Special Issue (2003), S1–S12

Advances in Geometry

On Lorentz–Minkowski geometry in real inner product spaces

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Dedicated to Adriano Barlotti on the occasion of his 80th birthday, in friendship

Let X be a real inner product space of finite or infinite dimension $\geqslant 2$, and let $\varrho \neq 0$ be a fixed real number. The following results will be presented in this note.

A. A surjective mapping $\sigma : X \to X$ preserving Lorentz–Minkowski distances 0 and ϱ in one direction must be a Lorentz transformation.

B. The causal automorphisms of X, $\dim X \geqslant 3$, are exactly the products $\delta\lambda$, where λ is an orthochronous Lorentz transformation and δ a dilatation $x \to \alpha x$, $\mathbb{R} \ni \alpha > 0$.

C. If $\varrho > 0$, there exist X and an *injective* $\sigma : X \to X$ preserving Lorentz–Minkowski distance ϱ, such that σ is not a Lorentz transformation. This result can be extended, mutatis mutandis, to Euclidean and Hyperbolic Geometry.

If X is finite-dimensional, result A is an immediate consequence of the following theorem of Benz–Lester ([4], [12], [13], [5]).

Theorem 1. *Suppose that X is a real inner product space of finite dimension $\geqslant 2$ and that $\varrho \neq 0$ is a fixed real number. If $\sigma : X \to X$ satisfies*

$$l(x, y) = \varrho \Rightarrow l(\sigma(x), \sigma(y)) = \varrho$$

for all $x, y \in X$, where $l(x, y)$ designates the Lorentz–Minkowski distance of x, y, then σ must be a Lorentz transformation.

Moreover, if X is finite-dimensional, statement B is a well-known theorem of Alexandrov–Ovchinnikova–Zeeman ([1], [2], [17], [5]).

It could be possible that Theorem 1 also holds true in the infinite-dimensional case provided that $\varrho < 0$. However, a proof, if it exists, is not yet known. Result C shows that Theorem 1 cannot be extended to the infinite-dimensional case if $\varrho > 0$, not even in the injective case.

1 Notation

Let X be a real inner product space of arbitrary finite or infinite dimension $\geqslant 2$, i.e. a real vector space equipped with a fixed inner product

$$\tau : X \times X \to \mathbb{R}, \quad \tau(x, y) =: xy,$$

satisfying $x^2 := xx > 0$ for all $x \neq 0$ of X. Notice that X need not be complete, i.e. that X need not be a real Hilbert space. Take a fixed $t \in X$ with $t^2 = 1$ and define $t^\perp := \{x \in X \mid xt = 0\}$. Observe $X = t^\perp \oplus \mathbb{R}t$. We hence get the uniquely determined decomposition

$$x =: \bar{x} + x_0 t$$

with $\bar{x} \in t^\perp$ and $x_0 \in \mathbb{R}$ for every $x \in X$. Define

$$l(x, y) := (\bar{x} - \bar{y})^2 - (x_0 - y_0)^2$$

to be the *Lorentz–Minkowski distance* of $x, y \in X$. The mapping $\lambda : X \to X$ is called a *Lorentz transformation* if, and only if,

$$l(x, y) = l(\lambda(x), \lambda(y))$$

holds true for all $x, y \in X$.

Remark. It might be noticed that the theory does not seriously depend on the chosen t ([6], p. 229).

The Lorentz transformations as defined before can explicitly be written by means of (proper or improper) Lorentz boosts and orthogonal transformations ([6], p. 221): For $p \in t^\perp$ with $p^2 < 1$ and $-1 \neq k \in \mathbb{R}$ with $k^2 \cdot (1 - p^2) = 1$ define for all $x \in X$,

$$A_p(x) := x_0 p + (\bar{x}p)t, \quad B_{p,k}(x) := x + kA_p(x) + \frac{k^2}{k+1} A_p^2(x).$$

Obviously, $k^2 \geqslant 1$. The mappings $A_p, B_{p,k}$ are linear and $B_{p,k} : X \to X$ is even bijective. Define also

$$B_{0,-1}(x) := \bar{x} - x_0 t.$$

$B_{p,k}$ is called a *Lorentz boost*, a *proper* one for $k \geqslant 1$, an *improper* one for $k \leqslant -1$. All Lorentz transformations λ of X are exactly given by

$$\lambda(x) = (B_{p,k}\omega)(x) + d$$

with a boost $B_{p,k}$, an orthogonal and linear mapping ω from X into X satisfying $\omega(t) = t$, and with an element d of X.

The following theorem was proved by Cacciafesta [9] in the case $\dim X < \infty$, and by Benz [7] in the general case.

Theorem 2. *If* $\dim X \geqslant 3$ *and if* $\sigma : X \to X$ *is bijective and satisfies*

$$l(x, y) = 0 \Rightarrow l(\sigma(x), \sigma(y)) = 0$$

for all $x, y \in X$, *then* σ *must be the product of a Lorentz transformation and a dilatation.*

Important partial results of Theorem 2 were proved by Alexandrov [1] and by Schröder [15], [16]. Schröder even studied the case of an arbitrary field instead of $\mathbb{R}$.

2 Proof of result A

Lemma 1. *Let* γ *be a real number and* $x \neq 0$ *be an element of* X. *Then there exist* $v \neq 0$ *in* X *and* α *in* $\mathbb{R}$ *with* $\bar{v}^2 = v_0^2$ *and*

$$(\bar{x} + \alpha\bar{v})^2 - (x_0 + \alpha v_0)^2 = \gamma. \tag{1}$$

Proof. Case 1: $x_0 \neq 0$. Take an element e in $t^{\perp}$ with $e^2 = 1$. Then $\bar{x}e \neq x_0$ or $\bar{x}e \neq -x_0$. Assume $\bar{x}e \neq \varepsilon x_0$ with $\varepsilon \in \mathbb{R}$ and $\varepsilon^2 = 1$. Now put $v := e + \varepsilon t$ and, by observing $\bar{x}\bar{v} \neq x_0 v_0$,

$$2\alpha(\bar{x}\bar{v} - x_0 v_0) := \gamma + x_0^2 - \bar{x}^2. \tag{2}$$

Hence (1) holds true.

Case 2: $x_0 = 0$. Hence $x \neq 0$ implies $\bar{x} \neq 0$. Now put $v := \bar{x} + \|\bar{x}\| \cdot t$ with $\|z\| := \sqrt{z^2}$ for $z \in X$, and define α by (2). Then also here (1) holds true. □

Lemma 2. *If* $p \neq q$ *are elements of* X *and if* $\gamma \in \mathbb{R}$, *there exists* $r \in X$ *satisfying*

$$l(r, p) = \gamma \quad \textit{and} \quad l(r, q) = 0. \tag{3}$$

Proof. Put $x := q - p$ and take elements v and α according to Lemma 1. Hence $r := q + \alpha v$ satisfies (3). □

Lemma 3. *Suppose that* $\varrho \neq 0$ *is a fixed real number and that* $\sigma : X \to X$ *satisfies*

$$l(x, y) = 0 \Rightarrow l(\sigma(x), \sigma(y)) = 0 \tag{4}$$

and

$$l(x, y) = \varrho \Rightarrow l(\sigma(x), \sigma(y)) = \varrho \tag{5}$$

for all $x, y \in X$. *Then* σ *must be injective.*

Proof. If $p \neq q$ are elements of X, take, in view of Lemma 2, $r \in X$ with $l(r,p) = \varrho$ and $l(r,q) = 0$. Hence, by (4), (5),

$$l(r',p') = \varrho \quad \text{and} \quad l(r',q') = 0 \tag{6}$$

where we put $z' = \sigma(z)$ for $z \in X$. Now (6) implies $p' \neq q'$. □

If $\dim X < \infty$, result A follows from Theorem 1. Suppose now that X is infinite-dimensional and that $\sigma : X \to X$ is surjective, satisfying (4) and (5) for all $x, y \in X$, where $\varrho \neq 0$ is a fixed real number. Hence, by Lemma 3, σ is injective, and thus bijective. Hence, by Theorem 2, there exists a Lorentz transformation $\lambda : X \to X$ and a real number $k \neq 0$ such that

$$\sigma(x) = k \cdot \lambda(x)$$

for all $x \in X$. Now (5) implies

$$l(x,y) = \varrho \Rightarrow \varrho = l(k\lambda(x), k\lambda(y))$$

for all $x, y \in X$, i.e. $l(x,y) = \varrho$ implies

$$\varrho = k^2 \cdot l(\lambda(x), \lambda(y)) = k^2 \cdot l(x,y) = k^2 \cdot \varrho.$$

Hence $k^2 = 1$, in view of $\varrho \neq 0$. If $k = 1$, we get $\sigma = \lambda$, and if $k = -1$, we obtain

$$\sigma(x) = -\lambda(x)$$

for all $x \in X$. But this is also a Lorentz transformation.

So we have proved

Theorem A. *Let $\varrho \neq 0$ be a fixed real number and let $\sigma : X \to X$ be a surjective mapping satisfying (4) and (5) for all $x, y \in X$. Then there exist a Lorentz boost $B_{p,k}$, a linear, bijective and orthogonal mapping $\omega : X \to X$ with $\omega(t) = t$, and an element d of X such that*

$$\sigma(x) = (B_{p,k}\omega)(x) + d$$

for all $x \in X$.

Remark. Theorem A holds true, as was shown, for all real inner product spaces X with $\dim X \geqslant 2$. If $X = \mathbb{R}^2$, if we put

$$xy := x_1 y_1 + x_2 y_2$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$ of X, and $t := (1/\sqrt{2}, 1/\sqrt{2})$, then

$$l(x,y) = -2(x_1 - y_1)(x_2 - y_2)$$

for all $x, y \in X$. Let f be a non-continuous bijection of $\mathbb{R}$, for instance $f(0) = 1$, $f(1) = 0$ and $f(x) = x$ otherwise, then

$$\sigma(x_1, x_2) := (f(x_1), x_2)$$

is a non-continuous bijection of X satisfying

$$l(x, y) = 0 \Leftrightarrow l(\sigma(x), \sigma(y)) = 0$$

for all $x, y \in X$ (Rätz [14]). Hence σ cannot be a Lorentz transformation, and it even cannot be a product of a Lorentz transformation and a dilatation. In the case $\dim X \geqslant 2$, the mapping $\sigma(x) = 2x$ is bijective, it satisfies (4), but not (5) for any given $\varrho \neq 0$. So it cannot be a Lorentz transformation.

3 Causal automorphisms

Let x, y be elements of X. Also in the infinite-dimensional case we put

$$x \leqslant y$$

if, and only if, $l(x, y) \leqslant 0$ and $x_0 \leqslant y_0$ hold true. A bijection $\sigma : X \to X$ is called a *causal automorphism* if, and only if,

$$x \leqslant y \Leftrightarrow \sigma(x) \leqslant \sigma(y)$$

for all $x, y \in X$.

The proof of Proposition 1 is not difficult.

Proposition 1. *Let x, y, z be elements of X and let k be a real number. Then the following statements hold true.*

(i) $x \leqslant x$,

(ii) $x \leqslant y$ *and* $y \leqslant x$ *imply* $x = y$,

(iii) $x \leqslant y$ *and* $y \leqslant z$ *imply* $x \leqslant z$,

(iv) $x \leqslant y$ *implies* $x + z \leqslant y + z$,

(v) $x \leqslant y$ *implies* $kx \leqslant ky$ *for* $k \geqslant 0$,

(vi) $x \leqslant y$ *implies* $kx \geqslant ky$ *for* $k < 0$.

Of course, $x < y$ stands for $x \leqslant y$ and $x \neq y$, $x \geqslant y$ for $y \leqslant x$, and $x > y$ for $y < x$.

Suppose that x, y are elements of X satisfying $x < y$. Then

$$[x, y] := \{z \in X \mid x \leqslant z \leqslant y\}$$

is called *ordered* if, and only if,

$$u \leqslant v \quad \text{or} \quad v \leqslant u$$

holds true for all $u, v \in [x, y]$.

Proposition 2. *Let x, y be elements of X with $x < y$. Then $l(x, y) = 0$ if, and only if, $[x, y]$ is ordered.*

Proof. a) Assume $l(x, y) = 0$ and $u \in [x, y]$, i.e.

$$x_0 \leqslant u_0 \leqslant y_0, \quad \|\bar{u} - \bar{x}\| \leqslant u_0 - x_0, \quad \|\bar{y} - \bar{u}\| \leqslant y_0 - u_0.$$

$l(x, y) = 0$ implies $\|\bar{y} - \bar{x}\| = y_0 - x_0$. Hence

$$y_0 - x_0 = \|\bar{y} - \bar{x}\| \leqslant \|\bar{y} - \bar{u}\| + \|\bar{u} - \bar{x}\| \leqslant y_0 - x_0, \tag{7}$$

and thus $\|\bar{y} - \bar{x}\| = \|\bar{y} - \bar{u}\| + \|\bar{u} - \bar{x}\|$. Since X is strictly convex, $\bar{y} - \bar{u}$, $\bar{u} - \bar{x}$ must be linearly dependent. Hence there exists $\alpha \in \mathbb{R}$ with

$$\bar{u} = \bar{x} + \alpha(\bar{y} - \bar{x}), \tag{8}$$

in view of $\bar{x} \neq \bar{y}$; observe that $\bar{x} = \bar{y}$ and $\|\bar{y} - \bar{x}\| = y_0 - x_0$ would imply $x = y$. Now (7), (8) yield

$$\|\bar{y} - \bar{x}\| = \|\bar{y} - \bar{u}\| + \|\bar{u} - \bar{x}\| = |1 - \alpha| \, \|\bar{y} - \bar{x}\| + |\alpha| \, \|\bar{y} - \bar{x}\|,$$

i.e. $1 = |1 - \alpha| + |\alpha|$, i.e. $0 \leqslant \alpha \leqslant 1$. Hence, with $\xi := y_0 - x_0$,

$$\xi = (1 - \alpha)\xi + \alpha\xi = \|\bar{y} - \bar{u}\| + \|\bar{u} - \bar{x}\| \leqslant (y_0 - u_0) + (u_0 - x_0) = \xi,$$

i.e. $\|\bar{y} - \bar{u}\| = y_0 - u_0$, $\|\bar{u} - \bar{x}\| = u_0 - x_0$, i.e. by (8),

$$u = x + \alpha(y - x).$$

Similarly, $v \in [x, y]$ implies

$$v = x + \beta(y - x), \quad 0 \leqslant \beta \leqslant 1.$$

Hence $u \leqslant v$ for $\alpha \leqslant \beta$, and $v \leqslant u$ for $\beta \leqslant \alpha$.

b) Assume that $[x, y]$ is ordered and that $l(x, y) \neq 0$. Hence, by $x < y$, we obtain $l(x, y) < 0$ and $x_0 \leqslant y_0$, i.e.

$$(\bar{y} - \bar{x})^2 < (y_0 - x_0)^2 \quad \text{and} \quad x_0 < y_0.$$

Choose $e \in t^{\perp}$ with $e^2 = 1$ and $\varepsilon \in \mathbb{R}$ with

$$0 < 2\varepsilon < (y_0 - x_0) - \|\bar{y} - \bar{x}\|, \tag{9}$$

and put

$$u := \frac{x+y}{2}, \quad v := \frac{x+y}{2} + \varepsilon e.$$

Observe $u_0 = v_0$ and $\bar{v} - \bar{u} = \varepsilon e$, i.e. $l(u, v) = \varepsilon^2 > 0$, i.e.

$$u \nleqslant v \quad \text{and} \quad v \nleqslant u. \tag{10}$$

Moreover,

$$u, v \in [x, y]. \tag{11}$$

In order to prove (11), we observe, first of all,

$$x_0 \leqslant u_0 \leqslant y_0 \quad \text{and} \quad x_0 \leqslant v_0 \leqslant y_0,$$

by $u_0 = v_0 = \frac{1}{2}(x_0 + y_0)$. Secondly,

$$l(x, u) = \frac{1}{4} l(x, y) = l(u, y),$$

i.e. $l(x, u) = l(u, y) < 0$. The triangle inequality yields

$$\left\| \frac{\bar{y} - \bar{x}}{2} \pm \varepsilon e \right\| \leqslant \left\| \frac{\bar{y} - \bar{x}}{2} \right\| + \varepsilon,$$

i.e. by (9),

$$\left\| \frac{\bar{y} - \bar{x}}{2} \pm \varepsilon e \right\| < \frac{y_0 - x_0}{2}.$$

Hence

$$\left(\frac{\bar{y} - \bar{x}}{2} \pm \varepsilon e \right)^2 < \left(\frac{y_0 - x_0}{2} \right)^2,$$

i.e. $l(x, v)$ and $l(v, y)$ are negative. Because of (10), (11), $[x, y]$ is not ordered, a contradiction. Hence $l(x, y) = 0$. □

A Lorentz transformation λ of X is called *orthochronous* if, and only if, it is also a causal automorphism.

Proposition 3. *The orthochronous Lorentz transformations λ are exactly given by all mappings*

$$\lambda(x) = (B_{p,k}\omega)(x) + d \tag{12}$$

with $\omega : X \to X$ *linear, orthogonal, bijective,* $\omega(t) = t$, $d \in X$, *and* $k \geqslant 1$.

Proof. a) Let λ be an arbitrary orthochronous Lorentz transformation, say

$$\lambda(x) = (B_{p,k}\omega)(x) + d.$$

Since λ is bijective, also $\omega : X \to X$ must be bijective. Moreover, $0 \leqslant t$ implies $\lambda(0) \leqslant \lambda(t)$, i.e.

$$d \leqslant kt + kp + d,$$

i.e. $0 \leqslant kp + kt$, i.e. $0 \leqslant k$, i.e. $1 \leqslant k$, in view of $k^2 \geqslant 1$.

b) Let λ be a mapping (12) with proper $B_{p,k}$ and bijective ω satisfying $\omega(t) = t$. We then have to prove

$$a \leqslant b \Leftrightarrow \lambda(a) \leqslant \lambda(b)$$

for all $a, b \in X$. This is clear for $\lambda(x) = x + d$, in view of Proposition 1 (iv). It is also clear for $\lambda(x) = \omega(x)$ because of

$$0 \geqslant l(a,b) = l(\lambda(a), \lambda(b)),$$
$$\omega(\bar{x} + x_0 t) = \omega(\bar{x}) + x_0 t,$$
$$\omega(\bar{x})t = \omega(\bar{x})\omega(t) = \bar{x}t = 0,$$

i.e. $\overline{\omega(x)} = \omega(\bar{x})$, and on account of the fact that ω^{-1} is linear and orthogonal as well, satisfying $\omega^{-1}(t) = t$.

Finally, we consider the case $\lambda(x) = B_{p,k}(x)$ with $k \geqslant 1$. Since $B_{p,k}^{-1} = B_{-p,k}$ we only have to prove

$$a \leqslant b \Rightarrow \lambda(a) \leqslant \lambda(b),$$

i.e. $0 \leqslant b - a \Rightarrow 0 \leqslant \lambda(b) - \lambda(a) = \lambda(b - a)$, i.e.

$$0 \leqslant x \Rightarrow 0 \leqslant \lambda(x).$$

Again, $0 \geqslant l(0, x) = l(\lambda(0), \lambda(x)) = l(0, \lambda(x))$. Since $z_0 = zt$ for all $z \in X$, we get

$$[B_{p,k}(x)]_0 = x_0(1 + k(k-1)) + k\bar{x}p =: R.$$

It remains to prove $R \geqslant 0$ in the case $0 \leqslant x_0$ and $\bar{x}^2 - x_0^2 = l(0, x) \leqslant 0$. If $\bar{x}p \geqslant 0$, we get $R \geqslant 0$ since $k \geqslant 1$. If $\bar{x}p < 0$, we observe

$$(\bar{x}p)^2 \leqslant \bar{x}^2 p^2 \leqslant x_0^2 \cdot 1 = x_0^2,$$

i.e. $-\bar{x}p = |\bar{x}p| \leqslant x_0$, i.e. $-x_0 \leqslant \bar{x}p$, i.e.

$$R \geqslant x_0(1 + k(k-1)) - kx_0 = x_0(k-1)^2 \geqslant 0. \qquad \square$$

Theorem B. *The causal automorphisms of* X, $\dim X \geqslant 3$, *are exactly given by all mappings*

$$\lambda(x) = \gamma \cdot (B_{p,k}\omega)(x) + d, \tag{13}$$

where $\gamma > 0$ *is a real number*, $B_{p,k}$ *a proper Lorentz boost*, ω *a linear*, *orthogonal*, *bijective mapping of* X *with* $\omega(t) = t$, d *an element of* X.

Proof. Observe that $\mu(x) := \gamma x$ defines a causal automorphism for a real constant $\gamma > 0$. Hence, by Proposition 3, (13) must be a causal automorphism as well.

Suppose now that $\lambda : X \to X$ is an arbitrary causal automorphism. If $x \neq y$ are elements of X with $l(x, y) = 0$, we may assume $x_0 \leqslant y_0$ without loss of generality, and hence $x < y$. Thus, by Proposition 2, $[x, y]$ is ordered. Since λ is a causal automorphism, also $[\lambda(x), \lambda(y)]$ must be ordered and $\lambda(x) < \lambda(y)$ holds true. Hence, by Proposition 2, $l(\lambda(x), \lambda(y)) = 0$. Now Theorem 2 implies that

$$\lambda(x) = m \cdot \lambda_1(x) \tag{14}$$

for all $x \in X$, where λ_1 is a Lorentz transformation and $m \neq 0$ a real constant. We may assume $m > 0$ without loss of generality, since otherwise we would consider

$$\lambda(x) = (-m) \cdot (-\lambda_1(x)),$$

by observing that also $x \to -\lambda_1(x)$ is a Lorentz transformation. Hence

$$x \to \frac{1}{m}\lambda(x)$$

is a causal automorphism, and thus, by (14), λ_1 is an orthochronous Lorentz transformation. In view of Proposition 3, we hence get (13) with the properties described in Theorem B. $\square$

4 Proof of result C

Let B be a set with $\mathrm{card}(B) \geqslant \aleph := \mathrm{card}(\mathbb{R})$ and define X to be the set of all functions

$$x : B \to \mathbb{R}$$

such that $\{b \in B \mid x(b) \neq 0\}$ is finite. We shall write x also in the form

$$x = \sum_{b \in B} x(b) \cdot b.$$

According to this notation, the element $b \in B$ is equal to the element x of X with $x(b) = 1$ and $x(b') = 0$ for all $b' \neq b$ in B.

Define $x + y, \lambda x, xy$ for $x, y \in X$, $\lambda \in \mathbb{R}$, by means of

$$x + y := \sum_{b \in B} [x(b) + y(b)]b, \quad \lambda x := \sum_{b \in B} [\lambda \cdot x(b)]b$$

and $xy = \sum_{b \in B} x(b)y(b)$. Then X is a real inner product space of dimension card(B) with basis B. If $\{b_1, \dots, b_n\}$ is a finite subset of B, then there exist exactly

$$\aleph \cdot \text{card}(B) = \text{card}(B)$$

elements of the form $r_1 b_1 + \dots + r_n b_n$ with real r_i, $i = 1, \dots, n$. Since the set of all finite subsets of B has also cardinality card(B), we get

$$\text{card}(X) = \text{card}(B).$$

Take a fixed $t \in B$. Hence $t^2 = 1$ and

$$\text{card}(X) = \text{card}(B \backslash \{t\}).$$

Therefore there exists a bijection $\mu : X \to B \backslash \{t\}$. Suppose that $\varrho > 0$ is a fixed real number and define

$$\sigma(x) := \sqrt{\frac{\varrho}{2}} \mu(x).$$

Hence $\sigma : X \to X$ must be injective. Writing again

$$x =: \bar{x} + x_0 t, \quad \bar{x} \in t^{\perp}, \ x_0 \in \mathbb{R},$$

for $x \in X$, let x, y be elements of X with

$$\varrho = l(x, y) = (\bar{x} - \bar{y})^2 - (x_0 - y_0)^2.$$

In view of $x \neq y$, the elements $b_1 := \mu(x)$, $b_2 := \mu(y)$ are distinct. Observe $b_1, b_2 \in B \backslash \{t\}$, i.e. $b_1, b_2 \in t^{\perp}$. Thus

$$l(\sigma(x), \sigma(y)) = \left(\sqrt{\frac{\varrho}{2}} b_1 - \sqrt{\frac{\varrho}{2}} b_2 \right)^2 = \varrho.$$

Of course, σ cannot be a Lorentz transformation, since $l(\sigma(x), \sigma(y)) = \varrho$ holds true for all distinct elements x, y of X. □

Remark. The method followed in this section can be applied to Euclidean and Hyperbolic Geometry in order to find *injective* mappings $\sigma : X \to X$ which are not distance preserving, but which preserve a fixed distance $\varrho > 0$. Non-injective mappings leaving invariant a fixed distance ϱ, but not all other distances, were given by Beckman and Quarles [3] in the euclidean case, and by Benz [8] in the hyperbolic case. In this connection observe theorems of Beckman, Quarles [3], Farrahi [10], Kuz'minyh [11], in which, in the finite-dimensional case, distance preserving mappings are characterized by the invariance of one single distance $\varrho > 0$.

Let B be a set with a cardinality $\geqslant \aleph$, and consider the real inner product space X as defined at the beginning of this section. In view of $\mathrm{card}(B) = \mathrm{card}(X)$, there exists a bijection

$$\mu : X \to B.$$

In the euclidean case with the distance notion

$$d(x, y) := \|x - y\|$$

for all $x, y \in X$, we define for a fixed $\varrho > 0$,

$$\sigma(x) := \frac{\varrho}{\sqrt{2}}\mu(x).$$

Hence $d(\sigma(x), \sigma(y)) = \sqrt{(\sigma(x) - \sigma(y))^2} = \varrho$ for $x \neq y$ with $x, y \in X$. So every distance $\neq 0$ goes over in distance ϱ.

In the hyperbolic case with the distance notion

$$h(x, y) \geqslant 0 \quad \text{and} \quad \cosh h(x, y) := \sqrt{1 + x^2}\sqrt{1 + y^2} - xy,$$

we define for a fixed $\varrho > 0$,

$$\sigma(x) := \sqrt{2}\sinh\frac{\varrho}{2} \cdot \mu(x).$$

Hence $\cosh h(\sigma(x), \sigma(y)) = 1 + 2\sinh^2\frac{\varrho}{2} = \cosh\varrho$, i.e.

$$h(\sigma(x), \sigma(y)) = \varrho$$

for every $x \neq y$ with $x, y \in X$. So also here every distance $\neq 0$ goes over in distance ϱ.

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Received 8 October, 2002

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