Special Issue (2003), S224–S231

Advances in Geometry

Dual partial quadrangles embedded in PG(3,q)

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Dedicated to Adriano Barlotti on the occasion of his 80th birthday

Abstract. The projective full embeddings of partial geometries are known. So are the projective full embeddings of semipartial and dual semipartial geometries in case of $\alpha > 1$. If $\alpha = 1$, a semipartial geometry is known as a partial quadrangle. No projective full embedding of a proper partial quadrangle is known. However besides a unique example for $q = 2$, there is one example known of a dual partial quadrangle fully embedded in a $\mathrm{PG}(3, q)$, any q. In this paper we will prove that if the dual of a proper partial quadrangle $\mathscr{S}$ is fully embedded in $\mathrm{PG}(3, q)$, then $\mu \leqslant q - \frac{q}{t+1}$. If equality holds, then $\mathscr{S}$ is uniquely defined.

1 Introduction

1.1 Definitions. An *embedding* of a point-line geometry in $\mathrm{PG}(n, q)$ is a representation of the geometry where the point set is a subset of the point set of $\mathrm{PG}(n, q)$, the line set is a subset of the line set of $\mathrm{PG}(n, q)$, and incidence is inherited from $\mathrm{PG}(n, q)$. We will always assume that n is the smallest dimension for which such an embedding in $\mathrm{PG}(n, q)$ exists. The geometry is *fully* embedded if the embedding has the additional property that for every line L of $\mathrm{PG}(n, q)$ which is also a line of the geometry, each point of $\mathrm{PG}(n, q)$ which is incident with L is also a point of the geometry.

A *semipartial geometry* [5] with parameters s, t, α, μ, also denoted by $\mathrm{spg}(s, t, \alpha, \mu)$, is a partial linear space $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ of order (s, t), such that for each anti-flag (x, L), the incidence number $\alpha(x, L)$, being the number of points on L collinear with x, equals 0 or a constant α ($\alpha > 0$) and such that for any two points which are not collinear, there are μ ($\mu > 0$) points collinear with both (μ-condition).

Remarks. 1) The point graph of a semipartial geometry is strongly regular. Besides the parameter μ, the other parameters of the graph are

$$v = 1 + \frac{(t+1)s(\mu + t(s - \alpha + 1))}{\mu},$$

$$k = (t+1)s.$$

$$\lambda = s - 1 + t(\alpha - 1).$$

2) A semipartial geometry with $\alpha = 1$ is called a *partial quadrangle* and is denoted by PQ(s, t, μ). It was introduced by Cameron [2] as a generalization of a generalized quadrangle. Semipartial geometries generalize at the same time the partial quadrangles and the partial geometries. See for instance [6] for more information on generalized quadrangles and [3, 4] for more information on partial and semipartial geometries.

For the rest of the paper we will be interested in the full embeddings in PG$(3, q)$ of the duals of partial quadrangles. The reason for this is, that for any q there is one proper partial quadrangle known (i.e. not being a generalized quadrangle) whose dual is fully embedded in PG$(3, q)$. We will construct this example in the next subsection.

There are quite some conditions known on the parameters of a partial quadrangle and hence of the dual partial quadrangles. We restrict ourselves to these conditions that will be used later in this paper; see [4] for more information. We will use again the standard notations, i.e., a dual partial quadrangle with parameters s, t, μ or of order (s, t) has $s + 1$ points on a line, while a point is incident with $t + 1$ lines; the parameter μ is the number of lines intersecting any two skew lines.

Theorem 1. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ be a proper dual partial quadrangle with parameters s, t, μ, then*

1. $|\mathscr{B}| = b = 1 + (s+1)t\left(1 + \frac{st}{\mu}\right)$;
2. *$s \geqslant t$, hence* $|\mathscr{P}| = v = \frac{b(s+1)}{t+1} \geqslant b$;
3. *either* $D = (t - 1 - \mu)^2 + 4((s+1)t - \mu)$ *is a square, or* $s = t = \mu = 1$ *and* $D = 5$ *in which case $\mathscr{S}$ is isomorphic to the pentagon*;
4. $\frac{2(s+1)t + (b-1)(t-1-\mu+\sqrt{D})}{2\sqrt{D}}$ *is an integer*,
5. *the Krein conditions on strongly regular graphs yield the following conditions, with $k = (s+1)t$, r and l the eigenvalues of the graph*:

$$(r+1)(k + r + 2rl) \leqslant (k + r)(l + 1)^2,$$
$$(l+1)(k + l + 2rl) \leqslant (k + l)(r + 1)^2.$$

1.2 A model of a dual partial quadrangle embedded in a projective space. Let $\mathscr{H}$ be a non-singular Hermitian variety in PG$(3, q)$, q a square, and let L be any line on $\mathscr{H}$. The incidence structure $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$, defined by taking as point set $\mathscr{P}$ the point set of $\mathscr{H} \backslash L$ and as line set $\mathscr{B}$ the set of lines of $\mathscr{H}$ minus all the lines concurrent with L, is the dual of a partial quadrangle PQ$(\sqrt{q} - 1, q, q - \sqrt{q})$ which is embedded in PG$(3, q)$. We will denote it by $H(3, q)^*$. Actually, it is commonly known that this geometry is isomorphic to the dual of the partial quadrangle obtained from the generalized quadrangle $Q^-(5, \sqrt{q})$ by deleting all lines incident with a given point and all points on these lines.

2 Some generalities on full embeddings of dual partial quadrangles in PG(3,q)

Lemma 1. *If $\mathscr{S}$ is a dual partial quadrangle of order (q,t) fully embedded in* $\mathrm{PG}(3,q)$, *then the $t+1$ lines of $\mathscr{S}$ through any point are coplanar.*

Proof. Let L be a line of $\mathscr{S}$. Assume that the plane π_i, $i=0,\ldots,q$, through L contains θ_i+1 lines of $\mathscr{S}$. Then $\sum_{i=0}^{q}\theta_i=(q+1)t$. However, for all i, $\theta_i+1\leqslant t+1$. Hence, for all i, $\theta_i=t$ and all $\theta_i+1=t+1$ lines are concurrent in π_i. □

Definitions. There are two types of planes.

Planes of type (a) contain $t+1$ lines of $\mathscr{S}$ through one point, called the center, together with some isolated points, that is, points of $\mathscr{S}$ in the plane but on no line of $\mathscr{S}$ in the plane. If p is the center, we denote the plane by $p^{\perp}$.

Planes of type (b) contain no line of $\mathscr{S}$ and some isolated points.

Lemma 2. *The number of isolated points in a plane of type* (a) *is a constant, say n. The number of isolated points in a plane of type* (b) *is a constant, say m. Moreover $m=qt+1+n$.*

Proof. Let π be a plane of type (a) containing n isolated points. Then, counting the number b of lines of $\mathscr{S}$ as lines intersecting π, it follows that $b=n(t+1)+(qt+1)(t+1)$. Hence all planes of type (a) contain n isolated points. On the other hand, assume π' is a plane of type (b). Then counting the number of lines of $\mathscr{S}$ as lines intersecting π' yields $b=m(t+1)$. Hence all planes of type (b) contain m isolated points. Moreover from both values of b it follows that $m=qt+1+n$. □

Lemma 3. *Assume L is a line of* $\mathrm{PG}(3,q)$ *which is not a line of $\mathscr{S}$. If $|L\cap\mathscr{P}|=k$, then there are k planes of type* (a) *through L.*

Proof. Assume there are l planes of type (a) through L. Then

$$l(qt+q+1+n-k)+(q+1-l)(m-k)+k=v=\frac{b(q+1)}{t+1}=m(q+1).$$

From this it follows that $k=l$. □

Corollary. *Each tangent to $\mathscr{S}$ is on exactly one plane of type* (a).

Lemma 4. *Assume $\mathscr{S}$ is a dual partial quadrangle with parameters q,t,μ, fully embedded in* $\mathrm{PG}(3,q)$, *then $n\geqslant t(q+1-\mu)$.*

Proof. Let p be a point of $\mathscr{S}$ and let L' be a line of $\mathscr{S}$ meeting $p^{\perp}$ in just one point collinear (in $\mathscr{S}$) with p. There are $(\mu-1)t$ lines of $\mathscr{S}$ meeting L' and meeting $p^{\perp}$ in just one point collinear with p but distinct from $L'\cap p^{\perp}$. Hence there are $qt-(\mu-1)t$ lines of $\mathscr{S}$ meeting L' and meeting $p^{\perp}$ in an isolated point. Two such lines meet $p^{\perp}$ in different isolated points. Hence $n\geqslant qt-(\mu-1)t$. □

Lemma 5. *If $\mathcal{S}$ is a proper dual partial quadrangle, fully embedded in* PG(3, q), *then* $\mu \leqslant q - \frac{q}{t+1}$.

Proof. From $b = (n + qt + 1)(t + 1)$ it follows that $n = \frac{b}{t+1} - qt - 1$. Hence from the previous lemma we get $n = \frac{b}{t+1} - qt - 1 \geqslant t(q + 1 - \mu)$. Using $b = 1 + (q + 1)t\left(1 + \frac{qt}{\mu}\right)$ we get

$$\mu + (q + 1)t(\mu + qt) - (\mu qt + \mu)(t + 1) \geqslant t(q + 1 - \mu)(t + 1)\mu;$$

that is,

$$(t + 1)\mu^2 - ((2q + 1)t + (q + 1))\mu + qt(q + 1) \geqslant 0.$$

Hence $\mu \leqslant \frac{qt}{t+1}$ or $\mu = q + 1$. We shall prove that this last case cannot occur. Let (p, L) be an antiflag of $\mathcal{S}$ and let L' be a line of $\mathcal{S}$ through p not meeting L. Since $\mu = q + 1$, there is a unique line of $\mathcal{S}$ through p meeting L. It follows that $\mathcal{S}$ is a generalized quadrangle, a contradiction. Hence $\mu \leqslant \frac{qt}{t+1}$. □

3 A characterization of $H(3, q^2)$*

In this section we assume that $\mathcal{S}$ is a proper dual partial quadrangle of order (q, t), fully embedded in PG(3, q) and such that μ is maximal, i.e., $\mu = q - \frac{q}{t+1} = \frac{qt}{t+1}$; hence $t + 1$ divides q.

Note by the way that indeed D is a square, as $\sqrt{D} = t + 1 + \frac{qt}{t+1} = t + 1 + \mu$.

In this case, $m = (q + 1)t + 1 + \mu$, $n = m - (qt + 1) = \mu + t$, $b = (q + 1)t(t + 2) + 1$ and $v = \frac{b(q+1)}{t+1}$.

Lemma 6. *If $\mathcal{S}$ is a proper dual partial quadrangle of order (q, t) and $\mu = \frac{qt}{t+1}$ which is fully embedded in* PG(3, q), *then for any two non-collinear points p and p′ of $\mathcal{S}$ the number $\mu(p, p')$ of points of $\mathcal{S}$ collinear with both p and p′, is either* 0, *t, or* $t + 1$. *Moreover $\mu(p, p') = 0$ if and only if p′ is an isolated point contained in the plane $p^\perp$ of type* (a) *with center p, or equivalently, p is an isolated point contained in the plane $p'^\perp$.*

Proof. Assume p' is an isolated point in $p^\perp$. Then clearly $\mu(p, p') = 0$. Assume that p' is a point of $\mathcal{S}$ not collinear in $\mathcal{S}$ with the point p of $\mathcal{S}$ and not contained in $p^\perp$. There can be at most $t + 1$ points of $\mathcal{S}$ collinear in $\mathcal{S}$ with both p and p', and they will all be contained in the projective line $p^\perp \cap p'^\perp$. Assume $\mu(p, p') \neq t + 1$. Then there is at least one point p'' of $\mathcal{S}$ collinear in $\mathcal{S}$ with p' but isolated in $p^\perp$. We denote the line $\langle p', p'' \rangle$ by N. We count the lines of $\mathcal{S}$ intersecting N and any of the $t + 1$ lines of $\mathcal{S}$ through p in two ways. As $\mathcal{S}$ is a dual partial quadrangle, there are $(t + 1)\mu = qt$ such lines. On the other hand, there are at most $(|N| - 1)t = qt$ such lines, hence there are exactly qt lines of $\mathcal{S}$ intersecting N and a line of $\mathcal{S}$ through p, hence $\mu(p, p') = t$. Note that this implies that there is exactly one line M of $\mathcal{S}$ through p, which has no point that is collinear with p', i.e., M intersects $p'^\perp$ in an isolated point of $p'^\perp$. □

Lemma 7. *Assume that p, p' and p'' are three mutually non-collinear points of $\mathscr{S}$ such that $\mu(p,p') = \mu(p,p'') = 0$, then $\mu(p',p'') = 0$ and p, p' and p'' are on one line of* PG$(3,q)$, *namely the line $p^\perp \cap p'^\perp = p^\perp \cap p''^\perp$.*

Proof. From the assumptions and from Lemma 6 follows that p' and p'' are isolated points in $p^\perp$, and that p is isolated in $p'^\perp$ as well as in $p''^\perp$. Assume $\mu(p',p'') \neq 0$, hence there exists a point r of $\mathscr{S}$ not contained in $p^\perp$ and collinear in $\mathscr{S}$ with p' and p''. However, this implies that $\mu(p,r) \leqslant t-1$, and so from Lemma 6 follows that $\mu(p,r) = 0$, which contradicts the fact that r is not contained in $p^\perp$. Hence $\mu(p',p'') = 0$ which implies that p' is contained in $p''^\perp$ and p'' is contained in $p'^\perp$. So the three points p, p' and p'' are on one line of PG$(3,q)$, namely the line $p^\perp \cap p'^\perp = p^\perp \cap p''^\perp$. □

Remark. From Lemma 7 follows that for p and p' points of $\mathscr{S}$ the relation $p \equiv p'$ if and only if either $p = p'$ or p not collinear (in $\mathscr{S}$) with p' but $\mu(p,p') = 0$, is an equivalence relation on the points of $\mathscr{S}$. Moreover, as the isolated points in a plane $p^\perp$ are n points on a projective line of $p^\perp$ containing p, it follows that $n \leqslant q$. From the proof of Lemma 7 also follows that any point r of $\mathscr{S}$ which is not a point of $p^\perp$, is collinear in $\mathscr{S}$ with at most one isolated point of $\mathscr{S}$ in $p^\perp$.

3.1 The case $n = q$. As $\mu = \frac{qt}{t+1}$, and as $n = \mu + t = q$, it follows that $q = t(t+1)$. Hence $q = 2$ and $t = 1$. So $v = 15$, $k = 10$, $\mu = 1$.

Note that $W(2)$ minus a spread is an example of an embedded dual partial quadrangle with $q = 2$, $t = 1$ satisfying $\mu = \frac{qt}{t+1}$ and $n = q$. It is quite easy to see that this is the only embedded dual partial quadrangle with these parameters.

3.2 The case $n < q - 1$. The condition $n \leqslant q-1$ is equivalent to $t+1 \leqslant \sqrt{q}$. Assume $n < q-1$, and so $t+1 < \sqrt{q}$. Since $q = p^h$ and $t+1 \mid q$ we can write $t+1 = p^l$. Moreover $l < h/2$.

We now check the divisibility conditions. Note that

$$\sqrt{D} = t+1+\mu = t+1+q-\frac{q}{t+1} = p^l + p^h - p^{h-l}.$$

Hence $\sqrt{D} = p^l + p^h - p^{h-l}$ divides $(q+1)t + \frac{(b-1)(t-1-\mu+\sqrt{D})}{2}$. Now one easily checks the following equalities.

$$b-1 = (q+1)t\left(1+\frac{qt}{\mu}\right) = (p^h+1)(p^l-1)(1+p^l),$$

and

$$\frac{t-1-\mu+\sqrt{D}}{2} = p^l - 1.$$

As $(q+1)t = (p^h+1)(p^l-1)$ one gets the following divisibility condition:

$$p^l + p^h - p^{h-l} \mid (p^h+1)(p^l-1) + (p^h+1)(p^{2l}-1)(p^l-1),$$

that is

$$p^l + p^h - p^{h-l} \mid (p^h+1)(p^l-1)p^{2l},$$

hence

$$p^l + p^h - p^{h-l} \mid p^{3l}(p^l + p^h - p^{h-l}) - p^l(p^{3l} - p^{2l} + p^l),$$

so

$$p^l + p^h - p^{h-l} \mid p^l(p^{3l} - p^{2l} + p^l).$$

Put $h = 2l + k$ with $k > 0$, then

$$p^l(p^{l+k} - p^k + 1) \mid p^{2l}(p^{2l} - p^l + 1).$$

As $k > 0$, $\gcd(p^{l+k} - p^k + 1, p^l) = 1$ and so

$$p^{l+k} - p^k + 1 \mid p^{2l} - p^l + 1.$$

Consequently $0 < k \leqslant l$. Suppose $k \neq l$. As

$$p^{2l} - p^l + 1 = (p^{l+k} - p^k + 1)p^{l-k} - p^{l-k} + 1,$$

this implies that

$$p^{l+k} - p^k + 1 \mid p^{l-k} - 1,$$

so

$$p^{l+k} - p^k + 1 \leqslant p^{l-k} - 1,$$

which is a contradiction. It follows that $k = l$, hence $h = 3l$ so $(t+1)^3 = q$.

Assume that $q = (t+1)^3$; then $n = \mu + t = q - \frac{q}{t+1} + t = t(t^2+2t+2)$, $\mu = (t+1)^2 t$. The line graph of $\mathscr{S}$ is a strongly regular graph with $b = (t+1)\cdot(n+qt+1) = (t+1)(t^4+4t^3+5t^2+3t+1)$ vertices, and parameters $k = (q+1)t = (t^3+3t^2+3t+2)t$, $\lambda = t-1$ and $\mu = (t+1)^2 t$.

One easily checks that by the Krein conditions (in this case $r = t$ and $l = -t(t+1)^2 - 1$) such a graph cannot exist.

3.3 The proof of the main result. From now on we may assume that $n = q - 1$, and that $\mathscr{S}$ is a dual partial quadrangle fully embedded in PG(3, q) with $t + 1 = \sqrt{q}$ lines through a point and moreover $\mu = \sqrt{q}(\sqrt{q} - 1)$. It has q^2 lines, $q\sqrt{q}(q+1)$ points

and as many planes of type (a). The $\sqrt{q}$ lines of $\mathscr{S}$ through a point p are the lines in the unique plane $p^\perp$ of type (a), which contains $q-1$ isolated points on a line of $\mathrm{PG}(3,q)$ not in $\mathscr{S}$. We will call the set consisting of a point p of $\mathscr{S}$ and the $q-1$ isolated points in the plane $p^\perp$ of type (a), a set of type 1 and will denote it by L_p. By Lemma 3 there are $\sqrt{q}(q+1)$ sets of type 1, and each such set L_p is in one plane of type (b) which contains $m = q\sqrt{q}$ isolated points. Every plane of type (b) that contains at least one set of type 1 will be called a plane of type (b_1). Note that in the model there are indeed two types of planes of type (b). On the one hand there are the $q+1$ planes through L (see Subsection 1.2 for the notations), and these are the planes of type (b_1); the $\sqrt{q}$ lines of $\mathscr{H}$ different from L that were omitted in such a plane, yield the sets of type 1 in that plane. The other planes of type (b) are the $q(q-\sqrt{q})(q+1)$ planes intersecting the Hermitian variety $\mathscr{H}$ in a Hermitian curve, and hence do not contain a set of type 1.

Let π be a plane of type (b_1). We will prove that the $q\sqrt{q}$ points of $\mathscr{S}$ in π can be partitioned in $\sqrt{q}$ sets of type 1. Let L_p be a set of type 1 in π and let r be a point of $\mathscr{S}$ in π not contained in L_p. We show that the line $\langle p, r\rangle$ of $\mathrm{PG}(3,q)$ intersects the set of points of $\mathscr{S}$ in π in $\sqrt{q}$ points. Indeed, if M is any line of $\mathscr{S}$ through p, then the plane $\langle M, r\rangle$ is a plane of type (a) with center say p'. If $p' = p$, then $\langle M, r\rangle$ would contain L_p, clearly a contradiction. So $p' \neq p$. The $\sqrt{q}$ lines of $\mathscr{S}$ through p' in $p'^\perp$ define $\sqrt{q}$ points of $\mathscr{S}$ on the line $\langle p, r\rangle$. Consider the plane $p''^\perp$ with $p'' \in L_p \backslash \{p\}$. The $\sqrt{q}$ lines of $\mathscr{S}$ through p'' intersect $p'^\perp$ in $\sqrt{q}$ points, one of which is necessarily on $L_{p'}$. It follows that any point of $L_{p'}$ (resp. L_p) is collinear (in $\mathscr{S}$) with exactly one point of L_p (resp. $L_{p'}$). It follows that $L_{p'} \cap \pi = \varnothing$. Hence $\langle p, r\rangle$ contains exactly $\sqrt{q}$ points of $\mathscr{S}$. From this follows that the q lines of $\mathrm{PG}(3,q)$ in π and through r intersecting L_p, all intersect the set of points of $\mathscr{S}$ in $\sqrt{q}$ points, and hence the line R of $\mathrm{PG}(3,q)$ in π and not intersecting L_p defines a set L_r of type 1 (as $|r^\perp \cap \pi| \neq \sqrt{q}$ we have that $r^\perp \cap \pi = R$). Hence the $q\sqrt{q}$ points of a plane of type (b_1) can be partitioned in $\sqrt{q}$ mutually disjoint sets of type 1; moreover these $\sqrt{q}$ sets define $\sqrt{q}$ lines of $\mathrm{PG}(3,q)$ in π through a point of $\mathrm{PG}(3,q)$ not in $\mathscr{S}$, which we will call a point of type I. From the above arguments follows that there are $q+1$ planes of type (b_1). Two distinct planes of type (b_1) do not have a point of $\mathscr{S}$ in common. Hence if π_1 and π_2 are distinct planes of type (b_1) and x_i is the point of type I in π_i, $i = 1, 2$, then $x_i \in \pi_j$ with $i, j \in \{1, 2\}$. Remark also that $x_1 \neq x_2$ as otherwise there would arise sets of type 1 which are contained in more than one plane of type (b_1). Now there easily follows that the $q+1$ points of type I are on a common line $[\infty]$ of $\mathrm{PG}(3,q)$ (not in $\mathscr{S}$).

We define now the following incidence structure $\mathscr{S}^* = (\mathscr{P}^*, \mathscr{B}^*, \mathrm{I}^*)$.

The set $\mathscr{P}^*$ is the set $\mathscr{P}$ union the set of $q+1$ points of type I. The set $\mathscr{B}^*$ is the set $\mathscr{B} \cup \{\overline{L}_p \mid p \in \mathscr{P}\} \cup \{[\infty]\}$ with $\overline{L}_p = L_p \cup \{x_p\}$, and x_p the point of type I defined by L_p. The incidence I* is the incidence of $\mathrm{PG}(3,q)$. One easily checks that $\mathscr{S}^*$ is a generalized quadrangle of order $(q, \sqrt{q})$ and as it is fully embedded in $\mathrm{PG}(3,q)$, it is the generalized quadrangle $H(3,q)$ [1]. Hence we have proved the following theorem.

Theorem 2. *If $\mathscr{S}$ is a dual partial quadrangle of order (q,t), fully embedded in* $\mathrm{PG}(3,q)$, *then $\mu \leqslant \frac{qt}{t+1}$. If equality holds, then either $q = 2$ and $\mathscr{S}$ is $W(2)$ minus a spread, or q is a square and $\mathscr{S} = H(3,q)^*$.*

Remark. The proofs heavily rely on the fact that the dimension of the projective space is three. Although there is no model of dual partial quadrangle known which is fully embedded in PG(n, q), $n \geqslant 4$, one should use other techniques if one wants to prove that no such examples exist.

Acknowledgment. The first author thanks the Department of Mathematics of the Università di Napoli, Federico II, while the second author thanks the Department of Pure Mathematics and Computer Algebra of Ghent University, for the kind hospitality they offered during the writing of this paper.

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Received 31 October, 2002

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