^n_{\mathbb{R}}$ if there exists a real hyperplane H such that $H(\mathbb{R}) \cap X(\mathbb{R}) = \emptyset$. In this case $X(\mathbb{R})$ is a real algebraic subvariety of $\mathbb{A}^n_{\mathbb{R}}(\mathbb{R}) = \mathbb{R}^n$ in the sense of [2]. Since the real hypersurface S of $\mathbb{P}^n_{\mathbb{R}}$ with equation $x_0^2 + \cdots + x_n^2 = 0$ has no real points, $X(\mathbb{R})$ is always contained in an affine open subset of $\mathbb{P}^n_{\mathbb{R}}$. More precisely the image of $X(\mathbb{R})$ by the 2-uple embedding is affine in $\mathbb{P}_{\mathbb{R}}^{1/2(n+1)(n+2)-1}$. We may wonder if $X(\mathbb{R})$ is already affine in $\mathbb{P}^n_{\mathbb{R}}$. Recall that X is linearly normal if the restriction map

$$H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(1)) \to H^0(X, \mathcal{O}(1))$$

is surjective. Let C be a connected component of $X(\mathbb{R})$. The component C is called a pseudo-line if the canonical class of C is nontrivial in $H_1(\mathbb{P}^n_{\mathbb{R}}(\mathbb{R}), \mathbb{Z}/2)$. Equivalently, C is a pseudo-line if and only if for each real hyperplane H, $H(\mathbb{R})$ intersects C in an odd number of points, when counted with multiplicities (see [9]). So a necessary condition for $X(\mathbb{R})$ to be affine in $\mathbb{P}^n_{\mathbb{R}}$ is that $X(\mathbb{R})$ has no pseudo-line.

Proposition 2.7. *Let $X \subseteq \mathbb{P}^n_{\mathbb{R}}$, $n \geqslant 2$, be a non-degenerate linearly normal curve of degree d such that $X(\mathbb{R})$ has no pseudo-line. If X satisfies one of the two following conditions*

(i) *X is rational or elliptic,*

(ii) *$g \geqslant 2$ and $d \geqslant 4g$,*

then $X(\mathbb{R})$ is affine in $\mathbb{P}^n_{\mathbb{R}}$.

Proof. A hyperplane section has even degree on each connected component of $X(\mathbb{R})$ and its degree $\geqslant 2M(X)$. The results follows from Corollary 2.5 and the linear normality. □

Example 2.8. Let X be an elliptic quartic curve in $\mathbb{P}^3_{\mathbb{R}}$ with only one real connected component. Then $X(\mathbb{R})$ is affine in $\mathbb{P}^3_{\mathbb{R}}$ since X satisfies the hypotheses of the proposition: X is a complete intersection and d is even (use Bezout's theorem).

Proposition 2.9. *Let $X \subseteq \mathbb{P}^n_{\mathbb{R}}$ be a non-degenerate curve of degree $d \leqslant 2n-1$ such that $X(\mathbb{R})$ has no pseudo-line and $g = d - n$. If $n \leqslant d \leqslant n+1$ or $d \leqslant \frac{4}{3}n$ then $X(\mathbb{R})$ is affine in $\mathbb{P}^n_{\mathbb{R}}$.*

Proof. Let H be a hyperplane section of X. By Clifford's inequality and since $d \leqslant 2n-1$, H is non-special (see Section 4). By Riemann–Roch, $g = d - \dim|H|$. Consequently $\dim|H| = n$ and X is linearly normal. The proof follows now from Proposition 2.7. □

Example 2.10. Let X be a smooth quartic curve in $\mathbb{P}^2_{\mathbb{R}}$. Then X is the canonical model of a curve of genus 3. By [4], X has always odd theta-characteristics that are in one-to-one correspondence with the real bitangent lines to X. Since the degree of X is 4, a real bitangent line to X intersects the curve $X_{\mathbb{C}}$ only at the two points of tangency. If these two points are non-real and switched by the complex conjugation, then $X(\mathbb{R})$ is affine in $\mathbb{P}^2_{\mathbb{R}}$. If the points are real, we may move the line to get a line which does not intersect $X(\mathbb{R})$, $X(\mathbb{R})$ is again affine in $\mathbb{P}^2_{\mathbb{R}}$. Notice that the conclusion cannot be deduced from Proposition 2.7.

3 Divisors with real support

This section is dedicated to the study of the invariant $N(X)$. We clearly have

Proposition 3.1. *If X is a real rational curve or a real elliptic curve, then*

$$N(X) = 1.$$

Hence, in the remainder of this section we will assume that $g > 1$, and use the invariant e defined by:

$$e = \begin{cases} \frac{1}{2}(g-s) & \text{if } g-s \text{ even,} \\ \frac{1}{2}(g-s+1) & \text{if } g-s \text{ odd.} \end{cases}$$

Let us state the principal result of this section:

Theorem 3.2. *Any complete linear system of degree $\geqslant s-1+g$ contains a divisor whose non-real part has degree $\leqslant 2e$.*

Proof. Let D be a divisor of degree $d \geqslant s-1+g$. We will prove that D is linearly equivalent to an effective divisor, whose non-real part has degree $\leqslant 2e$.

Let P be a real point and $\alpha = [D - dP] \in J(\mathbb{R})$. We fix $R_1, \dots, R_{g-2e}$ in $g-2e$ distinct components among $C_1, \dots, C_s$. To simplify the proof, we set $R_i \in C_i$. Let us denote $\beta = \alpha + \sum_{i=1}^{g-2e} [P - R_i]$. Consider the restriction to $\mathrm{Pic}^0(X)$ of the morphism c defined in Section 1, then it clearly induces an isomorphism $J(\mathbb{R})/J(\mathbb{R})_0 \simeq (\mathbb{Z}/2)^{s-1}$. Hence there exist $P_{g-2e+1}, \dots, P_{g-2e+s-1} \in X(\mathbb{R})$ such that

$$\beta = \sum_{j=1}^{s-1} [P_{g-2e+j} - P] + \beta_0,$$

with $\beta_0 \in J(\mathbb{R})_0$.

By Riemann–Roch, the natural map $(S^g X)(\mathbb{R}) \to \mathrm{Pic}^g(X)$ is surjective, $S^d X$ denoting the symmetric d-fold product of X over $\mathbb{R}$. Moreover if $[D'] = [D'']$ in $\mathrm{Pic}^d(X)$, then $\deg_{C_i}(D') \equiv \deg_{C_i}(D'') \bmod 2$ for $i = 1, \dots, s$. Let $u \in (\mathbb{Z}/2)^s$ such that $c_i(u) = 1$ for $i = 1, \dots, g-2e$ and $c_{g-2e+1}(u) = 0$. Consequently, if $[D'] \in U(g; u)$, then D' is linearly equivalent to the effective divisor

$$\sum_{i=1}^{g-2e} P_i + \sum_{i=1}^{e} Q_i,$$

where

1) $P_i \in C_i$, $1 \leqslant i \leqslant g-2e$ and,

2) Q_i is either a non-real point or a sum of two real points contained in the same connected component of $X(\mathbb{R})$, $i = 1, \ldots, e$.

The translation by $-[\sum_{i=1}^{g-2e} R_i] - 2e[P]$ is a bijection between $U(g;u)$ and $J(\mathbb{R})_0 = U(0;0,\ldots,0)$, hence

$$\beta_0 + \left[\sum_{i=1}^{g-2e} R_i\right] + 2e[P] = \sum_{i=1}^{g-2e}[P_i] + \sum_{i=1}^{e}[Q_i].$$

Finally,

$$\alpha = \sum_{i=1}^{s-1+g-2e}[P_i - P] + \sum_{i=1}^{e}[Q_i - 2P]$$

and the proof is done. □

The above theorem allows to give an upper bound for M-curves or $(M-1)$-curves.

Corollary 3.3. *Let X be an M-curve or an $(M-1)$-curve. Then*

$$N(X) \leqslant s - 1 + g.$$

In [6], it is shown that $N(X) \leqslant 2g-1$ for M-curves. Following the method used in [6], we will now show that the result of Theorem 3.2 may be improved in the case $s \equiv g+1 \bmod 2$.

Let $s \geqslant 2$. By Theorem 3.2, we already know that for every complete linear system $|D|$ of degree $\geqslant s-1+g$, there exists $D' \in |D|$ such that the non-real part of D' has degree $\leqslant 2e$. We would like to extend the result to linear systems of degree $g+d$, $0 \leqslant d \leqslant s-2$, under certain conditions on the invariant δ.

Proposition 3.4. *Assume* $\deg(D) = g+d$ *for* $d \in \{0,\ldots,s-2\}$. *If* $\delta(D) \geqslant s - d - \frac{1}{2}(1-(-1)^{s-g})$, *then there exists $D' \in |D|$ such that the non-real part of D' has degree $\leqslant 2e$.*

Proof. The proof depends on the parity of $s-g$.

First, assume that $s-g$ is odd. For $i = 1,\ldots,s$, let $u_i \in (\mathbb{Z}/2)^s$ such that $c_j(u_i) = 1 - \delta_{i,j}$ (δ is Kronecker's symbol). By Riemann–Roch, any divisor in $U(g,u_i)$ is linearly equivalent to an effective divisor whose non-real part has degree $\leqslant 2e$. We translate D by $-D''$, with D'' a totally real effective divisor of degree d such that $[D - D''] \in U(g, u_i)$ for a i. We have $\delta(D) = g + d \bmod 2$. Hence there exists $k \in \mathbb{Z}$ such that $\delta(D) + 2k = g + d$. Moreover $g + d = s - d - 1 \bmod 2$, hence $g + d = s - d - 1 + 2r$, with $r \in \mathbb{Z}$. By a closer look at these identities, we see that k and r are non-negative. Consequently

$$\delta(D) = 2(r - k) + s - d - 1. \tag{1}$$

By the hypothesis $\delta(D) \geqslant s - d - 1$. Hence $l = r - k \geqslant 0$ and by (1),

$$(s - \delta(D) - 1) + 2l = d. \tag{2}$$

We remark that $s - \delta(D)$ corresponds to the number of connected components C of $X(\mathbb{R})$ where $\deg_C(D)$ is even. If $s \neq \delta(D)$, then we choose a component C_i such that $\deg_{C_i}(D)$ is even, and by (2), we take as D'' a divisor that cuts out schematically a point on the components $C_j \neq C_i$ where $\deg_{C_j}(D)$ is even, and a point with multiplicity $2l$ on C_i. Then $[D - D''] \in U(g, u_i)$. If $s = \delta(D)$, then $d = 2l - 1$ is odd, and we take $D'' = dP_1$, with $P_1 \in C_1$. Again $[D - D''] \in U(g, u_1)$.

Second, assume that $s - g$ is even.

The situation is simpler since we know that any divisor in $U(g, u)$, with $u = (1, \ldots, 1) \in (\mathbb{Z}/2)^s$, is linearly equivalent to an effective divisor whose non-real part has degree $\leqslant 2e$. So we translate D by $-D''$ with D'' a totally real effective divisor of degree d, such that $[D - D''] \in U(g, u)$. By the same arguments as before,

$$\delta(D) = 2(r - k) + s - d, \tag{3}$$

for some non-negative integers r and k. If we assume that $\delta(D) \geqslant s - d$, then $l = r - k \geqslant 0$, and by (3),

$$(s - \delta(D)) + 2l = d. \tag{4}$$

Again $s - \delta(D)$ corresponds to the number of connected components C of $X(\mathbb{R})$ where $\deg_C(D)$ is even. For D'', we take the sum of any real point with multiplicity $2l$, with a divisor whose support consists of a unique point in each of the component C of $X(\mathbb{R})$, where $\deg_C(D)$ is even. □

Corollary 3.5. *Assume $s - g$ is odd and $s \geqslant 2$. Any complete linear system of degree $\geqslant s - 2 + g$ contains a divisor whose non-real part has degree $\leqslant 2e$.*

Proof. Using the previous proposition, we only have to prove that if D is a divisor of degree $g + s - 2$, then $\delta(D) \geqslant 1$. If $\delta(D) = 0$, then $g + s - 2$ must be even, contradicting the hypotheses. □

Let us state a nice consequence of the previous results:

Theorem 3.6. *Let X be an M-curve or an $(M-1)$-curve. Then $N(X) \leqslant 2g-1$.*

Equivalently, the theorem says that, for an M-curve or an $(M-1)$-curve, the natural map $X(\mathbb{R})^{2g-1} \to \mathrm{Pic}^{2g-1}(X)$ is surjective.

$(M-2)$-curves and unramified real curves in odd-dimensional projective spaces. Let X be real curve and $D \in \mathrm{Div}(X)$. For $D = \sum n_i P_i - \sum m_j Q_j$, with n_i and m_j positive, and the sum taken over distinct closed points of X, we define $D_{\mathrm{red}} = \sum P_i - \sum Q_j$. We also define the weight of D to be the natural number $w(D) = \deg(D - D_{\mathrm{red}})$. If $X \subseteq \mathbb{P}^n_{\mathbb{R}}$, $n \geqslant 1$, is non-degenerate, we say that X is unramified if for each hyperplane H of $\mathbb{P}^n_{\mathbb{R}}$, we have $w(H \cdot X) \leqslant n-1$.

The corresponding notion of an unramified complex algebraic curve in complex projective space is well understood. Indeed, any unramified complex algebraic curve is a rational normal curve and conversely [3]. Over $\mathbb{R}$, the situation is different and Huisman has given the following conjecture (see [9] Conjecture 3.6):

Conjecture. *Let $n \geqslant 3$ be an odd integer and $X \subseteq \mathbb{P}^n_{\mathbb{R}}$ be a non-degenerate real algebraic curve of positive genus. If X is unramified, then X is an M-curve.*

We relate this conjecture and the invariant N studied in this paper.

Theorem 3.7. *Let X be an $(M-2)$-curve. Assuming the above conjecture, we get*:

(i) *$N(X) \leqslant 3g-1$, if g is even, and*

(ii) *$N(X) \leqslant 3g$, if g is odd.*

Proof. Let $P \in X(\mathbb{R})$ and $\alpha \in J(\mathbb{R})$. Recall that $s = g-1$ and that $C_1, \dots, C_{g-1}$ denote the connected components of $X(\mathbb{R})$. We may assume that $P \in C_1$.

Assume g is even. Let $D = P_2 + \cdots + P_{g-1} + Q$ be an effective divisor with $P_i \in C_i$ for $i = 2, \dots, g-1$, and Q be a non-real point. In fact $[D] \in U(g; 0, 1, \dots, 1)$. Let $D' = D + (g+1)P$. Then D' is very ample and the linear system $|D'|$ allows us to embed X in $\mathbb{P}^{g+1}_{\mathbb{R}}$. Using the above conjecture, X is not unramified. Consequently there is an hyperplane H of $\mathbb{P}^{g+1}_{\mathbb{R}}$ such that $H \cdot X = \sum_{i=1}^r n_i R_i + \sum_{j=1}^t m_j Q_j$, where the sum is taken over distinct points. The R_i are real points and the Q_j are non-real points. Moreover,

$$\sum_{i=1}^{r} n_i + 2\sum_{j=1}^{t} m_j = 2g+1 \tag{5}$$

and

$$w(H \cdot X) = \sum_{i=1}^{r} (n_i - 1) + 2\sum_{j=1}^{t} (m_j - 1) \geqslant g+1. \tag{6}$$

Since $\deg_{C_i}(D')$ is odd for $i = 1, \dots, s$, each connected component of $X(\mathbb{R})$ is a pseudo-line. It follows that

$$w(H \cdot X) = \deg((H \cdot X) - (H \cdot X)_{\text{red}}) \leqslant (2g + 1) - (g - 1) = g + 2. \tag{7}$$

Using (5), (6) and (7), we get $g + 2 \geqslant -(r + 2t) + 2g + 1 \geqslant g + 1$ and

$$g \geqslant (r + 2t) \geqslant g - 1.$$

Since $r \geqslant g - 1$, we have $t = 0$ and D' is linearly equivalent to a totally real effective divisor. By Theorem 3.2, there are $2g - 4$ points $P_1, \dots, P_{2g-4} \in X(\mathbb{R})$ and Q a non-real point or a sum of two real points contained in the same connected component of $X(\mathbb{R})$, such that

$$\alpha = \left(\sum_{i=1}^{2g-4} [P_i - P] \right) + ([Q] - [2P]).$$

Moreover, looking at the proof of Theorem 3.2, we may choose $P_i \in C_i$ for $i = 2, \dots, g - 1$. Writing $\alpha = \alpha + (g + 1)[P] - (g + 1)[P]$, the statement follows from the above construction.

If we assume that g is odd, the proof is similar but for $D' = D + (g + 2)P$ here. □

Assuming the above conjecture, we may also obtain a more general result.

Proposition 3.8. *Let X be a real curve such that $s \leqslant g - 1$. Any complete linear system of degree $\geqslant s + 2g + \frac{1}{2}(1 - (-1)^g)$ contains a divisor whose non-real part has degree $\leqslant 2e - 2$.*

The above method allows us to have a description of a linear system with one non-real point less. Unfortunately this method is rigid; a repetition of this method does not give a description of a linear system with only real points.

4 Clifford's inequality and linear systems on real hyperelliptic curves

In this section, we study the family of special linear systems on real algebraic curves. Let $D \in \operatorname{Div}(X)$ and K be the canonical divisor. If $\ell(K - D) > 0$, D is said to be special. If not, D is said to be non-special. By Riemann–Roch, if $\deg(D) > 2g - 2$ then D is non-special. The classical Clifford inequality states that the dimension of a nonempty special complete linear system on a curve is bounded by half of its degree (see [5] Theorem 5.4, p. 343). We now recall the Clifford inequality for real curves given by Huisman ([8] Theorem 3.1).

Theorem 4.1. *Let $D \in \operatorname{Div}(X)$ be an effective divisor of degree d. The following statements hold.*

(i) *If* $d + \delta(D) < 2s$, *then* $\dim|D| \leqslant \frac{1}{2}(d - \delta(D))$.

(ii) *If* $d + \delta(D) \geqslant 2s$, *then* $\dim|D| \leqslant d - s + 1$.

A real hyperelliptic curve is a real curve X such that $X_{\mathbb{C}}$ is hyperelliptic, i.e. $X_{\mathbb{C}}$ has a g_2^1 (a linear system of dimension 1 and degree 2). As always, we assume that $X(\mathbb{R}) \neq \emptyset$ and moreover that $g \geqslant 2$.

Lemma 4.2. *Let X be a real hyperelliptic curve. Then X has a unique g_2^1.*

Proof. By [5] Proposition 5.3, $X_{\mathbb{C}}$ has a unique g_2^1. Let D be an effective divisor of degree 2 on $X_{\mathbb{C}}$ satisfying $|D| = g_2^1$. Since this unique g_2^1 is also complete and X is defined over $\mathbb{R}$, we have $|\bar{D}| = g_2^1$. Let $P \in X(\mathbb{R})$, we also denote by P the corresponding closed point of $X_{\mathbb{C}}$. Since $\ell_{\mathbb{C}}(D - P) > 0$, we may assume that $D = P + Q$ with Q a closed point of $X_{\mathbb{C}}$. Then $[P + Q] = [P + \bar{Q}]$ in $\mathrm{Pic}(X_{\mathbb{C}})$ and $Q = \bar{Q}$, since $X_{\mathbb{C}}$ is not rational. Hence $D = \bar{D}$ and since $\ell(D) = \ell_{\mathbb{C}}(D)$, the proof is done. □

This g_2^1 induces an involution, denoted by $\imath$, on the closed points of X. A real hyperelliptic curve X is said to be respected by the involution (we will abbreviate by r.b.i.), if for any real point P, P and $\imath(P)$ belong to the same real connected component. Most real hyperelliptic curves are r.b.i.

Proposition 4.3. *Let X be a real hyperelliptic curve such that X is not r.b.i. Then X is given by the real polynomial equation $y^2 = f(x)$, where f is a monic polynomial of degree $2g + 2$, with g odd, and where f has no real roots. In particular, the number of connected components of $X(\mathbb{R})$ is* 2.

Proof. Using the g_2^1, we easily see that an affine model of X is given by the real equation $y^2 = f(x)$, with $\deg(f) = 2g + 2$. Since X is not r.b.i., f cannot have a real root. We may assume that f is monic since $X(\mathbb{R}) \neq \emptyset$. If g is even, then $s = 1$ ([4] Proposition 6.3), contradicting the hypotheses. If g is odd, then $X(\mathbb{R})$ has 2 connected components exchanged by $\imath$. □

We give now the Clifford inequality for real hyperelliptic curves which are r.b.i.

Theorem 4.4. *Let X be a real hyperelliptic curve that is r.b.i. and let $D \in \mathrm{Div}(X)$ be an effective and special divisor of degree d. Then*

$$\dim|D| \leqslant \frac{1}{2}(d - \delta(D)).$$

Proof. The classical Clifford inequality allows us to assume that $\delta(D) \geqslant 1$. We may further assume that $C_1, \ldots, C_{\delta(D)}$ are the connected components of $X(\mathbb{R})$, where the degree of D is odd. Let $D' \leqslant D$ be the greatest effective common subdivisor of D and $\imath(D)$ with the property that $|D'_r| = \frac{\deg(D'_r)}{2} g_2^1$, D'_r denoting the real part of D'. Write

$D'' = D - D'$. Since X is r.b.i., then $\delta(D') = 0$. So, there are real points $P_1, \dots, P_{\delta(D)}$ such that $P_1 + \cdots + P_{\delta(D)} \leqslant D''$ and $P_i \in C_i$, $i = 1, \dots, \delta(D)$. We remark that

a) $d + \delta(D) \leqslant 2g - 2$ since D is special ([9] Theorem 2.3),

b) $\imath(P_i) \notin \mathrm{Supp}(D'')$ or P_i is a fixed point for $\imath$ such that $2P_i$ is not a subdivisor of D'', $i = 1, \dots, \delta(D)$.

Let ω be a global differential form on X, such that $\mathrm{div}(\omega) \geqslant D$. Then $\mathrm{div}(\omega) \geqslant D + \imath(P_1) + \cdots + \imath(P_{\delta(D)})$, since $K = (g-1)g_2^1$ and $d + \delta(D) \leqslant 2g - 2$. Hence $\ell(K - D) = \ell(K - (D + \imath(P_1) + \cdots + \imath(P_{\delta(D)})))$, and $D + \imath(P_1) + \cdots + \imath(P_{\delta(D)})$ is also special. By Riemann–Roch,

$$\dim|D| - \dim|K - (D + \imath(P_1) + \cdots + \imath(P_{\delta(D)}))| = d - g + 1, \tag{8}$$

and

$$\dim|D + \imath(P_1) + \cdots + \imath(P_{\delta(D)})| - \dim|K - (D + \imath(P_1) + \cdots + \imath(P_{\delta(D)}))| \\ = d + \delta(D) - g + 1. \tag{9}$$

Since $D + \imath(P_1) + \cdots + \imath(P_{\delta(D)})$ is effective and special, by the classical Clifford inequality, we get

$$\dim|D + \imath(P_1) + \cdots + \imath(P_{\delta(D)})| \leqslant \frac{1}{2}(d + \delta(D)).$$

Replacing in (9), we have

$$\dim|K - (D + \imath(P_1) + \cdots + \imath(P_{\delta(D)}))| \leqslant \frac{1}{2}(d + \delta(D)) - (d + \delta(D)) + g - 1 \\ = g - 1 - \frac{1}{2}(d + \delta(D)). \tag{10}$$

Finally, combining (8) and (10), we get

$$\dim|D| \leqslant \frac{1}{2}(d - \delta(D)).$$ □

Theorem 4.5. *Let X be a real hyperelliptic curves that is not r.b.i. and let $D \in \mathrm{Div}(X)$ be an effective and special divisor of degree d. Then*

$$\dim|D| \leqslant \frac{1}{2}(d - \delta(D)),$$

except when $|D| = rg_2^1$, with $0 < r < g - 1$ and r odd, in which case $\dim|D| = r = \frac{1}{2}d$.

Proof. We recall that under these hypotheses $s = 2$. If $\delta(D) = 0$, the classical Clifford inequality applies.

So let us assume $\delta(D) = 1$. As in the previous proof, we write $D = D' + D''$, with D' the greatest effective common subdivisor of D and $\imath(D)$, and D'' effective. Since $\delta(D) = 1$ and since the two connected components of $X(\mathbb{R})$ are exchanged by $\imath$, we easily see that there exists a real point P in the support of D''. Repeating the proof of the previous theorem, we get the result.

Now, if $\delta(D) = 2$, then the above arguments give the proof, except when D is invariant by $\imath$. In this case $|D| = rg_2^1$, r is odd and $\dim|D| = r$. □

We give some applications of the previous theorems. We know that Castelnuevo's inequality is one of the consequences of the (complex) Clifford inequality (see [3] corollary p. 251). Hence, we obtain a Castelnuevo inequality for real hyperelliptic curves.

Proposition 4.6. *Let $n \geqslant 2$ be an integer and $X \subseteq \mathbb{P}^n_{\mathbb{R}}$ be a non-degenerate real hyperelliptic curve r.b.i. Let d be the degree of X and δ be the number of pseudo-lines of X. Assume $d < 2n + \delta$. Then*

$$g \leqslant d - n,$$

with equality holding if and only if X is linearly normal.

Proof. Let H be a hyperplane section of X. Then $\dim|H| \geqslant n > \frac{1}{2}(d - \delta(H))$ by the hypotheses. Theorem 4.4 says that H is non-special and by Riemann–Roch,

$$g = d - \dim|H| \leqslant d - n.$$

Clearly, the previous inequality becomes an equality if and only if the map $H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(1)) \hookrightarrow H^0(X, \mathcal{O}(1))$ is an isomorphism. □

Proposition 4.7. *Let X be a real hyperelliptic curve r.b.i. Let $D = P_1 + \cdots + P_r$, $0 \leqslant r \leqslant g$, such that $P_1, \ldots, P_r \in X(\mathbb{R})$ and such that no two of them belong to the same connected component of $X(\mathbb{R})$. Then $\ell(D) = 1$.*

Remark 4.8. In the previous proposition, if X is any real algebraic curve and $r = s$, by Theorem 4.1, we can only say that $\ell(D) \leqslant 2$.

Let us set some more notations. For $d \geqslant 0$, let $S^d X$ denote the symmetric d-fold product of X over $\mathbb{R}$. We have a natural map $\varphi_d : (S^d X)(\mathbb{R}) \to \mathrm{Pic}^d(X)$. Write $W_d(\mathbb{R}) = \mathrm{Im}(\varphi_d)$ for the real part of the subvariety W_d of $\mathrm{Pic}^d(X_{\mathbb{C}})$ (see [1]), and $\theta(\mathbb{R}) \subseteq J(\mathbb{R})$ for the real part of the theta divisor.

Proposition 4.9. *Let X be a real hyperelliptic curve r.b.i. such that $s \geqslant g - 1$. If $u \in (\mathbb{Z}/2)^s$ satisfies $\sum_{i=1}^s c_i(u) \geqslant g - 1$, then $W_{g-1}(\mathbb{R}) \cap U(g-1; u)$ does not contain any singularity of W_{g-1}.*

Proof. If $\sum_{i=1}^{s} c_i(u) \geqslant g$, then $W_{g-1}(\mathbb{R}) \cap U(g-1;u) = \emptyset$. Let $[D] \in W_{g-1}(\mathbb{R}) \cap U(g-1;u)$, where $u \in (\mathbb{Z}/2)^s$ satisfies $\sum_{i=1}^{s} c_i(u) = g-1$. Then $\ell(D) > 0$ and we may assume that D is effective. By [1] Corollary 4.5, p. 190, the singular points of W_{g-1} correspond to the complete linear systems of dimension $\geqslant 1$ and degree $g-1$. By Theorem 4.4, $\dim|D| = 0$, hence the result. □

We may extend the previous proposition in any degree.

Proposition 4.10. *Let X be a real hyperelliptic curve r.b.i., and let d be a non-negative integer $\leqslant s$. If $u \in (\mathbb{Z}/2)^s$ satisfies $\sum_{i=1}^{s} c_i(u) \geqslant d$, then $W_d(\mathbb{R}) \cap U(d;u)$ does not contain any singularity of W_d.*

We prove a result similar to Theorem 3.1 in [7].

Proposition 4.11. *Let X be a real hyperelliptic curve r.b.i. which is an $(M-2)$-curve. Then $C_1 \times \cdots \times C_{g-1}$ is homeomorphic to the real part of the theta divisor contained in the neutral component $J(\mathbb{R})_0$ of the real part of the Jacobian.*

Proof. By Proposition 4.9, $C_1 \times \cdots \times C_{g-1}$ is homeomorphic to the part of $W_{g-1}(\mathbb{R})$ contained in $U(g-1;1,\dots,1)$. We may easily find a theta-characteristic $\kappa \in \mathrm{Pic}^{g-1}(X)$ (i.e. $2\kappa = [K]$) such that $\kappa \in U(g-1;1,\dots,1)$. By Riemann's theorem (see [1] p. 27), $W_{g-1} = \theta + \kappa$ and the proof is straightforward. □

Remark 4.12. Most of the results of this section are also valid for any real algebraic curve with $s \geqslant g$ (see [8] and Theorem 4.1).

The following result states a remarkable property of some special linear systems.

Proposition 4.13. *Let X be a real hyperelliptic curve r.b.i. Let $D \in \mathrm{Div}(X)$ be a special effective divisor of degree d satisfying* $\dim|D| = \frac{1}{2}(d - \delta(D))$. *Then $|D|$ contains a totally real divisor.*

Proof. Firstly, we assume that $d \leqslant g$. A consequence of the geometric version of the Riemann–Roch theorem is that any complete g_d^r on $X_{\mathbb{C}}$ is of the form

$$rg_2^1 + P_1 + \cdots + P_{d-2r},$$

where no two of the P_i are conjugate under $\imath$. Hence the complete linear system $|D|$ on $X_{\mathbb{C}}$ is of this form, with $r = \frac{1}{2}(d - \delta(D))$. Since $D = \overline{D}$, we have $D' = P_1 + \cdots + P_{d-2r} \in \mathrm{Div}(X)$. It follows that $|D|$ is of the form

$$\frac{1}{2}(d - \delta(D))g_2^1 + D',$$

where D' is an effective divisor of degree $\delta(D)$. Any divisor in g_2^1 is linearly equivalent to $P + \imath(P)$ where $P \in X(\mathbb{R})$. Since X is r.b.i., we have $\delta(D') = \delta(D)$ and D' is totally real.

Secondly, let $d > g$. The residual divisor $K - D$ is of degree $2g - 2 - d < g - 2$. Since the degree of K is even on each connected component of $X(\mathbb{R})$, $K - D$ is also special and satisfies $\delta(K - D) = \delta(D)$. So, $\dim|K - D| = \frac{1}{2}(d - \delta(D)) - d + g - 1 = \frac{1}{2}(2g - 2 - d - \delta(D)) = \frac{1}{2}(\deg(K - D) - \delta(K - D))$. We may apply the first part of the proof to $K - D$ to obtain that

$$[K - D] = \left[\frac{1}{2}(2g - 2 - d - \delta(D))(P + \imath(P))\right] + [P_1 + \cdots + P_{\delta(D)}], \qquad (11)$$

where $P, P_1, \ldots, P_{\delta(D)} \in X(\mathbb{R})$ and no two of the P_i are conjugate under $\imath$. Since $|K| = (g - 1)g_2^1$, then $[K] = [(g - 1)(P + \imath(P))]$. From (11), we get

$$[D] = \left[\frac{1}{2}(d + \delta(D))(P + \imath(P))\right] - [P_1 + \cdots + P_{\delta(D)}].$$

Then $[D] - [\imath(P_1) + \cdots + \imath(P_{\delta(D)})] = [\frac{1}{2}(d - \delta(D))(P + \imath(P))]$ and the proof is done. □

5 Existence of special linear systems of dimension r on real curves

Curves are classified by their genus. But we may further subdivide them according to whether or not they possess complete g_d^r, i.e. complete linear systems of degree d and dimension $r \geqslant 1$, for various d and r. For complex curves, we may find numerous results on this subject, it is a part of the Brill–Noether theory. This section deals with these problems for real curves.

5.1 Complete linear systems of dimension r on real curves.

Definition 5.1. For X a real curve and r a positive integer, we set:

(i) $\rho_{\mathbb{C}}(X, r) = \inf\{d \in \mathbb{N} \mid X_{\mathbb{C}} \text{ has a complete } g_d^r\}$.

(ii) $\rho_{\mathbb{R}}(X, r) = \inf\{d \in \mathbb{N} \mid X \text{ has a complete } g_d^r\}$.

For $g \geqslant 0$ and $r > 0$, we set:

(iii) $\rho_{\mathbb{C}}(g, r) = \sup\{\rho_{\mathbb{C}}(X, r) \mid X \text{ is a curve of genus } g\}$.

(iv) $\rho_{\mathbb{R}}(g, r) = \sup\{\rho_{\mathbb{R}}(X, r) \mid X \text{ is a curve of genus } g\}$.

Remark 5.2. It is easy to check that the $g^r_{\rho_{\mathbb{R}}(X,r)}$ and $g^r_{\rho_{\mathbb{C}}(X,r)}$ are necessarily complete.

Using the Riemann–Roch formula, it is easy to show that $\rho_{\mathbb{R}}(0, r) = \rho_{\mathbb{C}}(0, r) = r$, and that $\rho_{\mathbb{R}}(1, r) = \rho_{\mathbb{C}}(1, r) = r + 1$. In the remainder of the section we will assume that $g \geqslant 2$.

Remarks 5.3. If $r > g-1$, by the classical Clifford inequality, a complete g_d^r is non-special and $\rho_{\mathbb{R}}(X,r) = g+r$. From now on, we will also assume that $r \leqslant g-1$ in order to deal with special linear systems. By the classical Clifford inequality, and since there exist non-special linear systems of degree $g+r$, we have

$$2r \leqslant \rho_{\mathbb{R}}(X,r) \leqslant g+r.$$

Since the canonical divisor is invariant by the complex conjugation, by the previous inequality, we have the equalities

$$\rho_{\mathbb{R}}(X,g-1) = \rho_{\mathbb{R}}(g,g-1) = \rho_{\mathbb{C}}(X,g-1) = \rho_{\mathbb{C}}(g,g-1) = 2g-2.$$

From the classical theory of special linear systems (see [1] Theorem 1.1 p. 206, Theorem 1.5 p. 214), we may see $\rho_{\mathbb{C}}(g,r)$ as the smallest integer d such that the Brill–Noether number $\rho(g,r,d) = g-(r+1)(g-d+r)$ is non-negative.

Now, we state the principal result of this section.

Theorem 5.4. *Let X be a real curve and let r be an integer such that $1 \leqslant r \leqslant g-1$. Then*

(i) $\rho_{\mathbb{R}}(X,r) \leqslant \rho_{\mathbb{R}}(g,r) \leqslant g+r-1$, *and*

(ii) $\rho_{\mathbb{C}}(X,r) \leqslant \rho_{\mathbb{R}}(X,r) \leqslant 2\rho_{\mathbb{C}}(X,r) - 2r$.

Proof. For (i), let $D \in \mathrm{Div}(X)$ be an effective divisor of degree $g-1-r$. Choosing D general, we have $\ell(D) = 1$. Then the residual divisor $K - D \in \mathrm{Div}(X)$ and satisfies $\ell(K-D) = 1-(g-1-r)+g-1 = r+1$, and we get the first assertion.

As for (ii), clearly $\rho_{\mathbb{C}}(X,r) \leqslant \rho_{\mathbb{R}}(X,r)$, since for any divisor $D \in \mathrm{Div}(X)$ we have $\ell(D) = \ell_{\mathbb{C}}(D)$. It remains to show that

$$\rho_{\mathbb{R}}(X,r) \leqslant 2\rho_{\mathbb{C}}(X,r) - 2r.$$

Let $d = \rho_{\mathbb{C}}(X,r)$ and let $D \in \mathrm{Div}(X_{\mathbb{C}})$ be an effective divisor of degree d such that $\dim_{\mathbb{C}}|D| = r$. Let $P_1,\dots,P_r$ be real points of X. We also denote by $P_1,\dots,P_r$ the corresponding closed points of $X_{\mathbb{C}}$. We may choose $P_1,\dots,P_r$ such that $\ell(P_1+\cdots+P_r) = 1$ and $\ell_{\mathbb{C}}(D-P_1-\cdots-P_r) = 1$. Moreover, we may assume that $D = D''+D'$, where:

1) D'' is an effective divisor of degree u satisfying $\overline{D}'' = D''$, and having $P_1,\dots,P_r$ in its support.

2) D' is an effective divisor such that there is no nonzero effective divisor $\leqslant D'$ invariant by the complex conjugation.

If $D' = 0$, then $D \in \mathrm{Div}(X)$ and $\rho_{\mathbb{C}}(X,r) = \rho_{\mathbb{R}}(X,r)$. So, assume $D' \neq 0$ and let $l = \dim|D''|$. If $r = l$, then $|D|$ has a base point, but then $X_{\mathbb{C}}$ has a complete g_k^r

with $k < d$, hence a contradiction. It follows that $r > l$. Since X is a real curve, $\dim_{\mathbb{C}}|D'' + \bar{D}'| = r$. We may find suitable nonzero effective divisors $D'_1, \ldots, D'_{r-l}$ such that

1) $D' = D'_1 + \cdots + D'_{r-l}$, and
2) $\ell_{\mathbb{C}}(D'' + D'_1 + \cdots + D'_i) = \ell_{\mathbb{C}}(D'') + i$, $i = 1, \ldots, r - l$.

Let $\{1, f_1, \ldots, f_r\}$ be a base of $H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + \bar{D}'))$ and $g_1, \ldots, g_{r-l} \in H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'))$ such that

1) $g_1 \in H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'_1)) \backslash H^0(X_{\mathbb{C}}, \mathcal{O}(D''))$, and
2) $g_i \in H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'_1 + \cdots + D'_i)) \backslash n H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'_1 + \cdots + D'_{i-1}))$, $i = 2, \ldots, r - l$.

Claim. $\ell_{\mathbb{C}}(D'' + D' + \bar{D}') \geqslant 2r + 1 - l$. More precisely, we show, by induction on i, that $1, f_1, \ldots, f_r, g_1, \ldots, g_i$ are linearly independent in the vector space $H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'_1 + \cdots + D'_i + \bar{D}'))$, i.e. $\ell_{\mathbb{C}}(D'' + D'_1 + \cdots + D'_i + \bar{D}') \geqslant r + 1 + i$.

For $i = 1$, $1, f_1, \ldots, f_r, g_1 \in H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'_1 + \bar{D}'))$ and $1, f_1, \ldots, f_r$ are linearly independent. If g_1 were a linear combination of $1, f_1, \ldots, f_r$, then g_1 would be a global section of $\mathcal{O}(D'' + \bar{D}')$ and also $\operatorname{div}_\infty(g_1) \leqslant D'' + \bar{D}'$. By the construction of g_1, $\operatorname{div}_\infty(g_1) \leqslant D'' + D'_1$. Since $\bar{D}'$ and D'_1 have distinct supports, we would have $\operatorname{div}_\infty(g_1) \leqslant D''$. This is a contradiction.

Assume now that $1, f_1, \ldots, f_r, g_1, \ldots, g_{i-1}$ $(r - l > i > 1)$ are linearly independent and that g_i would be a linear combination of $1, f_1, \ldots, f_r, g_1, \ldots, g_{i-1}$. Arguing as in the case $i = 1$, the pole divisor of g_i would be $\leqslant D'' + \bar{D}' + D'_1 + \cdots + D'_{i-1}$. By the construction of g_i, and since $\bar{D}'$ and D'_i have distinct supports, we would obtain $\operatorname{div}_\infty(g_i) \leqslant D'' + D'_1 + \cdots + D'_{i-1}$, contradicting the fact that $g_i \notin H^0(X_{\mathbb{C}}, \mathcal{O}(D'' + D'_1 + \cdots + D'_{i-1}))$. This ends the proof of the claim.

Since $D'' + D' + \bar{D}'$ is invariant by the complex conjugation, we get $\ell(D'' + D' + \bar{D}') = \ell_{\mathbb{C}}(D'' + D' + \bar{D}') \geqslant 2r + 1 - l$. Let $P'_1, \ldots, P'_{r-l}$ be suitable real points. Then $\ell(D'' + D' + \bar{D}' - P'_1 - \cdots - P'_{r-l}) \geqslant r + 1$, and X has at least one complete $g^r_{2d-u-r+l}$. To get the second assertion of the theorem, it is sufficient to prove that $r \leqslant u - l$. Since $\dim_{\mathbb{C}}|P_1 + \cdots + P_r| = 0$ and $D'' = P_1 + \cdots + P_r + E$, with $E \in \operatorname{Div}(X)$ an effective divisor of degree $u - r$, we get $l = \dim_{\mathbb{C}}|P_1 + \cdots + P_r + E| \leqslant u - r$. □

Let us mention some consequences of Theorem 5.4.

Corollary 5.5. *Let X be a real hyperelliptic curve and r be an integer such that $1 \leqslant r \leqslant g - 1$. Then $\rho_{\mathbb{R}}(X, r) = 2r$.*

Proof. Since $\rho_{\mathbb{C}}(X, r) = 2r$, using the first inequality of the theorem, the result follows. □

Corollary 5.6. *Let X be a real curve of genus g which is not hyperelliptic, and r*

be an integer such that $1 \leqslant r < g-1$. *Then* $\rho_{\mathbb{R}}(X,r) > 2r$. *In particular, we have* $\rho_{\mathbb{R}}(3,1) = 3$.

Proof. The proof is clear by Clifford's inequality and by the existence of real non-hyperelliptic curves of genus 3. □

Corollary 5.7. *Let X be a real curve. Then the map* $\varphi_g : (S^g X)(\mathbb{R}) \to \mathrm{Pic}^g(X)$ *is not injective.*

Proof. If $D \in \mathrm{Div}(X)$ of degree d satisfies $\ell(D) = 2$, then the fiber of φ_d at $[D]$ is one dimensional and the map φ_d is not injective. Theorem 5.4 asserts the existence of a g^1_g on X, hence we get the result. □

Remark 5.8. In Theorem 5.4, we get two upper bounds for $\rho_{\mathbb{R}}$, one of them depending on $\rho_{\mathbb{C}}$, but not the other. It is interesting to compare these two bounds. The invariant $\rho_{\mathbb{C}}$ is given by the Theorems 1.1 p. 206, 1.5 p. 214, in [1]. For X a general curve of genus g and $1 \leqslant r \leqslant g-1$:

$$\rho_{\mathbb{C}}(X,r) = g + r - \left[\frac{g}{r+1}\right].$$

($[\frac{g}{r+1}]$ is the integral part of $\frac{g}{r+1}$). We thus obtain

$$\rho_{\mathbb{R}}(X,r) \leqslant \min\left\{g + r - 1, 2g - 2\left[\frac{g}{r+1}\right]\right\}.$$

Assume $\frac{g}{r+1} \in \mathbb{N}$. We see that $2g - 2\frac{g}{r+1} = g + r - 1$ if $r = 1$ or $r = g-1$, and that $2g - 2\frac{g}{r+1} > g + r - 1$ if not. Hence $\rho_{\mathbb{R}}(g,r) \leqslant g + r - 1$ is the best upper bound we may find at this moment. As we have seen for hyperelliptic curves, the second inequality of Theorem 5.4 gives a smaller upper bound for $\rho_{\mathbb{R}}(X,r)$ only when X is a special curve.

5.2 Complete linear systems of dimension 1 on real curves. The following theorem is a refinement of Theorem 5.4 in dimension 1.

Proposition 5.9. *Let X be a real curve of genus* $g \geqslant 2$. *If* $X_{\mathbb{C}}$ *has exactly an odd number of* $g^1_{\rho_{\mathbb{C}}(X,1)}$, *then* $\rho_{\mathbb{R}}(X,1) = \rho_{\mathbb{C}}(X,1)$.

Proof. We have $\rho_{\mathbb{C}}(X,1) \geqslant 2$ since X is not rational. Let $d = \rho_{\mathbb{C}}(X,1)$. Assume that $X_{\mathbb{C}}$ has exactly $2n+1$ distinct g^1_d, $n \in \mathbb{N}$. Let D'_i, $i = 1, \dots, 2n+1$, some effective divisors on $X_{\mathbb{C}}$ such that the linear systems $|D'_i|$ are the $2n+1$ distinct g^1_d. We may clearly assume that, for every i, $D'_i = P + D_i$, with P a closed point of $X_{\mathbb{C}}$ satisfying $\bar{P} = P$ and with $D_i \in \mathrm{Div}(X_{\mathbb{C}})$ an effective divisor of degree $d-1$. Since X is real, the linear systems $|\bar{D}'_i|$ are also g^1_d. Consequently, there exists $k \in \{1, \dots, 2n+1\}$ such that $|D'_k| = |\bar{D}'_k|$, since an involution acting on a finite set with an odd number of elements has a fixed point. Hence $P + D_k$ is linearly equivalent to $P + \bar{D}_k$. Consequently, either $D_k = \bar{D}_k$ and we have the claim, or $|D_k|$ is a g^1_{d-1}, but then d is not minimal. □

Corollary 5.10. *Let X be a general real curve of genus* 6. *Then* $\rho_{\mathbb{C}}(X,1) = \rho_{\mathbb{R}}(X,1) = 4$.

Proof. Since $X_{\mathbb{C}}$ has five g^1_4 (see [3] p. 299), the proof follows from the above proposition. □

5.3 A real Brill–Noether number. In the context, a natural question one can ask is about the existence of a real curve X of genus $g \geqslant 2$ with $\rho_{\mathbb{R}}(X,1) = g$. Such an existence, for any $g \geqslant 2$, would show that $\rho_{\mathbb{R}}(g,1) = g$.

If $d < g$ and X is a real curve of genus g having a g^1_d, then, adding $(g-1-d)$ general real points to this g^1_d, we get a complete g^1_{g-1}. By [1] Corollary 4.5, p. 190, the singularities of $W_{g-1} = \varphi_{g-1}(S^{g-1}X_{\mathbb{C}})$ are the complete g^k_{g-1} with $k > 0$, where $\varphi_{g-1} : S^{g-1}X_{\mathbb{C}} \to \mathrm{Pic}^{g-1}(X_{\mathbb{C}})$; $(P_1, \ldots, P_{g-1}) \mapsto [P_1 + \cdots + P_{g-1}]$ is the natural map. By Riemann's singularity theorem (see [1] p. 226), the singular part of the theta divisor $\theta \subseteq J(\mathbb{C})$ is a translation (by a theta-characteristic) of the singular part of W_{g-1}. Recall that a real curve always admits real theta-characteristics [4]. Hence we may reformulate the previous question asking if there exist real curves of genus $g \geqslant 2$ with $\theta(\mathbb{R})$ non-singular. We state the following conjecture:

Conjecture 1. *Let $g \geqslant 2$ be an integer. There exists a real curve X of genus g such that the singularities of the theta divisor $\theta \subseteq J(\mathbb{C})$ are not real.*

Proposition 5.11. *The above conjecture holds for* $2 \leqslant g \leqslant 4$.

Proof. The conjecture holds trivially for genus 2 curves, and genus 3 curves since there exist non-hyperelliptic real curves of genus 3.

Following Gross and Harris [4], we may show that the conjecture holds for genus 4 curves. Let X be a real trigonal curve (i.e. $\rho_{\mathbb{C}}(X,1) \leqslant 3$) of genus 4 which is non-hyperelliptic. Its canonical model lies on a unique real quadric surface $S \subseteq \mathbb{P}^3_{\mathbb{R}}$. For a general X, S is smooth and then has two different rulings. For some X, these two rulings are complex and switched by the complex conjugation. Then $X_{\mathbb{C}}$ has only two g^1_3 induced by these two rulings and $\rho_{\mathbb{C}}(X,1) = 3$. By this, we conclude that $\rho_{\mathbb{R}}(X,1) = 2\rho_{\mathbb{C}}(X,1) - 2 = 4$. □

Let $g, r \in \mathbb{N}$ satisfying $g \geqslant 2$ and $1 \leqslant r \leqslant g-1$. If $d - (g+r-1) \geqslant 0$, then Theorem 5.4 says that any real curve of genus g has a complete g^r_d. We may wonder if this condition is optimal.

Conjecture 2. *Let $g \geqslant 2$ and $1 \leqslant r \leqslant g-1$. The real Brill–Noether number is* $\rho_{\mathbb{R}}(g,r,d) = d - (g+r-1)$, *i.e. if* $d - (g+r-1) < 0$, *then there exists a real curve X of genus g such that X has no g^r_d.*

Remark that Conjecture 2 implies Conjecture 1.

6 Linear systems with base points on real curves

This section is devoted to the problem of finding lower bounds for N. We prove that this problem is related to the existence of special linear systems of dimension 1 and small degree, i.e. the subject of the previous section.

Proposition 6.1. *Let X be a real curve of genus $g \geqslant 2$. Assume that X has a complete g_d^r, with $d \leqslant g - 1$ and $r \geqslant 1$. Then $N(X) > 2g - d$.*

Proof. Let $D \in \mathrm{Div}(X)$ be an effective divisor of degree $d \leqslant g - 1$ such that $\dim|D| = r \geqslant 1$. It is a special divisor. Let $D' \in \mathrm{Div}(X)$ be an effective divisor of degree $2g - 2 - d$ such that $D + D'$ is the canonical divisor. Let Q be a non-real point of X such that $\ell(D - Q) = \ell(D) - 2$. By Riemann–Roch,

$$\ell(D' + Q) = 2g - 2 - d + 2 - g + 1 + \ell(D - Q) = g - d + 1 + r - 1 = \ell(D').$$

Hence Q is a base point of $|D' + Q|$ and consequently the divisor $D' + Q$ of degree $2g - d$ is not linearly equivalent to a totally real effective divisor. Clearly $N(X) > 2g - d$. □

The existence of linear systems of small degree on real curves is studied in the previous section. One of the results, is that a real curve has always a complete g_g^1, hence

Corollary 6.2. *Let X be a real curve such that $g \geqslant 2$. Then $N(X) \geqslant g + 1$.*

From the previous section and Proposition 6.1, we obtain

Corollary 6.3. *Let X be a real curve of genus $g \geqslant 2$. If $\theta \subseteq J(\mathbb{C})$ has a real singularity, then $N(X) \geqslant g + 2$.*

If X is hyperelliptic, by Lemma 4.2, X has also a g_2^1 and we may state the following result (use Theorem 3.6):

Corollary 6.4. *Let X be a real hyperelliptic curve of genus $g \geqslant 2$. Then $N(X) \geqslant 2g - 1$. If furthermore X is an M-curve or an $(M - 1)$-curve, then $N(X) = 2g - 1$.*

Remark 6.5. Since there exist real hyperelliptic M-curves of any genus, the previous corollary gives a large family of curves for which the invariant N is explicitely calculated.

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J.-Ph. Monnier, Département de Mathématiques, Université d'Angers, 2, Bd. Lavoisier, 49045 Angers cedex 01, France
Email: monnier@tonton.univ-angers.fr