Elliptic curves and rank-2 vector bundles on the prime Fano threefold of genus 7

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(Communicated by R. Miranda)

Abstract. According to Mukai, any prime Fano threefold X of genus 7 is a linear section of the spinor tenfold in the projectivized half-spinor space of Spin(10). It is proven that the moduli space of stable rank-2 vector bundles with Chern classes $c_1 = 1$, $c_2 = 5$ on a generic X is isomorphic to the curve of genus 7 obtained by taking an orthogonal linear section of the spinor tenfold. This is an inverse of Mukai's result on the isomorphism of a non-abelian Brill–Noether locus on a curve of genus 7 to a Fano threefold of genus 7. An explicit geometric construction of both isomorphisms and a similar result for K3 surfaces of genus 7 are given.

2000 Mathematics Subject Classification. 14J30

1 Introduction

The study of moduli spaces of stable vector bundles on Fano threefolds of indices 1 and 2 is quite a recent topic. The *index* of a Fano threefold X is the maximal integer v dividing K_X in the Picard group of X. The results known so far include the description of one component of moduli of rank-2 vector bundles on each one of the following four Fano threefolds: the cubic [22], [14], [6], [2], the quartic [15], the prime Fano threefold of genus 9 [16] and the double solid of index 2 [34]. It turns out that the flavour of the results one can obtain depends strongly on the index. In the index-2 case, the answers are given in terms of the Abel–Jacobi map of the moduli of vector bundles into the intermediate Jacobian $J(X)$, defined by the second Chern class c_2, and the techniques originate from Clemens, Griffiths, and Welters. For the cubic X_3, the moduli space $M_{X_3}(r; c_1, c_2)$ with invariants $r = 2$, $c_1 = 0$, $c_2 = 5$ is of dimension 5 and is identified with an open subset of the intermediate Jacobian. For the double solid Y_2 of index 2, Tikhomirov found a 9-dimensional component of $M_{Y_2}(2; 0, 3)$ whose Abel–Jacobi map is quasi-finite onto an open subset of the theta-divisor of $J(Y_2)$, and its degree is 84 [23].

In the index-1 case, the Abel–Jacobi map does not bring much new information about the moduli spaces investigated up to now. For the quartic threefold X_4, we proved in [15] that $M_{X_4}(2; 1, 6)$ has a component of dimension 7 with a 7-dimensional Abel–Jacobi image in the 30-dimensional intermediate Jacobian $J(X_4)$. One can conclude from here about the geometry of this component only that its Kodaira dimension is positive.

In the present paper, we consider one more index-1 case: we determine the moduli space $M_X = M_X(2; 1, 5)$ for a generic prime Fano threefold $X = X_{12}$ of genus 7.

Following the classical terminology, we call the Fano threefolds X_{2g-2} of index 1 and degree $2g - 2$ with Picard number 1 *prime Fano threefolds of genus g*. They have been classified, up to deformations, by Iskovskikh [18], [19]. There is only one moduli family of threefolds X_{2g-2} for every $g = 2, \ldots, 12$, $g \neq 11$ (see Table 12.2 in [19]). Mukai [27] proved that X_{2g-2} is a linear section of some projective homogeneous space Σ_{2g-2} for $7 \leqslant g \leqslant 10$. In the case $g = 7$, $\Sigma = \Sigma_{12}$ is the spinor tenfold in $\mathbb{P}^{15}$. It is self-dual, that is, the dual variety $\check{\Sigma} \subset \check{\mathbb{P}}^{15}$, formed by the hyperplanes in $\mathbb{P}^{15}$ tangent to Σ, is isomorphic to Σ via some projectively linear map identifying $\mathbb{P}^{15}$ with its dual $\check{\mathbb{P}}^{15}$. Thus, to a linear section $X = \mathbb{P}^{7+k} \cap \Sigma$ of Σ of dimension $2 - k$ we can associate the orthogonal linear section $\check{X} := \check{\mathbb{P}}^{7-k} \cap \check{\Sigma}$ of dimension $2 + k$, where $\check{\mathbb{P}}^{7-k} = (\mathbb{P}^{7+k})^{\perp} \subset \check{\mathbb{P}}^{15}$. For $k = 1$, we obtain a curve linear section $\Gamma = \check{X}$, which is a canonical curve of genus 7. Our main result is the following statement.

Theorem 1.1. *Let $X = X_{12}$ be a generic prime Fano threefold of genus 7. Then M_X is isomorphic to the curve $\Gamma = \check{X}$.*

We prove also similar statements in the cases $k = 0$, where $X, \check{X}$ are generic K3 surfaces of degree 12, and $k = -1$, where X is a curve and $\check{X}$ is a threefold. In the latter case, one should take the non-abelian Brill–Noether locus of rank-2 vector bundles on X with canonical determinant and 5 linearly independent global sections on the role of M_X. For $k = 0$, Mukai [26], [31] proved that M_X is another K3 surface of degree 12 (M_X represents the so-called Fourier–Mukai transform of X; see [11]). We make this statement more precise by identifying M_X with the orthogonal K3 surface $\check{X}$ via an explicit map ρ_X having a beautiful geometric construction.

For $k = -1$, Mukai [30] proved that M_X is a Fano threefold of degree 12. Again, we show that this Fano threefold is isomorphic to the orthogonal linear section of the spinor tenfold, and our Main Theorem represents the inverse of this result.

Iliev–Ranestad [16] obtained similar results for the 1-, 2- and 3-dimensional linear sections of the symplectic Grassmannian $\Sigma_{16} \subset \mathbb{P}^{13}$, but in their case the dual of Σ_{16} is a singular quartic hypersurface in $\check{\mathbb{P}}^{13}$, so the moduli spaces (or the non-abelian Brill–Noether locus in dimension 1) that they consider are isomorphic to linear sections of this quartic hypersurface.

Our construction of the map $\rho_X : \check{X} \to M_X$ is very simple: for any $w \in \check{X}$ the corresponding hyperplane $\mathbb{P}^{14}_w$ in $\mathbb{P}^{15}$ is tangent to Σ along a projective space $\mathbb{P}^4$, and the linear projection π_w of $X \cap \mathbb{P}^{14}_w$ into $\mathbb{P}^9$ with center $\mathbb{P}^4$ has its image inside the Grassmannian $G(2, 5) \subset \mathbb{P}^9$. It turns out that the pullback of the universal rank-2 bundle from $G(2, 5)$ to X is stable and its class belongs to M_X. This defines the image of w

in M_X. It is not so obvious that the thus defined map ρ_X is nontrivial. In Proposition 5.4 we prove that its image is an irreducible component of M_X. This enables us to conclude the proof of the fact that ρ_X is an isomorphism in the cases where the irreducibility and smoothness of M_X are already known from the work of Mukai: $k = -1$ (Proposition 5.6) and $k = 0$ (Proposition 5.7). For the case $k = 1$, we prove in Proposition 5.9 that every vector bundle $\mathscr{E} \in M_X$ is globally generated and is obtained by Serre's construction from a normal elliptic quintic contained in X. The irreducibility of M_X, equivalent to that of the family of elliptic quintics in X, is reduced to the known irreducibility in the K3 case. The smoothness of M_X is proved separately by using Takeuchi–Iskovskikh–Prokhorov birational maps $\Phi_p : X \dashrightarrow Y = Y_5$ and $\Psi_q : X \dashrightarrow Q$, where $p \in X$ is a point, $q \subset X$ a conic, Y_5 the Del Pezzo threefold of degree 5 and Q the three-dimensional quadric hypersurface.

Takeuchi [33] has undertaken a systematic study of birational transformations of Fano varieties that can be obtained by a blow up with center in a point p, a line ℓ or a conic q followed by a flop and a contraction of one divisor. Iskovskikh–Prokhorov [19] have extended Takeuchi's list, in particular, they found the two birational transformations for X_{12} mentioned above. The techniques of proofs are those of Mori theory, based on the observation that the Mori cone of X_{12} blown up at a point, line or conic is an angle in $\mathbb{R}^2$, hence there are exactly two extremal rays to contract, the first one giving the initial 3-fold, the second one defining the wanted birational map. But before one can contract the second extremal ray, one has to make a flop. We describe in detail the structure of Φ_p, Ψ_q (Theorems 6.3 and 6.5). The last contraction in both cases blows down one divisor onto a curve of genus 7. Thus, we have 3 curves of genus 7 associated to X: the orthogonal linear section Γ, and Γ', Γ'' coming from the birational maps. We prove that the three curves are isomorphic. We also identify the flopping curves for Φ_p: they are the 24 conics passing through p, and their images are the 24 bisecants of Γ'.

The ubiquity of the maps Φ_p, Ψ_q is in that they provide a stock of well-controlled degenerate elliptic quintics: the ones with a node at p are just the proper transforms of the unisecant lines of Γ' in Y and the reducible ones having q as one of components are nothing else but the proper transforms of the exceptional curves contracted by Ψ_q into points of Γ'', that is, they are parametrized by Γ''. The smoothness of M_X follows from the existence, among the zero loci of sections of any vector bundle $\mathscr{E} \in M_X$, of a nodal quintic with a node at p such that the normal bundle of the corresponding unisecant of Γ' is $\mathcal{O} \oplus \mathcal{O}$ (see the proofs of Proposition 7.1 and Lemma 7.3). The family of lines on Y is well known (see for example [13], [8]). In particular, Y contains a rational curve C_6^0 which is a locus of points z such that there is a unique line in Y passing through z, and the normal bundle of this line is $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. Hence our proof of smoothness does not work in the case when Γ' meets C_6^0. We prove that in this case a generic deformation Γ_t' of Γ' does not meet C_6^0 (Lemma 7.6) and that Γ_t' corresponds to some birational map $\Phi_{p_t} : X_t \dashrightarrow Y$ of the same type. This explains why we state our Main Theorem only for generic X. We conjecture that the conclusion of the Theorem is true for any smooth 3-dimensional linear section $\Sigma \cap \mathbb{P}^8$.

In Section 2, we give a definition of the spinor tenfold Σ, represent it as one of

the two components of the family of maximal linear subspaces of an 8-dimensional quadric, and introduce the notion of the pure spinor associated to a point of Σ.

In Section 3, we study some properties of linear sections X of Σ, in particular, the projections π_w to the Grassmannian $G(2,5)$ in $\mathbb{P}^9$, defined by the points $w \in \check{X}$, and prove that any linear embedding of X into $G(2,5)$, under some additional restrictions, is always given by such a projection (the case $\dim X = 1$ is postponed until Section 5).

In Section 4, we list standard facts about the moduli space M_X ($\dim X = 3$) and the Hilbert scheme of elliptic quintics on X; we show that any $\mathscr{E} \in M_X$ is obtained by Serre's construction from a "quasi-elliptic" quintic and that the fibers of Serre's construction over M_X are projective spaces $\mathbb{P}^4$.

In Section 5, we define the map $\rho_X : \check{X} \to M_X$ in all the three cases $\dim X = 1, 2, 3$, and prove that its image is a component M_X^0 of M_X. We prove that ρ_X is an isomorphism for $\dim X = 1, 2$. For $\dim X = 3$ we obtain the following more precise version of the result of the previous section: any $\mathscr{E} \in M_X$ is globally generated and is obtained by Serre's construction from a *smooth* elliptic quintic.

In the Sections 6 and 7, $\dim X = 3$. In Section 6, we provide some basic properties of the families of lines and conics on X, in particular, we prove the irreducibility of the family of conics, and we describe the structure of the two Takeuchi–Iskovskikh–Prokhorov birational maps. We show that the vector bundles constructed from the stock of quasi-elliptic quintics generated by these maps belong to M_X^0, and we deduce the isomorphism of the three curves $\Gamma, \Gamma', \Gamma''$.

Section 7 is devoted to the proof of Theorem 1.1.

2   Spinor tenfold

The spinor tenfold Σ_{12}^{10} is a homogeneous space of the complex spin group $\mathrm{Spin}(10)$, or equivalently, that of $SO(10) = \mathrm{Spin}(10)/\{\pm 1\}$. It can be defined as the unique closed orbit of $\mathrm{Spin}(10)$ in the projectivized half-spinor representation of $\mathrm{Spin}(10)$ on $\mathbb{P}^{15}$. We will recall an explicit description of Σ_{12}^{10} and some of its properties, following essentially [4], [28], [32].

Let $\mathrm{Alt}_5(\mathbb{C}) \cong \mathbb{C}^{10}$ be the space of skew-symmetric complex 5×5 matrices. For $\hat{A} \in \mathrm{Alt}_5(\mathbb{C})$ denote by $\vec{\mathrm{Pf}}(\hat{A}) \in \mathbb{C}^5$ the 5-vector with coordinates $\vec{\mathrm{Pf}}(\hat{A})_i = (-1)^i \mathrm{Pf}_i(\hat{A})$, $i = 1, \dots, 5$, where Pf_i are the codimension 1 Pfaffians of an odd-dimensional skew-symmetric matrix.

Definition 2.1. The Spinor tenfold $\Sigma = \Sigma_{12}^{10} \subset \mathbb{P}^{15}$ is the closure of the image $j(\mathrm{Alt}_5(\mathbb{C}))$ under the embedding

$$j : \mathbb{C}^{10} \cong \mathrm{Alt}_5(\mathbb{C}) \to \mathbb{P}^{15}, \quad \hat{A} \mapsto (1 : \hat{A} : \vec{\mathrm{Pf}}(\hat{A})). \tag{1}$$

We will write homogeneous coordinates in $\mathbb{P}^{15}$ in the form $(u : \hat{X} : \vec{y})$, where $u \in \mathbb{C}$, $\hat{X} = (x_{ij}) \in \mathrm{Alt}_5(\mathbb{C})$ and $\vec{y} = (y_1, \dots, y_5) \in \mathbb{C}^5$.

The map j parameterizes the points of the open subset $j(\mathrm{Alt}_5(\mathbb{C})) = \Sigma \cap (u \neq 0)$ and

$$(u : \hat{X} : \vec{y}) \in \Sigma \Leftrightarrow u\vec{y} = \vec{\mathrm{Pf}}(X) \quad \text{and} \quad \hat{X}\vec{y} = 0. \tag{2}$$

Writing down the components of the above matrix equations, we obtain the defining equations of Σ, or the generators of the homogeneous ideal of $\Sigma \subset \mathbb{P}^{15}$:

$$q_1^+ = uy_1 + x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25}$$
$$q_2^+ = uy_2 - x_{13}x_{45} + x_{14}x_{35} - x_{34}x_{15}$$
$$q_3^+ = uy_3 + x_{12}x_{45} - x_{14}x_{25} + x_{24}x_{15}$$
$$q_4^+ = uy_4 - x_{12}x_{35} + x_{13}x_{25} - x_{23}x_{15}$$
$$q_5^+ = uy_5 + x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14}$$
$$q_1^- = \phantom{-x_{12}y_1} x_{12}y_2 + x_{13}y_3 + x_{14}y_4 + x_{15}y_5$$
$$q_2^- = -x_{12}y_1 \phantom{+x_{23}y_3} + x_{23}y_3 + x_{24}y_4 + x_{25}y_5$$
$$q_3^- = -x_{13}y_1 - x_{23}y_2 \phantom{+x_{34}y_4} + x_{34}y_4 + x_{35}y_5$$
$$q_4^- = -x_{14}y_1 - x_{24}y_2 - x_{34}y_3 \phantom{+x_{45}y_5} + x_{45}y_5$$
$$q_5^- = -x_{15}y_1 - x_{25}y_2 - x_{35}y_3 - x_{45}y_4$$

An important property of the spinor tenfold is its self-duality [7]:

Lemma 2.2. *The projectively dual variety $\Sigma^\vee \subset \mathbb{P}^{15\vee}$, consisting of all the hyperplanes in $\mathbb{P}^{15}$ that are tangent to Σ, is projectively equivalent to Σ.*

This follows also from the self-duality of the half-spinor representation of $\mathrm{Spin}(10)$ and the fact that $\mathrm{Spin}(10)$ has only two orbits in $\mathbb{P}^{15}$: the spinor tenfold and its complement [12].

There is an alternative interpretation of the spinor tenfold Σ: it is isomorphic to each one of the two families of 4-dimensional linear subspaces in a smooth 8-dimensional quadric $Q^8 \subset \mathbb{P}^9$. In other words, it is a component of the Grassmannian $G_q(5, 10) = \Sigma^+ \sqcup \Sigma^-$ of maximal isotropic subspaces of a nondegenerate quadratic form q in $\mathbb{C}^{10}$. The varieties $\Sigma, \Sigma^\vee$ can be simultaneously identified with Σ^+, Σ^- respectively in such a way that the duality between $\Sigma, \Sigma^\vee$ is given in terms of certain incidence relations between the four-dimensional linear subspaces of the quadric $Q^8 = \{q = 0\}$.

Namely, denote by $\mathbb{P}_c^4$ the subspace of Q^8 corresponding to a point $c \in \Sigma^\pm$, and let, for example, $c \in \Sigma^+$. Then we have:

$$\Sigma^+ = \{d \in G_Q(5, 10) \mid \dim(\mathbb{P}_c^4 \cap \mathbb{P}_d^4) \in \{0, 2, 4\}\}, \tag{3}$$
$$\Sigma^- = \{d \in G_Q(5, 10) \mid \dim(\mathbb{P}_c^4 \cap \mathbb{P}_d^4) \in \{-1, 1, 3\}\}, \tag{4}$$

where the dimension equals -1 if and only if the intersection is empty. Furthermore, if we denote by $\mathbb{P}^{14}_w$ the hyperplane in $\mathbb{P}^{15}$ represented by a point $w \in \check{\mathbb{P}}^{15} := (\mathbb{P}^{15})^\vee$, and by H_w the corresponding hyperplane section $\mathbb{P}^{14}_w \cap \Sigma$ of $\Sigma = \Sigma^+$, then for any $w \in \Sigma^- = \Sigma^\vee$, we have

$$H_w = \{c \in \Sigma^+ : \mathbb{P}^4_c \cap \mathbb{P}^4_w \neq \varnothing\} = \{\mathbb{P}^4 \subset Q : \dim(\mathbb{P}^4 \cap \mathbb{P}^4_w) \text{ is odd and} \geqslant 0\} \quad (5)$$

For future use, we will describe explicitly the identifications of $\Sigma, \Sigma^\vee$ with Σ^+, Σ^-. Let V be a $2v$-dimensional $\mathbb{C}$-vector space ($v = 5$ in our applications) with a non-degenerate quadratic form q and $(\cdot, \cdot)$ the associated symmetric bilinear form. Fix a pair of maximal isotropic subspaces U_0, U_∞ of V such that $V = U_0 \oplus U_\infty$. The bilinear form $(\cdot, \cdot)$ identifies U_0 with the dual of U_∞. Let Σ^+, resp. Σ^- be the component of $G_q(v, V)$ that contains U_0, resp. U_∞. Let $S = \bigwedge^\bullet U_\infty$ be the exterior algebra of U_∞; it is called the spinor space of (V, q) and its elements are called spinors. The even and the odd parts of S

$$S^+ = \bigwedge{}^{\text{even}} U_\infty, \quad S^- = \bigwedge{}^{\text{odd}} U_\infty$$

are called half-spinor spaces. To each maximal isotropic subspace $U \in \Sigma^+ \cup \Sigma^-$ one can associate a unique, up to proportionality, nonzero half-spinor $s_U \in S^+ \cup S^-$ such that $\varphi_u(s_U) = 0$ for all $u \in U$, where $\varphi_u \in \text{End}(S)$ is the Clifford automorphism of S associated to u:

$$\varphi_u(v_1 \wedge \cdots \wedge v_k) = \sum_i (-1)^{i-1}(u_0, v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k + u_\infty \wedge v_1 \wedge \cdots \wedge v_k,$$

if $u = u_0 + u_\infty$, $u_0 \in U_0$, $u_\infty \in U_\infty$.

The element s_U is called the pure spinor associated to U. The map $U \mapsto [s_U] \in \mathbb{P}(S^\pm)$ is the embedding of $\Sigma^\pm$ into the projective space $\mathbb{P}^{2^{v-1}-1}$ from which we started our description of the spinor tenfold (Formula (1), $v = 5$). The duality between S^+, S^- is given by the so called fundamental form β on S, for which S^+, S^- are maximal isotropic 2^{v-1}-dimensional subspaces of S:

$$\beta(\xi, \xi') = (-1)^{p(p-1)/2}(\xi \wedge \xi')_{\text{top}}$$

where $\deg \xi = p$ and $(s)_{\text{top}}$ denotes the $\bigwedge^v U_\infty$-component of a spinor $s \in \bigwedge^\bullet U_\infty$.

Describe now the spinor embedding in coordinates. Let $U \in \Sigma_+$. Then the intersection $U \cap U_\infty$ is always even-dimensional and generically $U \cap U_\infty = 0$. Choose a basis $e_1, \ldots, e_v$ of U_∞ in such a way that $U \cap U_\infty = \langle e_{v-2k}, e_{v-2k+1}, \ldots, e_v \rangle$. Let $e_{-1}, \ldots, e_{-v}$ be the dual basis of U_0. Then U possesses a basis $u_1, \ldots, u_v$ of the following form: $u_i = e_{-i} + \sum_{j=1}^{v-2k-1} a_{ij} e_j$ for $i = 1, \ldots, v - 2k - 1$, and $u_i = e_i$ for $i = v - 2k, \ldots, v$, where (a_{ij}) is a skew-symmetric matrix of dimension $v - 2k - 1$. The pure spinor associated to U is given by the following formula:

$$s_U = \exp(\alpha) \wedge e_{v-2k} \wedge e_{v-2k+1} \wedge \cdots \wedge e_v, \quad \alpha = -\sum_{j=1}^{v-2k-1} a_{ij} e_i \wedge e_j \quad (6)$$

Here the exponential is defined by

$$\exp \alpha = \sum_{i=1}^{[v/2]} \frac{\alpha \wedge \cdots \wedge \alpha}{i!} = \sum_{I \subset \{1,\ldots,v-2k-1\}} \mathrm{Pf}_I(A) e_I$$

where $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ if $I = \{i_1, \ldots, i_p\}$ and $\mathrm{Pf}_I(A)$ is, up to the sign, the Pfaffian of the submatrix of A consisting of its rows and columns with numbers $i_1, \ldots, i_p$ (so that only even values of p can yield nonzero terms). The coordinates in $\mathbb{P}^{15}$ used in Formulas (1) and (2) are the ones corresponding to the basis $1, e_i \wedge e_j, e_i \wedge e_j \wedge e_k \wedge e_l$ of $S^+ = \bigwedge^{\mathrm{even}} U_\infty$.

3   Linear sections of the spinor tenfold

Mukai [27] has observed that a nonsingular section of the spinor tenfold $\Sigma_{12}^{10} \subset \mathbb{P}^{15}$ by a linear subspace $\mathbb{P}^{7+k}$ for $k = -1, 0$, resp. 1 is a canonical curve, a K3 surface, resp. a prime Fano threefold of degree 12. He has proven that a generic canonical curve of genus 7, a generic K3 surface of degree 12 and any nonsingular prime Fano threefold X_{12} (with Picard group $\mathbb{Z}$) are obtained as linear sections of Σ_{12}^{10} in a unique way modulo the action of $\mathrm{Spin}(10)$.

Definition 3.1. For a given linear section X of Σ, we denote its orthogonal linear section by $\check{X}$ and call it the *dual* of X. In particular, the dual of a Fano linear section X_{12} is a canonical curve $\Gamma = \Gamma_{12}^7 = \check{X}_{12}$ (the superscript being the genus, and the subscript the degree), and the dual of a K3 linear section S is another K3 surface $\check{S}$ of degree 12.

Lemma 3.2. *Let $\mathbb{P}^{7+k}$ with $k = -1, 0$ or 1 be a linear subspace in $\mathbb{P}^{15}$, transversal to Σ. Then the orthogonal complement $(\mathbb{P}^{7+k})^\perp = \check{\mathbb{P}}^{7-k}$ is transversal to $\Sigma^\vee$. Thus there is a natural way to associate to a linear section of Σ which is a Fano threefold, a K3 surface, resp. a canonical curve, the orthogonal linear section of $\Sigma^\vee$, which is a canonical curve, a K3 surface, resp. a Fano threefold of degree 12.*

Proof. Assume that $c \in X = \mathbb{P}^{7+k} \cap \Sigma$ is a singular point. We can represent $\mathbb{P}^{7+k}$ as the intersection of $8 - k$ hyperplanes, so that $X = \Sigma \cap \mathbb{P}^{14}_{u_0} \cap \cdots \cap \mathbb{P}^{14}_{u_{7-k}}$. As c is a singular point, we can replace the u_i by some linear combinations of them in such a way that $c \in \Pi^4_{u_0} = \mathrm{Sing}\, H_{u_0}$. We can even represent X as the intersection $\Sigma \cap \mathbb{P}^{14}_{u_0} \cap \cdots \cap \mathbb{P}^{14}_{u_{7-k}}$ with $u_i \in \check{X}$, since the span of the dual section $\check{X} = \check{\mathbb{P}}^{7-k} \cap \Sigma^\vee$ is the whole $\check{\mathbb{P}}^{7-k}$.
By reflexivity of tangent spaces, $c \in \Pi^4_{u_0}$ implies that $T_{u_0}\Sigma^\vee \subset H_c = \mathbb{P}^{14}_c \cap \Sigma^\vee$. We can complete c to a sequence $c = c_0, \ldots, c_{7+k}$ in such a way that $\check{X} = \Sigma^\vee \cap \mathbb{P}^{14}_{c_0} \cap \cdots \cap \mathbb{P}^{14}_{c_{7+k}}$, and the fact that $\mathbb{P}^{14}_{c_0}$ contains the tangent space $T_{u_0}\Sigma^\vee$ implies that u_0 is a singular point of $\check{X}$. We have proven that if X is singular, then $\check{X}$ is. By the symmetry of the roles of X and $\check{X}$, the converse is also true. $\square$

For future reference, we will cite the following lemma on plane sections of Σ. As

Σ is an intersection of quadrics, every nonempty section of it by a plane $\mathbb{P}^2$ is either a 0-dimensional scheme of length $\leqslant 4$, or a conic (possibly reducible), or a line, or a line plus a point, or the whole plane. Mukai proves that the case of length 4 is impossible:

Lemma 3.3. Σ *has no 4-secant 2-planes.*

Proof. This is Proposition 1.16 of [28]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.4. *For any* $w \in \Sigma^-$, *the singular locus of* H_w *is a projective space* $\mathbb{P}^4$, *linearly embedded into* $\mathbb{P}^{15}$, *and it consists only of points of multiplicity* 2. *Denote this* $\mathbb{P}^4$ *by* Π_w^4, *and its complement* $H_w \backslash \Pi_w^4$ *by* U_w. *Then the linear projection* $\mathrm{pr}_w : U_w \to \mathbb{P}^9$ *with center* Π_w^4 *is surjective onto the Grassmannian* $G(2,5) \subset \mathbb{P}^9$ *and induces on* U_w *the structure of the universal vector subbundle of* $\mathbb{C}^5 \times G(2,5)$ *of rank* 3.

Proof. The statement about the multiplicity of H_w at the tangency locus follows from Formulas (3)–(5) and Proposition 2.6 in [28], saying that $\mathrm{mult}_v U_w = \frac{1}{2}(\dim \mathbb{P}_v^4 \cap \mathbb{P}_w^4 + 1)$.

The fact that the tangency locus of $\mathbb{P}_w^{14}$ is a linearly embedded $\mathbb{P}^4$ follows from a quite general observation, which one can refer to as the reflexivity property of the tangent spaces (see [20]): Let $Y \subset \mathbb{P}^N$, $Y^* \subset \mathbb{P}^{N\vee}$ be dual to each other, $\dim Y = n$, $\dim Y^* = n^*$. Then for any nonsingular point $[H] \in Y^*$ representing a hyperplane H in $\mathbb{P}^N$, the latter is tangent to Y along the linear subspace Π of dimension $N - n^* - 1$, consisting of all the points $[h] \in \mathbb{P}^N$ such that $T_{[H]} Y^* \subset h$ (a point $[h] \in \mathbb{P}^N$ represents a hyperplane $h \subset \mathbb{P}^{N\vee}$). In our case $N = 15$, $n = n^* = 10$, so the tangency locus Π is $\mathbb{P}^4$. Now write down the projection with center Π in coordinates. By homogeneity of $\Sigma_{12}^{10\vee}$, we can choose coordinates $(u : \hat{X} : \vec{y})$ in such a way that $w = (1 : \vec{0} : \vec{0})$ (in dual coordinates), so that the equation of the hyperplane section is

$$H_w = \Sigma \cap (u = 0).$$

In these coordinates, $H_w \subset \mathbb{P}_w^{14}$ is defined by the restrictions of the Equations (2) for $\Sigma \subset \mathbb{P}^{15}$:

$$\vec{0} = \vec{\mathrm{Pf}}(\hat{X}) \quad \text{and} \quad \hat{X}\vec{y} = 0.$$

Therefore either $\mathrm{rk}\,\hat{X} = 2$ or $\hat{X} = \hat{0}$, and

$$H_w = U_w \cup \Pi_w^4,$$

where

$$U_w = \{(0 : \hat{X} : \vec{y}) \in H_c : \mathrm{rk}\,\hat{X} = 2, \vec{y} \in \ker \hat{X}\},$$

and

$$\Pi_c^4 = H_c - U_c = \{(0 : \hat{X} : \vec{y}) \in H_c : \text{rk}\,\hat{X} = 0\} = \{(0 : \hat{0} : \vec{y}) : \vec{y} \in \mathbb{C}^5\}.$$

The constraint $\text{rk}\,\hat{X} = 2$ cuts out exactly the Grassmannian $G(2, 5)$, and $\vec{y} \in \ker \hat{X}$ defines the universal kernel bundle of rank 3 on it. This proves our assertion. $\qquad \square$

Now let $X = \Sigma \cap \mathbb{P}^{7+k}$ for $k = -1, 0$ or 1 be a general linear section of the spinor tenfold, and $\check{X} = \check{\mathbb{P}}^{7-k} \cap \Sigma^\vee$ its dual. For any $w \in \check{X}$, let $\text{pr}_w : U_w \to G(2, 5)$ be the linear projection of Lemma 3.4. We have $X \subset H_w$, and the nonsingularity of X implies that $X \cap \Pi_w^4 = \varnothing$, that is $X \subset U_w$. Let $\pi_w = \text{pr}_w|_X$. Note that X is a *linear* section of U_w and the fibers of pr_w are linear subspaces in U_w, so the fibers of π_w are also linear subspaces. They are obviously 0-dimensional if X is a curve ($k = -1$). As $\text{Pic}(X) = \mathbb{Z}$ for $k = 0, 1$, they are 0-dimensional in these cases as well, and hence π_w is a linear isomorphism of X onto its image in $G(2, 5)$. Moreover, $X = \mathbb{P}^{7+k} \cap U_w$, hence $\langle X \rangle = \mathbb{P}^{7+k}$ does not meet Π_w^4 and $\pi_w(\langle X \rangle) = \langle \pi_w(X) \rangle$ is of dimension $7 + k$.

We will now investigate an arbitrary linear embedding of X into $G(2, 5)$. To this end, we will need Mukai's description of the embedding of X into the spinor tenfold. Let us forget that our X is a linear section of Σ and construct a spinor embedding of it in a functorial way. Consider X as a projectively normal subvariety of some projective space $\mathbb{P}^{7+k}$ and denote by $\mathscr{I}_X$ the ideal sheaf of X in this projective space. According to [27], [28], the vector space $V = H^0(\mathbb{P}^{7+k}, \mathscr{I}_X(2))$ is 10-dimensional, the subspace $U_p = H^0(\mathbb{P}^{7+k}, \mathscr{I}_X(2 - 2p)) \subset V$ is 5-dimensional for any $p \in X$, and this yields a map $\eta_X : X \to G(5, V)$, $p \mapsto U_p$. There is only one quadratic relation between the elements of V (Theorem 4.2, [28]) providing a quadratic form q_V on V, and all the spaces U_p are maximal isotropic with respect to q_V. Thus the image of η_X lies on one of the spinor varieties $\Sigma^\pm$ in $G(5, V)$ associated to the quadratic form q_V. Mukai proves (Theorem 0.4, ibid.) that η_X is an isomorphism onto its image. Let us declare this spinor variety to be Σ^+, and denote the image of X by X^+. Then $\check{X}$ is naturally embedded into Σ^-, with image X^-, and we can use the incidence Formulas (3), (4) and (5).

Lemma 3.5. *Let X be as above, and $i : X \hookrightarrow G(2, 5)$ a projective linear embedding, $U = H^0(\langle G \rangle, \mathscr{I}_G(2))$ the 5-dimensional space of quadrics passing through G. Assume that the natural map $i^* : U \to V$ is injective and that $i^*(U)$ is maximal isotropic with respect to q_V. Then there exists $w \in X^-$ such that the map $i \circ \eta_X^{-1} : X^+ \hookrightarrow G(2, 5)$ and the restriction $\pi_w : X^+ \to G(2, 5)$ of the projection pr_w defined in Lemma 3.4 are equivalent under the action of $\text{PGL}(5)$ on $G(2, 5)$.*

Proof. Consider $G = G(2, 5)$ in its Plücker embedding in $\mathbb{P}^9$ and identify U with its image in V. We have $\dim U = 5$ and $Z_p = H^0(\langle G \rangle, \mathscr{I}_G(2 - 2p)) \subset U$ is 2-dimensional for every $p \in Y$. This defines a linear isomorphism $\zeta : G \to G(2, U)$. Thus, the original embedding $i : X \hookrightarrow G(2, 5)$ is equivalent to the map $\zeta_X := \zeta \circ i : X \hookrightarrow G(2, U)$, sending a point $p \in X$ to the 2-plane $Z_p = U \cap U_p$. By (4), $w = [U] \in \Sigma^-$, by (5), $X^+ \subset H_w$, and we obtain the linear projection $\pi_w : X^+ \to G(2, 5)$. Let us complete $U = U_\infty$ to a decomposition $V = U_0 \oplus U_\infty$ of V into the direct sum of maximal isotropic subspaces. Then, as in the proof of Proposition 5.2(i),

$w = s_{U_\infty} \in \Sigma^-$ and $\pi_w(p)$ is the Plücker image $\xi = \xi_{U_p \cap U_\infty}$ of the 2-plane $Z_p = U \cap U_\infty$. This ends the proof. $\qquad\qquad\square$

Lemma 3.6. *Let X be a nonsingular Fano 3-fold ($k = 1$) or a K3 surface of genus 7 with Picard number 1 ($k = 0$). Then for any linear embedding $i : X \hookrightarrow G(2,5)$ such that the map $i^* : U = H^0(\langle G \rangle, \mathscr{I}_G(2)) \to V = H^0(\langle X \rangle, \mathscr{I}_X(2))$ is injective, U is a maximal isotropic subspace of V with respect to the quadratic form q_V.*

Proof. Assume that U is not isotropic. Then q_V defines a 3-dimensional quadric Q in $\mathbb{P}(U)$. In the notation from the proof of Lemma 3.5, the isotropy of the 5-spaces U_p implies that the projective lines $\mathbb{P}(U \cap U_p)$ ($p \in X$) are all contained in Q. Thus the map $p \mapsto \mathbb{P}(U \cap U_p)$, projectively equivalent to i, transforms X isomorphically onto a subvariety of the family of lines $G(1; Q)$ on the 3-dimensional quadric Q.

Let $k = 1$, that is X is a Fano threefold. If Q is nonsingular, $G(1; Q) \simeq \mathbb{P}^3$, and this is absurd, as $X \not\simeq \mathbb{P}^3$. If Q is of rank 4, then the family of lines on Q has two components, each one of which is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$; this is absurd because X does not contain any plane. The cases of smaller rank lead also to contradictions, hence U is isotropic.

The argument is similar for the case of a K3 surface: if $\mathrm{rk}\, Q = 5$, then $X \subset \mathbb{P}^3$, which is absurd, and if $\mathrm{rk}\, Q = 4$, then X has a pencil of curves defined by the $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, but the generic K3 surface has no pencils of curves. $\qquad\square$

Similar statements hold also in the case $k = -1$, but the proof uses vector bundle techniques and is postponed until Section 5.

4   Elliptic quintics and rank-2 vector bundles on X_{12}

Let $X = X_{12} = \mathbb{P}^8 \cap \Sigma$ be a Fano 3-dimensional linear section of the spinor tenfold Σ. An *elliptic quintic* in X is a nonsingular irreducible curve $C \subset X$ of genus 1 and of degree 5. We will also deal with degenerate "elliptic" quintics, which we will call just *quasi-elliptic quintics*. A quasi-elliptic quintic is a locally complete intersection curve C of degree 5 in X, such that $h^0(\mathcal{O}_C) = 1$ and the canonical sheaf of C is trivial: $\omega_C = \mathcal{O}_C$. A reduced quasi-elliptic quintic will be called a *good quintic*.

Lemma 4.1. *Let $C \in X$ be a quasi-elliptic quintic. Then $\langle C \rangle = \mathbb{P}^4$, where the angular brackets denote the linear span.*

Proof. Assume that $C \subset \mathbb{P}^3$. Then a general section of C by a plane $\mathbb{P}^2 \subset \mathbb{P}^3$ is a 0-dimensional scheme of length 5. This contradicts Lemma 3.3. Hence $\dim\langle C \rangle \geqslant 4$.

To prove the opposite inequality, it suffices to show that $h^0(C, \mathcal{O}_C(1)) \leqslant 5$. This follows from the Serre duality and the Riemann–Roch formula. $\qquad\square$

Starting from any quasi-elliptic quintic $C \subset X$, one can construct a rank-2 vector bundle $\mathscr{E}$ with Chern classes $c_1(\mathscr{E}) = 1$, $c_2(\mathscr{E}) = 5$. It is obtained as the middle term of the following nontrivial extension of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_C(1) \to 0, \tag{7}$$

where $\mathcal{I}_C = \mathcal{I}_{C,X}$ is the ideal sheaf of C in X. One can easily verify (see [22] for a similar argument) that, up to isomorphism, there is a unique nontrivial extension (7), thus C determines the isomorphism class of $\mathcal{E}$. This way of constructing vector bundles is called Serre's construction. The vector bundle $\mathcal{E}$ has a section s whose scheme of zeros is exactly C. Conversely, for any section $s \in H^0(X, \mathcal{E})$ such that its scheme of zeros $(s)_0$ is of codimension 2, the vector bundle obtained by Serre's construction from $(s)_0$ is isomorphic to $\mathcal{E}$. As $\det \mathcal{E} \simeq \mathcal{O}_X(1)$, we have $\mathcal{E} \simeq \mathcal{E}^\vee(1)$.

The proofs of the following three lemmas are standard; see, for example, [22], where similar facts are proved for elliptic quintics in a cubic threefold.

Lemma 4.2. *For any quasi-elliptic quintic $C \subset X$, the associated vector bundle $\mathcal{E}$ possesses the following properties*:

(i) $h^0(\mathcal{E}) = 5$, $h^i(\mathcal{E}(-1)) = 0$ *for* $i \in \mathbb{Z}$, *and* $h^i(\mathcal{E}(k)) = 0$ *for* $i > 0$, $k \geqslant 0$.

(ii) $\mathcal{E}$ *is stable and the local dimension of the moduli space of stable vector bundles at* $[\mathcal{E}]$ *is at least* 1.

(iii) *The scheme of zeros* $(s)_0$ *of any nonzero section* $s \in H^0(X, \mathcal{E})$ *is a quasi-elliptic quintic*.

(iv) *If* s, s' *are two nonproportional sections of* $\mathcal{E}$, *then* $(s)_0 \neq (s')_0$. *This means that* $(s)_0$ *and* $(s')_0$ *are different subschemes of* X.

Lemma 4.3. *Let* $\mathcal{E}$ *be a vector bundle as in Lemma* 4.2, *C the scheme of zeros of any nonzero section of* $\mathcal{E}$, *and* $\mathcal{N}_{C/X}$ *its normal bundle. Then the following properties are equivalent*:

(i) $h^1(\mathcal{N}_{C/X}) = 0$;

(ii) $h^0(\mathcal{N}_{C/X}) = 5$;

(iii) $h^1(\mathcal{E}^\vee \otimes \mathcal{E}) = 1$;

(iv) $h^2(\mathcal{E}^\vee \otimes \mathcal{E}) = 0$.

If one of the properties (i)–(iv) *is true, then we have*:

(a) *The Hilbert scheme* Hilb_X^{5n} *of subschemes in X with Hilbert polynomial* $\chi(n) = 5n$ *is smooth and of dimension 5 at the point* $[C]$ *representing C.*

(b) *The moduli space* $M_X = M_X(2; 1, 5)$ *of stable vector bundles of rank 2 with Chern classes* $c_1 = 1$, $c_2 = 5$ *is smooth and of dimension 1 at the point* $[\mathcal{E}]$ *representing the vector bundle* $\mathcal{E}$.

(c) $[\mathcal{E}]$ *has a Zariski neighbourhood U in M_X with a universal vector bundle $\mathcal{E}$ over* $U \times X$, *and the projective bundle* $\mathbb{P}(\mathrm{pr}_{1*} \mathcal{E})$ *is isomorphic to an open subset H_U of* Hilb_X^{5n} *via a map that can be defined pointwise on the fiber over each point $t \in U$ by* $s \mapsto (s)_0$, *where* $s \in (\mathrm{pr}_{1*} \mathcal{E})_t = H^0(X, \mathcal{E}_t)$.

The map $[D] \mapsto [\mathcal{E}_D]$ *given by Serre's construction is a smooth morphism from H_U onto U with fiber* $\mathbb{P}^4 = \mathbb{P}H^0(\mathcal{E}_D)$.

Lemma 4.4. *Let $\mathscr{E}$ be a vector bundle as in Lemma 4.2, C the scheme of zeros of any nonzero section of $\mathscr{E}$, and $\mathscr{N}_{C/X}$ its normal bundle. Then $h^2(\mathscr{E}^\vee \otimes \mathscr{E}) \neq 0$ if and only if $\mathscr{N}_{C/X}$ is a decomposable vector bundle on C. In this case $\mathscr{N}_{C/X} \simeq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $h^2(\mathscr{E}^\vee \otimes \mathscr{E}) = h^1(\mathscr{N}_{C/X}) = 1$.*

Now we will exploit the restrictions of the vector bundles $\mathscr{E}$ to the hyperplane sections $H \subset X$, which are K3 surfaces of genus 7.

Lemma 4.5. *Let $\mathscr{E} \in M_X(2; 1, 5)$, H a general hyperplane section of X, and $\mathscr{E}_H = \mathscr{E}|_H$ the restriction of $\mathscr{E}$ to H. Then $\mathscr{E}_H$ is stable and has the following cohomology:* $h^0(\mathscr{E}_H) = 5$, $h^1(\mathscr{E}_H(n)) = 0$ *for all* $n \in \mathbb{Z}$, $h^2(\mathscr{E}_H(n)) = 0$ *for all* $n \geqslant 0$.

Proof. By [24], Theorem 3.1, $\mathscr{E}_H$ is Mumford–Takemoto semistable for general H. Hence $h^0(\mathscr{E}_H(-1)) = 0$, and the determinant of $\mathscr{E}_H(-1)$ being odd, the semistability is equivalent to the stability. We have $\chi(\mathscr{E}_H) = 5$ and $h^2(\mathscr{E}_H) = h^0(\mathscr{E}_H(-1)) = 0$, so $h^0(\mathscr{E}_H) \geqslant 5$. Let s be a section of $\mathscr{E}_H$ and $Z = (s)_0$ its scheme of zeros. It is 0-dimensional, because if it contained a curve from $|\mathscr{O}(k)|$, then $\mathscr{E}_H(-k)$ would have nonzero global sections. Thus s defines a Serre triple

$$0 \to \mathscr{O} \to \mathscr{E}_H \to \mathscr{I}_Z(1) \to 0 \tag{8}$$

which provides the equivalence

$$h^0(\mathscr{E}_H) = 5 + k \Leftrightarrow \dim\langle Z \rangle = 3 - k,$$

where the angular brackets denote the linear span. If we assume that $h^0(\mathscr{E}_H) > 5$, then $Z \subset \mathbb{P}^2$, which contradicts Lemma 3.3. Hence $h^0(\mathscr{E}_H) = 5$, $h^i(\mathscr{E}_H) = 0$ for $i > 0$. Twisting (8) by $\mathscr{O}(n)$, we deduce the remaining assertions. □

In fact, by the same arguments as above, one proves:

Lemma 4.6. *Let S be any nonsingular surface linear section of Σ_{12}^{10} by a subspace $\mathbb{P}^7$ with Picard group $\mathbb{Z}$. Then a rank-2 vector bundle E on S with Chern classes $c_1 = 1$, $c_2 = 5$ is stable if and only if it is obtained by Serre's construction from a zero-dimensional subscheme Z of S whose linear span is $\mathbb{P}^3$. The twists $E(n)$ of any such vector bundle on S have the same cohomology as $\mathscr{E}_H$ in Lemma 4.5.*

Lemma 4.7. *Let $\mathscr{E} \in M_X(2; 1, 5)$. Then $h^i(\mathscr{E}(-1)) = 0$ for all $i \in \mathbb{Z}$, and $\mathscr{E}$ can be obtained by Serre's construction from a quasi-elliptic quintic $C \subset X$. Hence $\mathscr{E}$ satisfies also the properties* (i), (iii), (iv) *of Lemma 4.2.*

Proof. The exact triple

$$0 \to \mathscr{E}(n-1) \to \mathscr{E}(n) \to \mathscr{E}_H(n) \to 0 \tag{9}$$

for generic H together with Lemma 4.5 implies that $h^2(\mathscr{E}(n-1)) = h^2(\mathscr{E}(n))$

and $h^1(\mathcal{E}(n-1)) \geqslant h^1(\mathcal{E}(n))$ for all $n \geqslant 0$. By the Kodaira vanishing theorem, $h^2(\mathcal{E}(n)) = 0$ for $n \gg 0$, hence $h^2(\mathcal{E}(n)) = 0$ for all $n \geqslant -1$. Now look at the same triple for $n = 0$. By stability and Serre duality, $h^0(\mathcal{E}(-1)) = h^3(\mathcal{E}(-1)) = 0$; and we have just proved that $h^2(\mathcal{E}(-1)) = 0$, which implies, by Serre duality, that $h^1(\mathcal{E}(-1)) = 0$. Hence $h^1(\mathcal{E}(n)) = 0$ for all $n \geqslant 0$. As $\chi(\mathcal{E}) = 5$, $h^0(\mathcal{E}) = 5$. Take any section $s \neq 0$ of $\mathcal{E}$. Its scheme of zeros $C = (s)_0$ is of codimension 2, because $h^0(\mathcal{E}(-1)) = 0$, so $\mathcal{E}$ is obtained by Serre's construction from C. $\qquad\square$

5   The map $\rho_X : \check{X} \to M_X$

Let $X = \mathbb{P}^{7+k} \cap \Sigma$ for $k = -1, 0$ or 1 be a nonsingular linear section of the spinor tenfold Σ, and $\check{X} = \check{\mathbb{P}}^{7-k} \cap \Sigma^\vee$ its dual. In the case $k = 0$, assume that X is sufficiently general, so that $\operatorname{Pic} X \simeq \mathbb{Z}$. In the case $k = -1$, assume that X is a generic curve of genus 7.

Let M_X be the moduli space $M_X(r; c_1, c_2)$ of stable vector bundles of rank $r = 2$ on X with Chern classes $c_1 = 1$, $c_2 = 5$ in the cases when X is K3 or Fano ($k = 0, 1$), and the non-abelian Brill–Noether locus $W_{r,K}^\alpha$ of stable vector bundles on X of rank $r = 2$ with canonical determinant K and with at least $\alpha = 5$ global sections in the case when X is a canonical curve ($k = -1$). In Proposition 5.2, we will construct a natural morphism $\rho = \rho_X : \check{X} \to M_X$.

Lemma 5.1. *Let $i : \Gamma \to G(2,5)$ be an embedding of a generic canonical curve of genus 7, linear with respect to the Plücker coordinates and such that $\langle i(\Gamma) \rangle = \mathbb{P}^6$. Let $\mathcal{Q}_G$ be the universal quotient rank-2 bundle on $G = G(2,5)$ and $\mathcal{E} = i^* \mathcal{Q}_G$. Then $\mathcal{E}$ is stable and $h^0(\mathcal{E}) = 5$.*

Proof. Assume that $h^0(\mathcal{E}) < 5$. Then there is a section of $\mathcal{Q}_G$ vanishing identically on the image of Γ. The zero loci of the sections of $\mathcal{Q}_G$ are the sub-Grassmannians $G(2,4)$ in G, so i embeds Γ into some $G(2,4) \subset G$. This is absurd, because the linear span of $G(2,4)$ is $\mathbb{P}^5$, but by hypothesis, that of $i(\Gamma)$ is $\mathbb{P}^6$. Thus, $h^0(\mathcal{E}) \geqslant 5$ and the restriction map $i^* : H^0(G, \mathcal{Q}_G) \to H^0(\Gamma, \mathcal{E})$ is injective. Denote by W the image of i^*. The initial embedding i is projectively equivalent to the map

$$\varphi_W : x \in \Gamma \mapsto W_x^\perp = \{u \in W^\vee \,|\, u(s) = 0 \ \text{for all} \ s \in W_x\} \in G(2, W^\vee),$$

where $W_x = \{s \in W \,|\, s(x) = 0\}$ is of codimension 2 for any $x \in \Gamma$.

Assume that $\mathcal{E}$ is non-stable. Then there is an exact triple

$$0 \to L_1 \to \mathcal{E} \to L_2 \to 0,$$

in which L_1, L_2 are line bundles, $L_2 = \omega_\Gamma \otimes L_1^{-1}$ and $\deg L_1 \geqslant 6$. Remark first that the case $h^0(L_2) = 0$ is impossible. Indeed, in this case all the sections of $\mathcal{E}$ are those of the line subbundle L_1 and the subspaces W_x are of codimension 1.

Assume now that $h^0(L_2) = 1$. Then either $W \subset H^0(\Gamma, L_1)$ and this brings us to a contradiction as above, or the map $W \to H^0(\Gamma, L_2)$ is surjective. In the latter case

the Plücker image of $\varphi_W(\Gamma)$ is contained in a linear subspace $\mathbb{P}^3$ of $\mathbb{P}^9$ of the form $\langle e_1 \wedge e_5, e_2 \wedge e_5, e_3 \wedge e_5, e_4 \wedge e_5 \rangle$, where $e_1, \ldots, e_5$ is a basis of $W^\vee$ such that e_5 generates the image of the natural inclusion $H^0(\Gamma, L_2)^\vee \to W^\vee$. This is absurd, because φ_W is projectively equivalent to i and the linear span of $i(\Gamma)$ is $\mathbb{P}^6$.

Hence $h^0(L_2) \geqslant 2$. As Γ has no g_4^1, we have $\deg L_2 \geqslant 5$. Hence $\deg L_2 = 5$ or 6.

1st *case*: $\deg L_1 = \deg L_2 = 6$. By Riemann–Roch, $h^0(L_1) = h^0(L_2)$. As $h^0(\mathscr{E})$ must be at least 5, we have $h^0(L_i) \geqslant 3$, $i = 1, 2$. Therefore Γ has two (possibly coincident) g_6^2's. By [1], Theorem V.1.5, the expected dimension of the family G_d^r of linear series g_d^r on a curve of genus g is $\rho_{g,d,r} = g - (r+1)(g-d+r)$, and $G_d^r = \varnothing$ for a generic curve of genus g if $\rho_{g,d,r} < 0$. Hence a generic Γ of genus 7 has no g_6^2's.

2nd *case*: $\deg L_1 = 7$, $\deg L_2 = 5$. If $h^0(L_2) \geqslant 3$, then Γ has a g_5^2, which is impossible by the same argument as above. So, $h^0(L_1) = 3$, $h^0(L_2) = 2$. The Bockstein morphism $\delta : H^0(L_2) \to H^1(L_1)$ being given by the cup-product with the extension class $e \in H^1(L_1 \otimes L_2^{-1}) = H^0(L_2^{\otimes 2})^\vee$, the vanishing of δ implies that of e. Hence $\mathscr{E} = L_1 \oplus L_2$. Let s_1, s_2, s_3 be a basis of $H^0(L_1)$ and t_1, t_2 that of $H^0(L_2)$. Then φ_W can be given in appropriate Plücker coordinates by $x \mapsto (s_1(x)e_1 + s_2(x)e_2 + s_3(x)e_3) \wedge (t_1(x)e_4 + t_2(x)e_5)$. Thus if we fix $t = (t_1 : t_2)$ we will get a plane $\mathbb{P}_t^2 = \langle e_1, e_2, e_3 \rangle \wedge (t_1 e_4 + t_2 e_5)$ in $G(2,5)$ which meets $\varphi_W(\Gamma)$ in 5 points in which $(t_1(x) : t_2(x)) = (t_1 : t_2)$. This contradicts Lemma 3.3.

According to [3], the non-abelian Brill–Noether loci $W_{2,K}^\alpha$ on a generic curve of genus $g \leqslant 8$ are empty if and only if their expected dimension $d = 3g - 3 - \alpha(\alpha+1)/2$ is negative. Hence in our case $W_{2,K}^6 = \varnothing$ and $h^0(\mathscr{E}) = 5$. $\qquad\square$

Proposition 5.2. *Denote by* $\mathrm{pr}_w : U_w \to G(2,5)$ *for any* $w \in \check{X}$ *the linear projection of Lemma 3.4. We have* $X \subset U_w = H_w \backslash \Pi_w^4$. *Let* $\pi_w = \mathrm{pr}_w|_X$. *It is an isomorphism of* X *onto its image in* $G(2,5)$. *Define a rank-2 vector bundle* $\mathscr{E} = \mathscr{E}_w$ *on* X *as the pullback of the universal quotient rank-2 bundle on the Grassmannian:* $\mathscr{E}_w := \pi_w^* \mathscr{Q}_{G(2,5)}$. *Then* $\mathscr{E}_w$ *is stable and the map* $\rho = \rho_X : w \mapsto [\mathscr{E}_w]$ *is a morphism from* $\check{X}$ *to* M_X. *Any vector bundle* $\mathscr{E}$ *in the image of* ρ *possesses the following properties.*

(i) $h^0(\mathscr{E}) = 5$ *and if* $k = 0$ (*resp.* $k = 1$), *then* $\mathscr{E}$ *is obtained by Serre's construction from a l. c. i. 0-dimensional subscheme* Z *of length 5 such that* $\langle Z \rangle = \mathbb{P}^3$ (*resp. from a quasi-elliptic quintic* $C \subset X$).

(ii) $\mathscr{E}$ *is globally generated and, for a generic section* s *of* $\mathscr{E}$, $(s)_0$ *is a smooth elliptic quintic if* $k = 1$ *and a subset of 5 distinct points if* $k = 0$.

(iii) *Let* $k = 1$, *that is* X *is a Fano threefold. Then the family of singular curves* $(s)_0$ ($s \in H^0(\mathscr{E})$) *is a divisor in* $\mathbb{P}H^0(\mathscr{E})$. *For generic* $p \in X$, *there are at most three curves* $(s)_0$ *which are singular at* p.

Proof. In the case $k = -1$, the wanted assertion is an immediate consequence of Lemma 5.1. Consider now the case $k = 1$. It implies easily the one of $k = 0$ by taking hyperplane sections. We will first prove Part (i), and the stability of $\mathscr{E}_w$ will follow from Lemma 4.2.

(i) The sections of $\mathscr{Q}_{G(2,5)}$ are in a natural one-to-one correspondence with linear forms ℓ on the 5-dimensional vector space W, if we think of $G(2,5)$ as the variety of 2-planes Π in W, the fiber $\mathscr{Q}_t$ of $\mathscr{Q}_{G(2,5)}$ at $t \in G(2,5)$ being just $\Pi^\vee$. Let s_ℓ be the

section of $\mathcal{2}_{G(2,5)}$ associated to ℓ; denote by the same symbol the induced section of $\mathrm{pr}_w^* \mathcal{2}_{G(2,5)}$ and by s its restriction to X.

Let us choose coordinates $(u : x_{ij} : y_k)$ in $\mathbb{P}^{15}$ in such a way that the equations of $\{s_\ell = 0\}$ acquire a very simple form. First of all, as in the proof of Lemma 3.4, we choose the origin at w, so that $w = (1 : \vec{0} : \vec{0})$. In a coordinate free form, we fix an identification of Σ with Σ^+, as in Section 2, corresponding to a decomposition $V = U_0 \oplus U_\infty$ of a 10-dimensional vector space V endowed with a nondegenerate quadratic form into the direct sum of maximal isotropic subspaces, and choose $w = s_{U_\infty} \in \Sigma^-$. Then by (6), the $\bigwedge^2 U_\infty$ component of the pure spinor s_U associated to any maximal isotropic $U \subset V$, $[U] \in U_w = H_w \backslash \Pi_w^4$, is just the Plücker image $\xi = \xi_{U \cap U_\infty}$ of the 2-plane $U \cap U_\infty$; in the notation of (6), $\xi = e_4 \wedge e_5$. Thus the above 5-space W used for the definition of $G(2,5)$ is naturally identified with U_∞. So, if we choose coordinates $(x_1, \ldots, x_5)$ in U_∞ in such a way that $\ell = x_5$, we obtain the following equations for the zero locus of s_ℓ in the Plücker coordinates associated to $(x_1, \ldots, x_5)$: $x_{15} = x_{25} = x_{35} = x_{45} = 0$. To these, one should add the equation $u = 0$ of H_w and the 10 quadratic ones for $\Sigma = \Sigma^+$. Five of the latter ones are trivially satisfied under the above linear constraints, so finally we obtain the following system of equations for the closure Z_ℓ of $\{s_\ell = 0\} \subset U_w$ in $\mathbb{P}^{15}$:

$$u = x_{15} = x_{25} = x_{35} = x_{45} = 0$$

$$q_5^+ = x_{12} x_{34} - x_{13} x_{24} + x_{23} x_{14} = 0$$

$$q_1^- = \qquad\quad x_{12} y_2 + x_{13} y_3 + x_{14} y_4 = 0$$

$$q_2^- = -x_{12} y_1 \qquad\quad + x_{23} y_3 + x_{24} y_4 = 0$$

$$q_3^- = -x_{13} y_1 - x_{23} y_2 \qquad\quad + x_{34} y_4 = 0$$

$$q_4^- = -x_{14} y_1 - x_{24} y_2 - x_{34} y_3 \qquad\quad = 0$$

The five quadratic equations are just (up to sign) the quadratic Pfaffians of the skew-symmetric matrix

$$M = \begin{bmatrix} 0 & -y_1 & y_2 & -y_3 & y_4 \\ y_1 & 0 & x_{34} & x_{24} & x_{23} \\ -y_2 & -x_{34} & 0 & x_{14} & x_{13} \\ y_3 & -x_{24} & -x_{14} & 0 & x_{12} \\ -y_4 & -x_{23} & -x_{13} & -x_{12} & 0 \end{bmatrix}.$$

By an obvious linear change of variables, we see that the quadratic Pfaffians of M define the 6-dimensional Grassmannian $G(2,5)$ in the projective space $\mathbb{P}^9$ with coordinates $y_1, y_2, y_3, y_4, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$, and in taking into account the coordinate y_5 missing in all the equations, we conclude that Z_ℓ is the 7-dimensional cone over $G(2,5)$ with vertex $(0 : \ldots : 0 : 1) \in \mathbb{P}^{15}$.

It is well known that the degree of $G(2,5)$ is 5 and that its curve linear sections are quintics with trivial canonical bundle [10], so the same property is true for Z_ℓ,

if one considers only complete intersection linear sections (that is, defined by 6 equations). Adding to the above equations of Z_ℓ the 6 linear equations of X in H_w, we obtain the wanted zero locus $(s)_0$ of $s = s_\ell|_X$ as a linear section of Z_ℓ of expected dimension 1. If it is indeed a curve, we are done. It cannot contain a surface, because the degree of the surface would not exceed 5, but the Picard group of X is generated by the class of hyperplane section which is of degree 12. Finally, s cannot be identically zero. Indeed, assume the contrary. The map π_w projects X linearly and isomorphically onto its image $\overline{X}$ in $G(2,5)$, and the fact that $s \equiv 0$ means that X is contained in the zero locus of s_ℓ. The latter is the Schubert subvariety $\sigma_{11}(L)$, where L denotes the hyperplane $\ell = 0$ in V, that is the Grassmannian $G(2,4)$ of vector planes contained in L. This is impossible, because X cannot be represented as a hypersurface in a 4-dimensional quadric. This proves the wanted assertion about the loci $(s)_0$ and that $H^0(G(2,5), \mathcal{Q}_{G(2,5)})$ is mapped injectively into $H^0(X, \mathcal{E}_w)$.

(ii) $\mathcal{Q}_{G(2,5)}$ is globally generated, hence so is $\mathcal{E}_w = \pi_w^* \mathcal{Q}_{G(2,5)}$. The smoothness of the zero locus of the generic section follows then by Bertini's Theorem.

(iii) Consider X as a subvariety of $G(2,5)$. Let us verify that for any $p \in X$, there is a Grassmannian $G(2,4) = \sigma_{11}(L)$ passing through p and such that its intersection with X is not transversal at p, that is, $\dim T_p X \cap T_p G(2,4) > 1$. To this end, choose a basis $e_1, \ldots, e_5$ of $\mathbb{C}^5$ in such a way that $p = [e_1 \wedge e_2]$. We may assume that $\sigma_{11}(L_0) \cap X$ is a smooth elliptic quintic, where $L_0 = \langle e_1, e_2, e_3, e_4 \rangle$ and that $T_p X$ is not contained in the span $\mathbb{P}^5$ of $\sigma_{11}(L_0)$. Assume also that there is no line on $\sigma_{11}(L_0)$ through p whose tangent direction coincides with that of the elliptic quintic; the case when there is one is treated similarly. Under this assumptions we can represent a basis of $T_p X$ in the form $(e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_5 + a_{14} e_1 \wedge e_4 + a_{23} e_2 \wedge e_3 + a_{24} e_2 \wedge e_4, e_2 \wedge e_5 + b_{14} e_1 \wedge e_4 + b_{23} e_2 \wedge e_3 + b_{24} e_2 \wedge e_4)$, where a_{ij}, b_{ij} are constants. Any L such that $p \in \sigma_{11}(L)$ is given by the equation $\alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 = 0$. Then $T_p \sigma_{11}(L)$ is spanned by four bivectors $e_i \wedge v_j$, $1 \leqslant i, j \leqslant 2$, where (v_1, v_2) is a basis of the vector plane $\{\alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 = 0\} \subset \langle e_3, e_4, e_5 \rangle$. For example, if $\alpha_5 \neq 0$, then one can choose $v_1 = -\alpha_5 e_3 + \alpha_3 e_5$, $v_2 = -\alpha_5 e_4 + \alpha_4 e_5$. It is an easy exercise to check that the 7×8 matrix of components of the vectors generating $T_p X + T_p \sigma_{11}(L)$ is of rank < 6 for at least one value of $(\alpha_3 : \alpha_4 : \alpha_5) \in \mathbb{P}^2$, and if the number of such values is finite, then it is at most three. Since a generic curve $(s)_0 \in \mathbb{P}H^0(\mathcal{E})$ is smooth, the family of singular ones is at most three-dimensional. We have seen that the subfamily Z_p of curves $(s)_0 \in \mathbb{P}H^0(\mathcal{E})$ singular at p is nonempty for any $p \in X$, hence, by a dimension count, Z_p is finite for generic p. This ends the proof. $\qquad \square$

Part (ii) of the proposition implies the following corollary.

Corollary 5.3. *In the case $k = 1$, the family of elliptic quintics in X is nonempty.*

For instance, we have not even verified that the morphism $\rho : \check{X} \to M_X$ is nonconstant. This follows from the next lemma.

Proposition 5.4. *The image of ρ is an irreducible component M_X^0 of M_X.*

Proof. It suffices to prove that the image of ρ is open. Let $\mathscr{E}_0$ be a vector bundle on X in the image of ρ. Then $\mathscr{E}_0$ is generated by global sections and the natural quotient map $H^0(X, \mathscr{E}_0) \otimes \mathscr{O}_X \to \mathscr{E}_0$ defines a linear embedding of X into the Grassmannian $G(2, 5)$ of 2-dimensional quotients of $\mathbb{C}^5 = H^0(X, \mathscr{E}_0)$. This is an open property and it will be verified for a vector bundle $\mathscr{E}$ in a neighborhood of $\mathscr{E}_0$ in M_X. Also the conditions in the hypotheses of Lemmas 3.5, 3.6 are open, so, in the cases when X is either a K3 surface or a Fano threefold ($k = 0$ or 1), the embedding of X into $G(2, 5)$ by global sections of $\mathscr{E}$ is given, up to a linear change of coordinates, by the projection π_w for some $w \in \check{X}$, and hence $\mathscr{E}$ is in the image of ρ. The case $k = -1$ follows in the same manner from Lemma 5.5 below. $\qquad\square$

Lemma 5.5. *Let $X = \Gamma$ be a generic canonical curve of genus* 7. *Under the hypotheses of Lemma* 5.1, *suppose in addition, as in Lemma* 3.6, *that the map $i^* : U = H^0(\langle G \rangle, \mathscr{I}_G(2)) \to V = H^0(\langle X \rangle, \mathscr{I}_X(2))$ is injective. Then U is a maximal isotropic subspace of V with respect to the quadratic form q_V.*

Proof. Assume that U is not isotropic. As in the proof of Lemma 3.6, q_V defines a 3-dimensional quadric Q in $\mathbb{P}(U)$ and the isotropy of the 5-spaces U_p implies that the projective lines $\mathbb{P}(U \cap U_p)$ are all contained in Q. Let $\mathscr{E} = i^*(\mathscr{Q}_G)$ be the pullback of the universal quotient rank-2 vector bundle on $G = G(2, 5)$. The fiber of $\mathscr{E}$ over $p \in X$ is canonically identified with the dual of the 2-plane $U \cap U_p$. By Lemma 5.1, it is stable and $h^0(X, \mathscr{E}) = 5$. We can now apply Proposition 4.1 of [3], which yields the injectivity of the modified Petri map $\mathrm{Sym}^2(H^0(X, \mathscr{E})) \to H^0(X, \mathrm{Sym}^2(\mathscr{E}))$. Further, the authors of [3] prove in the claim on p. 267 that the injectivity of the modified Petri map is equivalent to the following property: there is no quadric Q in $\mathbb{P}(H^0(X, \mathscr{E})^*)$ containing all the lines $\mathbb{P}(\mathscr{E}_x)$ for $x \in X$. This ends the proof. $\qquad\square$

Proposition 5.6. *Let $k = -1$, that is, X is a generic canonical curve of genus* 7. *Then M_X is a Fano threefold of genus* 7 *and $\rho_X : \check{X} \to M_X$ is an isomorphism of Fano threefolds.*

Proof. Mukai [30] has proved that M_X is a Fano threefold of genus 7 with Picard number 1. By Proposition 5.4, ρ_X is surjective. It is easy to see that any non-constant morphism between two Fano threefolds of genus 7 with Picard number 1 is an isomorphism. Indeed, let $f : X_1 \to X_2$ be such a morphism. The fact that $\mathrm{Pic}\, X_1 \simeq \mathbb{Z}$ implies that f is finite of degree $\delta \geqslant 1$. Suppose that $\delta > 1$. As X_1, X_2 are smooth, the ramification divisors $\Delta_i \subset X_i$ are smooth surfaces. Let H_i be a hyperplane section of X_i. We have, for some positive integers $d > 1$, $e \geqslant 1$, the following relations:

$$f^* H_2 \sim d H_1, \quad K_{X_i} \sim -H_i, \quad f^* \Delta_2 \sim \Delta_1, \quad K_{X_1} \sim f^* K_{X_2} + (e-1)\Delta_1.$$

One deduces immediately the relations

$$\Delta_1 \sim \frac{d-1}{e-1} H_1, \quad \Delta_2 \sim \frac{d-1}{d} \cdot \frac{e}{e-1} H_2.$$

As the Δ_i are integer multiples of H_i, we conclude that $d = e$ and $\Delta_i \sim H_i$ ($i = 1, 2$). Hence $\text{Pic}(X_2 \backslash \Delta_2) = 0$, and this contradicts the fact that f induces a non-ramified covering of $X_2 \backslash \Delta_2$. Hence $\delta = 1$ and f is an isomorphism. $\square$

Proposition 5.7. *If $k = 0$, that is, X is a K3 surface, then $M_X = M_X^0$ is a K3 surface; in particular, it is irreducible and nonsingular. Moreover, if X is generic, then $\rho_X: \check{X} \to M_X$ is an isomorphism of K3 surfaces.*

Proof. The first assertion is due to Mukai (Proposition 4.4 in [26], or Theorem 6.1.8 in [11]). The fact that ρ_X is an isomorphism follows, as above, from Proposition 5.4 and the fact that there are no surjective morphisms between K3 surfaces that are not isomorphisms. $\square$

Corollary 5.8. *If $k = 0$, that is, X is a K3 surface with Picard number 1, then any $\mathscr{E} \in M_X$ is globally generated, and for generic $s \in H^0(X, \mathscr{E})$, the zero locus (s) is a set of 5 distinct points which span $\mathbb{P}^3$.*

Proof. By Proposition 5.7, M_X is the image of ρ and the wanted assertions follow from Proposition 5.2. $\square$

Proposition 5.9. *Let $k = 1$, that is, X is a generic prime Fano threefold of degree 12. Then any $\mathscr{E} \in M_X$ is globally generated and the curve (s) is a (smooth) elliptic quintic for generic $s \in H^0(X, \mathscr{E})$.*

Proof. In the setting of Lemma 4.7, we deduce from the restriction isomorphism $H^0(X, \mathscr{E}) \to H^0(H, \mathscr{E}_H)$ and from Corollary 5.8 that for generic $s \in H^0(X, \mathscr{E})$, the curve $C_s = (s)_0$ is reduced and may be singular only at a finite set of points $T \subset X$ where $\mathscr{E}$ is not generated by global sections. If the restriction $\mathscr{E}_H$ of $\mathscr{E}$ to some K3 linear section H of X through a point $x \in T$ were stable, we would apply the same argument to see that $\mathscr{E}_H$, and hence $\mathscr{E}$ itself is globally generated, which would be a contradiction. Hence T possesses the property that for any $x \in T$ and for any non-singular K3 linear section H passing through x, $\mathscr{E}_H$ is unstable. By Lemma 4.7 and the restriction exact sequence (9), $h^0(\mathscr{E}_H(-1)) = 0$, hence $\mathscr{E}_H$ is stable for any non-singular K3 linear section H of X such that $\text{Pic}(H) \simeq \mathbb{Z}$. By Theorem 5.4 of [25], this condition is verified for a very general H ("very general" means "in the complement of a union of countably many Zarisky closed subsets"). But the proof in [25] actually implies a subtler result in dimension 3:

Lemma 5.10 (Moishezon). *Let V be a nonsingular projective 3-fold and $\{H_t\}_{t \in \mathbb{P}^1}$ a Lefschetz pencil of hyperplane sections of V such that $h^{2,0}(H_t) > h^{2,0}(V)$ for the non-singular members H_t of the pencil. Assume also that the base locus of the pencil is a nonsingular curve. Then the restriction map $\text{Pic}(V) \to \text{Pic}(H_t)$ is an isomorphism for a very general $t \in \mathbb{P}^1$.*

It is easy to see that through any point $x \in X$ passes a Lefschetz pencil of K3 linear

sections satisfying the hypothesis of the lemma. Hence $\mathscr{E}$ is globally generated and T is empty. $\square$

6   Takeuchi–Iskovskikh–Prokhorov birational maps

The general setting for the constructions of the two birational maps that we will discuss is as follows: let X be a Fano 3-fold, and $Z \subset X$ a point or a nonsingular curve. Let X' be the blowup of Z. Assume that the linear system $|-nK_{X'}|$ for some $n > 0$ defines a nontrivial birational morphism. Then it has only finitely many positive-dimensional fibers, which are curves, there exists a flop $X' \dashrightarrow X^+$ centered on these curves, and the resulting variety X^+ possesses an extremal ray that can be contracted down onto some variety Y. Sometimes Y is a 3-fold, and this is the case studied by Takeuchi–Iskovskikh–Prokhorov. In what follows, $X = X_{12}$ is a prime Fano threefold of genus 7, Z will be either a point, or a conic in X.

In order to identify the flopping curves we will need several facts about lines and conics in X.

Lemma 6.1. *The family of lines on X is parametrized by an equidimensional reduced curve $\tau(X)$. A line $\ell \subset X$ is a regular (resp. singular) point of $\tau(X)$ if and only if $\mathscr{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ (resp. $\mathscr{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$). For generic X, $\tau(X)$ is a nonsingular curve.*

Every line on X meets only finitely many other lines. The union of lines $R(X) = \bigcup_{v \in \tau(X)} \ell_v$ is a surface from the linear system $|\mathcal{O}_X(7)|$, and a generic line meets 8 other lines on X.

Proof. All the assertions, except for the nonsingularity of $\tau(X)$ for generic X, follow from [17], Proposition 1, [18], Proposition 2.1, (iv), Proposition 2.4, (iii), Lemma 2.6, and Theorem 3.1, (vii). Though the statement of the latter Theorem does not assert that the number of lines meeting the given one is 8, the proof gives this value (p. 808).

Let us prove the nonsingularity of $\tau(X)$ for generic X. By [32], 6.12, the family of lines $\tau(\Sigma)$ on the spinor tenfold is a nonsingular irreducible 15-dimensional variety which can be identified with the family of planes $\mathbb{P}^2$ contained in the 8-dimensional quadric Q^8. The identification is done as follows: the line $\ell_{\mathbb{P}^2}^{\pm} \subset \Sigma^{\pm}$ corresponding to a plane $\mathbb{P}^2 \subset Q^8$ is the pencil $\{w \in \Sigma^{\pm} \mid \mathbb{P}_w^4 \supset \mathbb{P}^2\}$. The family of lines contained in the section of Σ by 7 generic hyperplanes is the scheme of zeros of a generic section of the rank-14 vector bundle $\mathscr{V} = 7\mathcal{Q}_{\tau(\Sigma)}$ which is the 7-uple direct sum of the universal rank-2 vector bundle. As $\mathcal{Q}_{\tau(\Sigma)} = \mathcal{Q}_{G(2,16)}|_{\tau(\Sigma)}$ and $\mathcal{Q}_{G(2,16)}$ is generated by global sections, the same is true for $\mathscr{V}$, so the generic section defines a nonsingular curve. $\square$

Lemma 6.2. *Let X be a generic Fano threefold of degree 12 with Picard number 1. Then the family of conics on X is parametrized by a generically reduced irreducible scheme $\mathscr{F}(X)$ of dimension 2 (the "Fano surface" of X). A generic conic C is nonsingular and $\mathscr{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. The number of conics passing through a generic point of X is finite.*

Proof. By Proposition 4.2.5 and Theorem 4.5.10 of [19] the scheme $\mathscr{F}(X)$ is generically reduced and equidimensional of dimension 2, and the number of conics passing through a generic point of X is finite. Let us prove that $\mathscr{F}(X)$ is indeed irreducible.

It is obvious that the family of Fano threefold linear sections of the spinor tenfold Σ that contain a fixed conic in Σ, if nonempty, is irreducible and of constant dimension 42. Hence the incidence variety I parametrizing the pairs (C, X), where $C \subset X$ is a conic and X is a Fano threefold linear section of Σ, is irreducible. The fiber of its projection $I \to G(9, 16)$ to the second factor over X is the Fano surface $\mathscr{F}(X)$. By a monodromy argument, to prove the irreducibility of $\mathscr{F}(X)$, it suffices to present a distinguished component of $\mathscr{F}(X)$ for generic X. To this end, we will construct a natural map $\mathrm{Sym}^2 \Gamma \to \mathscr{F}(X)$, where Γ is the dual of X, and the wanted distinguished component is the one containing the image of $\mathrm{Sym}^2 \Gamma$.

Identify Σ with Σ^+ and let $u, v \in \Gamma \subset \Sigma^-$ be two generic points and $p_{u,v} = \mathbb{P}^4_u \cap \mathbb{P}^4_v$. Then $\dim p_{u,v} \in \{0, 2, 4\}$. The dimension is definitely not equal to 4. By the description of lines in Σ given in Lemma 6.1, it is not 2, because otherwise $\ell^-_{p_{u,v}}$ would be a bisecant line of Γ contained in Σ^-, but this would contradict the fact that Γ is a linear section of Σ^-. So $p_{u,v}$ is a point. The family of all the $\mathbb{P}^4$'s in Q^8 in each one of the two components $\Sigma^\pm$ passing through a given point p is a 6-dimensional quadric $Q_p^{6\pm} \in \Sigma^\pm$ which can be identified with an orbit of $\mathrm{Spin}(8) \subset \mathrm{Spin}(10)$. We can complete $u = u_1$, $v = u_2$ to a family of 7 points $u_1, \ldots, u_7 \in \Gamma$ in such a way that X is the intersection of Σ^+ with 7 hyperplanes $\mathbb{P}^{14}_{u_i}$, $i = 1, \ldots, 7$. Denote H_{u_i} the hyperplane sections $\Sigma^+ \cap \mathbb{P}^{14}_{u_i}$.

It is easy to see that $Q^{6+}_{p_{u,v}} \subset H_u \cap H_v$. Indeed, if $\mathbb{P}^4_+ \in Q^{6+}_{p_{u,v}}$, then $u \in \mathbb{P}^4_+ \cap \mathbb{P}^4_+$, hence $\mathbb{P}^4_u \cap \mathbb{P}^4_+ \neq \varnothing$ and, by (4), $\mathbb{P}^4_+ \in H_u$. Similarly, $\mathbb{P}^4_+ \in H_v$. Hence $Q^{6+}_{p_{u,v}} \cap H_{u_3} \cap \cdots \cap H_{u_7} \subset X$ is a conic.

We have constructed a rational map $f : \Gamma^{(2)} \to \mathscr{F}(X)$, and this ends the proof. $\square$

We will use the symbol $\mathscr{C}^g_d[k]_Z$ to denote the family of all the connected curves of genus g, degree d, meeting k times a given subvariety Z. More precisely, let $Z \subset X$ be a nonsingular curve (resp. a point). Then $\mathscr{C}^g_d[k]_Z$ is the closure in the Chow variety of X of the family of reduced connected curves C of degree d such that length $(\mathscr{O}_X/(\mathscr{I}_C + \mathscr{I}_Z)) = k$ (resp. $\mathrm{mult}_Z C = k$) and $p_a(\tilde{C}) = g$, where $\tilde{C}$ is the proper transform of C in the blowup of Z in X.

Birational isomorphisms $\Phi_p : X_{12} \to Y_5$. These are birational isomorphisms from a given variety $X = X_{12}$ to a Del Pezzo threefold $Y = Y_5$ of degree 5 parametrized by a sufficiently general point p in X. The threefold Y_5 is defined as a smooth section of the Grassmannian $G(2, 5) \subset \mathbb{P}^9$ by a subspace $\mathbb{P}^6$. It is a Fano threefold of index two, that is $-K_Y$ is twice the class of hyperplane section of Y.

Theorem 6.3. *Let $X = X_{12} \subset \mathbb{P}^8$ be a smooth anticanonically embedded Fano 3-fold of index 1 and of degree 12, and let $p \in X$ be a sufficiently general point of X. "Sufficiently general" here means that it does not lie on a line in X and satisfies the Condition (4.1)(**) from [19]. Then the following assertions hold*:

(a) *The non-complete linear system $|\mathcal{O}_X(3-7p)|$ defines a birational isomorphism $\Phi_p = \varphi : X \to Y$, where $Y = Y_5$ is the Fano 3-fold of index 2 and of degree 5.*

(b) *The one-dimensional family $\mathscr{C}_7^0[3]_p$ of curves $C \subset X$ of degree 7, of genus 0 and such that $\mathrm{mult}_p(C) \geqslant 3$ sweeps out the unique effective divisor $M = M_p$ in the linear system $|\mathcal{O}_X(5-12p)|$.*

(c) *The birational map $\varphi : X \to Y$ can be represented as a product $\varphi = \tau \circ \kappa \circ \sigma^{-1}$, where $\sigma : X' \to X$ is the blowup of $p \in X$, $\kappa : X \to X^+$ is a flop over the double projection X'' of X from p, and $\tau : X^+ \to Y$ is a blow-down of the proper image $M^+ \subset X^+$ of M onto a canonical curve $\Gamma \subset Y \subset \mathbf{P}^6$ of degree 12 (and of genus 7). The extremal curves $C^+ \subset X^+$ contracted by τ are the strict transforms of the curves $C \in \mathscr{C}_7^0[3]_p$.*

(d) *The birational map $\psi = \varphi^{-1} : Y \to X$ is defined by the non-complete linear system $|\mathcal{O}_Y(12-7\Gamma)|$.*

(e) *The two-dimensional family $\mathscr{C}_7^0[12]_\Gamma$ of curves $C \subset Y$ of degree 7 and genus 0 meeting Γ 12 times sweeps out the unique effective divisor $N = N_\Gamma$ in the linear system $|\mathcal{O}_Y(5-3\Gamma)|$. The proper transform $N' \subset X'$ of N coincides with the exceptional divisor $\sigma^{-1}(p) \cong \mathbf{P}^2$ of σ.*

The extremal curves $C' \subset X'$ contracted by σ are the strict transforms of the curves $C \in \mathscr{C}_7^0[12]_\Gamma$.

(f) *Let $q_1, \ldots, q_e \subset X$ be all the conics on X which pass through p, $q_1', \ldots, q_e' \subset X'$ the proper transforms of $q_1, \ldots, q_e$ on X'. Let $l_1, \ldots, l_{e'}$ be all the bisecant lines to Γ which lie on Y, and $l_1^+, \ldots, l_e^+$ the proper transforms of $l_1, \ldots, l_{e'}$ on X^+. Then for generic $p \in X$ we have $e = e' = 24$, $q_1', \ldots, q_e' \subset M'$ (resp. $l_1^+, \ldots, l_e^+ \subset N^+$) are all the flopping curves of κ (resp. κ^{-1}), and κ transforms q_i' into l_i^+ for an appropriate ordering of the l_i ($i = 1, \ldots, e$). The map π_{2p}' (resp. π_Γ^+) is a small birational morphism contracting the curves q_i' (resp. l_i^+) into isolated ordinary double points. It is given by the linear system $|K_{X'}| = \sigma^* \mathcal{O}_X(1) - 2p$ (resp. $|K_{X^+}| = \tau^* \mathcal{O}_Y(2) - \Gamma|$) and its image $\overline{X} = \overline{Y}$ is a quartic threefold in $\mathbf{P}^4$ with only $e = 24$ singular points $\pi_{2p}'(q_i')$ (resp. $\pi_\Gamma^+(l_i^+)$), $i = 1, \ldots, e$.*

The statement of the theorem is illustrated by Diagram 1.

Proof. The Parts (a)–(e) are essentially an expanded version of Theorem 4.5.8, (ii) from Iskovskikh–Prokhorov [19]. According to loc. cit., the linear system of the divisor N is nothing but the proper transform of $|-5K_{X^+} - 2M^+|$, which coincides with our $|\mathcal{O}_X(5-12p)|$. Other values of α, β in various linear systems $|\mathcal{O}_X(\alpha - \beta p)|$ or $|\mathcal{O}_Y(\alpha - \beta\Gamma)|$ are extracted from the preliminary material of Chapter 4 in [19] (Lemmas 4.1.1–4.1.9).

Now turn to the proof of (f). By [19], Lemma 4.1.1, the small morphisms π_{2p}', π_Γ^+ are given by n-th multiples of the anticanonical linear system for some $n > 0$. By ibid., the proof of Proposition 4.5.10, $n = 1$ and the image is a quartic threefold. By Lemma 4.5.1, the image F of any flopping curve of κ in X belongs to one of the three types: a conic passing through p, a quartic with a double point at p, or a sextic with a triple point at p. But the second and the third types are impossible, because such curves F span $\mathbf{P}^3$, and hence F has a 3-dimensional family of 4-secant planes. It

$$q'_1, \ldots, q'_e \qquad\qquad\qquad\qquad l^+_1, \ldots, l^+_e$$

$$\cap \qquad\qquad\qquad\qquad\qquad\qquad \cap$$

$$M', N' \subset X' \xrightarrow[\text{flop } \kappa^{-1}]{\text{flop } \kappa} X^+ \supset M^+, N^+$$

maps: σ, π'_{2p}, π^+_Γ, τ

$$\varphi = |\mathcal{O}_X(3 - 7p)|$$

$$p \in X \xrightleftharpoons[\psi = |\mathcal{O}_Y(12 - 7\Gamma)|]{} Y \supset \Gamma$$

$$\cup \qquad\qquad\qquad\qquad\qquad\qquad \cup$$

$$q_1, \ldots, q_e \qquad\qquad\qquad\qquad l_1, \ldots, l_e$$

maps: π_{2p}, π_Γ

$$\overline{X} = \overline{Y}$$

extremal curves $\mathcal{C}^0_7[3]_p$ extremal curves $\mathcal{C}^0_7[12]_\Gamma$

extremal divisor   extremal divisor

$$M = M_p = |\mathcal{O}_X(5 - 12p)| \qquad\qquad N = N_\Gamma = |\mathcal{O}_Y(5 - 3\Gamma)|$$
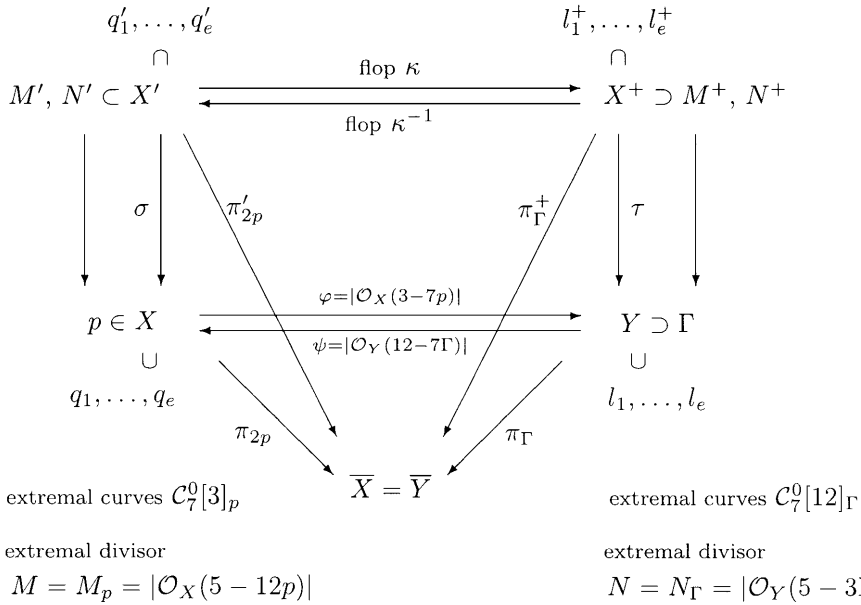
Diagram 1. The birational isomorphism between X_{12} and the Del Pezzo 3-fold $Y_5 = G(2,5) \cap \mathbf{P}^6$ defined by a point $p \in X_{12}$.

defines a 3-dimensional family of conics in X by Lemma 3.3, which contradicts the fact that the family of conics on X is 2-dimensional. Hence the only flopping curves in X are the conics passing through p.

By Lemma 6.2, all the conics passing through p have normal bundle $\mathcal{N}_{q_i/X} \simeq \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$. They also have different tangent directions at p (see Remark 6.6 below), so the curves q'_i are disjoint. Hence $\mathcal{N}_{q'_i/X'} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ and κ consists in blowing up all the curves q_i and blowing down the obtained exceptional quadrics $\mathbf{P}^1 \times \mathbf{P}^1$ along the other ruling. For such a one-storey flop, its defect $(N')^3 - (N^+)^3$ is just the number e of flopping curves. It can be determined from the second Formula (2.6.4) of [33]; one obtains $e = 24$.

To identify the images of q'_i in Y, remark that the images q^+_i of q'_i under κ are contained in N^+ and satisfy $(q^+_i)^2_{N^+} = (q_i \cdot N^+)_X = -1$. As $N^+ \sim 5\tau^* H - 3M^+$, where H is the hyperplane section of Y, we obtain the Equation 5 $\deg \tau(q^+_i) - 3 \operatorname{length}(\tau(q^+_i) \cap \Gamma) = -1$. Further, by the general definition of a flop, all the flopping curves have zero intersection number with the canonical class, so $(q^+_i \cdot K_{X^+}) = 0$, which gives the Equation 2 $\deg \tau(q^+_i) - \operatorname{length}(\tau(q^+_i) \cap \Gamma) = 0$. Thus $\deg \tau(q^+_i) = 1$, $\operatorname{length}(\tau(q^+_i) \cap \Gamma) = 2$, that is, $\tau(q^+_i)$ is a bisecant line to Γ. This ends the proof of the Theorem. □

The first application of this theorem is the following lemma.

Lemma 6.4. *Let p be a generic point of X. Then all the curves $C \in \mathcal{C}^0_5[2]_p$, except*

for finitely many of them, are irreducible good quintics with only one double point (node or cusp) at p, $\dim \mathscr{C}_5^0[2]_p = 1$, and there is a rational dominant map of curves $\pi : \mathscr{C}_5^0[2]_p \to \Gamma_{12}^7$ of degree $\leqslant 3$, where Γ_{12}^7 is the genus-7 curve from the statement of Theorem 6.3.

Proof. The map Φ_p transforms the curves C from $\mathscr{C}_5^0[2]_p$ into lines meeting Γ one time, so all of them are rational curves with a node or a cusp, except for finitely many ones which are transforms of lines in Y_5 meeting the flopping curves. The structure of the family of lines on Y_5 is well known (see e.g. [13], [8]). There is only one variety Y_5 up to projective equivalence, and it possesses a $SL(2)$-action with three orbits: $C_6^0, S_{10} \backslash C_6^0$ and $Y_5 \backslash S_{10}$, where C_6^0 is a rational normal sextic curve and S_{10} its tangential scroll (the subscript is the degree). The family of lines on Y_5 is parametrized by $\mathbb{P}^2$ and through any point of the k-th orbit for $k = 1, 2, 3$ (in the above order) pass exactly k lines. Define π by assigning to each $C \in \mathscr{C}_5^0[2]_p$ the point of intersection of $\Phi_p(C)$ with Γ_{12}^7. $\qquad \square$

Birational isomorphisms Ψ_q with a three-dimensional quadric. This is the family of birational maps $\Psi_q : X_{12} \to Q_2$ to the three-dimensional quadric $Q_2 \subset \mathbb{P}^4$ parametrized by a sufficiently general conic $q \subset X$.

Theorem 6.5. *Let $X = X_{12} \subset \mathbb{P}^8$ be a smooth anticanonically embedded Fano 3-fold of index 1 and of degree 12, and let $q \subset X$ be a sufficiently general conic in X. "Sufficiently general" here means that it satisfies the Conditions (4.1)(*), (**) from [19]. Then the following assertions hold:*

(a) *The non-complete linear system $|\mathscr{O}_X(2 - 3q)|$ defines a birational isomorphism $\Psi_q = \varphi : X \to Q$, where $Q = Q_2$ is a smooth three-dimensional quadric.*

(b) *The one-dimensional family $\mathscr{C}_3^0[2]_q$ of twisted cubics $C \subset X$ meeting twice the conic q sweeps out the unique effective divisor $M = M_q$ in the non-complete linear system $|\mathscr{O}_X(5 - 8q)|$.*

(c) *The birational map $\varphi : X \to Y$ can be represented as a product $\varphi = \tau \circ \kappa \circ \sigma^{-1}$, where $\sigma : X' \to X$ is the blowup along $q \subset X$, $\kappa : X \to X^+$ is a flop over the projection X'' of X from q, and $\tau : X^+ \to Y$ is the blow-down of the proper image $M^+ \subset X^+$ of M onto a curve $\Gamma \subset Q$ of degree 10 and of genus 7.*

The extremal curves $C^+ \subset X^+$ contracted by τ are the strict transforms of the curves $C \in \mathscr{C}_3^0[2]_q$.

(d) *The birational map $\psi = \varphi^{-1} : Q \to X$ is defined by the linear system $|\mathscr{O}_Q(8 - 3\Gamma)|$.*

(e) *The one-dimensional family $\mathscr{C}_3^0[8]_\Gamma$ of twisted cubics on Q intersecting the curve Γ in a divisor of degree 8 sweeps out the unique effective divisor $N = N_\Gamma$ in the linear system $|\mathscr{O}_Q(5 - 2\Gamma)|$. The proper transform $N' \subset X'$ of N coincides with the exceptional divisor $\sigma^{-1}(q) \cong \mathbf{P}^1 \times \mathbf{P}^1$ of σ.*

The extremal curves $C' \subset X'$ contracted by σ are the strict transforms of the curves $C \in \mathscr{C}_3^0[8]_\Gamma$.

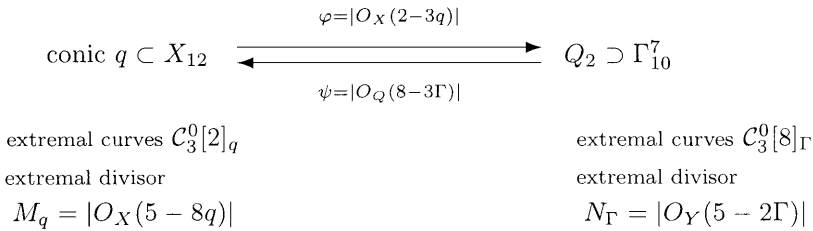The statement of the theorem is illustrated by Diagram 2.

$$\text{conic } q \subset X_{12} \quad \underset{\psi = |O_Q(8-3\Gamma)|}{\overset{\varphi = |O_X(2-3q)|}{\rightleftarrows}} \quad Q_2 \supset \Gamma_{10}^7$$

extremal curves $\mathcal{C}_3^0[2]_q$ extremal curves $\mathcal{C}_3^0[8]_\Gamma$

extremal divisor   extremal divisor

$$M_q = |O_X(5 - 8q)| \qquad\qquad\qquad N_\Gamma = |O_Y(5 - 2\Gamma)|$$

Diagram 2. The birational isomorphism between X_{12} and the quadric 3-fold $Q_2 \subset \mathbf{P}^4$ defined by a conic $q \subset X_{12}$.

Proof. See Iskovskikh–Prokhorov [19], Theorem 4.4.11, (iii).   □

Remark 6.6. According to [19], p. 66 (Remark) and Proposition 4.4.1, the Conditions (4.1)(∗)–(∗∗) will be satisfied for a smooth conic $q \subset X$ if:
   a) there is a finite number of lines on X which intersect q;
   b) there is a finite number of bisecant conics to q on X.
   It is easy to see that the property b) is automatically verified and in fact even a stronger condition is satisfied: there are no pairs of bisecant conics in X. Indeed, the union of two bisecant conics has a 3-dimensional family of 4-secant planes $\mathbf{P}^2$, each of which determines a conic contained in X by Lemma 3.3. This is absurd, because the family of conics in X is 2-dimensional. It seems very likely that the property a) is also true for any smooth conic on X.

   The second birational isomorphism also provides examples of good quintics in X:

Lemma 6.7. *Let q be a general conic in X. Then all but a finite number of curves of the form $q \cup C_3^0$ with $C_3^0 \in \mathscr{C}_3^0[2]_q$ are good quintics. The family of good quintics of this form is birationally parametrized by an open set of the genus-7 curve Γ_{10}^7 from Theorem 6.5.*

Proof. By Part (c) of the above theorem, the strict transforms of curves from $\mathscr{C}_3^0[2]_q$ with respect to σ are the extremal curves of the contraction τ, that is, they form the family of $\mathbf{P}^1$'s in M^+ contracted to the points of Γ.   □

Three curves of genus 7 associated to X. So far, we have associated to the Fano three-fold $X = X_{12}$ three curves of genus 7: the dual linear section $\check{X}$, the curve Γ_{12}^7 from Theorem 6.3, and Γ_{10}^7, the one from Theorem 6.5. Denote them by different symbols, say, Γ, Γ' and Γ'' respectively. We will see right now that Γ', Γ'' are isomorphic; the isomorphism with Γ will be deduced later from the birational isomorphisms of Γ, Γ'' to the same component of the moduli space $M_X = M_X(2; 1, 5)$ of vector bundles on X.

Lemma 6.8. *The intermediate Jacobian $J(X)$ is isomorphic to the Jacobians of both curves Γ', Γ'', so by Torelli's Theorem, $\Gamma' \simeq \Gamma''$.*

Proof. By [5], the intermediate Jacobian of the blowup $\tilde{Y}$ of a smooth curve D in a smooth projective 3-fold Y is isomorphic, as a p. p. a. v., to the product $J(Y) \times J(D)$. As $J(Q_2) = J(Y_5) = 0$ and the birational isomorphisms Φ_p, Ψ_q are obtained by blowing up or down only one nonrational curve and some sets of rational curves, we have $J(X) \simeq J(\Gamma') \simeq J(\Gamma'')$, so the assertion follows by Torelli's Theorem.   □

The good quintics from Lemmas 6.4 and 6.7 define some vector bundles of the type considered in Section 4. We will see that they fill an irreducible component of the moduli space.

Lemma 6.9. *Let $M_X = M_X(2; 1, 5)$ be as in Section 4. Let $p \in X$ be a generic point and $q \subset X$ a generic conic. Let $M_X(p)$, resp. $M_X(q)$ be the closure in M_X of the locus of vector bundles $\mathscr{E}_C$ on X obtained by Serre's construction from generic good quintics $C \in \mathscr{C}_5^0[2]_p$, resp. $C = q \cup C_3^0$ with $C_3^0 \in \mathscr{C}_3^0[2]_q$. Then $M_X(p), M_X(q)$ coincide with a 1-dimensional irreducible component M_X^1 of M_X which does not depend on p, q. Moreover, a generic vector bundle from M_X^1 can be obtained by Serre's construction from a smooth elliptic quintic, and $(M_X^1)_{\mathrm{red}}$ is birational to Γ''.*

Proof. Denote by M_X^1 the closure of the union of all the $M_X(p)$, where p runs over the open subset $U \in X$ for which the conditions of Theorem 6.3 are satisfied. A generic $\mathscr{E}_0$ in any component of M_X^1 is obtained by Serre's construction from a good quintic, so by Lemma 4.2, the family of curves $(s)_0$ of zeros of all the sections s of $\mathscr{E}_0$ is $\mathbb{P}^4 = \mathbb{P}H^0(\mathscr{E}_0)$. By Proposition 5.9, the generic curve $(s)_0$ is nonsingular. As the locus of points of $\mathbb{P}^4$ corresponding to singular quintics is non-empty, it is a hypersurface in $\mathbb{P}^4$, hence it is 3-dimensional. The family $\bigcup_{p \in U} \mathscr{C}_5^0[2]_p$ being 4-dimensional, the subfamily of these quintics contained in $\mathbb{P}H^0(\mathscr{E}_0)$ for generic $\mathscr{E}_0$ is 3-dimensional; if we fix now such a generic $\mathscr{E}_0$, then the subfamily of quintics with only one node at p is nonempty for any $p \in U$ and forms a curve for generic p. In particular, $\mathscr{E}_0 \in M_X(p)$ for any $p \in U$. Hence $M_X^1 = M_X(p)$ for generic p. Further, M_X^1 is a union of components of M_X. Indeed, the same argument as above shows that any $\mathscr{E} \in M_X$ close to a generic $\mathscr{E}_0 \in M_X^1$ also belongs to M_X^1. Finally, by Lemma 6.4, M_X^1 is birational to the family of lines in Y_5 meeting Γ', so it dominates Γ' with degree at most 3.

Let us show now that $M_X(q)$ for a generic conic q is contained in M_X^1. Indeed, the structure of Ψ_q as described in Theorem 6.5 implies easily that the two points $p, p' \in q$ of the intersection $C_3^0 \cap q$ are movable on q, when C_3^0 runs over $\mathscr{C}_3^0[2]_q$, so for generic q and generic C_3^0, any of them is a generic point of X. Thus the curve $C = C_3^0 \cup q$ for generic $C_3^0 \in \mathscr{C}_3^0[2]_q$ is an element of $\mathscr{C}_5^0[2]_p$, where p satisfies the hypotheses of Theorem 6.3. Let $\mathscr{E} \in M_X(q)$ be the vector bundle associated to C. By Lemma 4.2, the family of curves $(s)_0$ of zeros of all the sections s of $\mathscr{E}$ is $\mathbb{P}^4 = \mathbb{P}H^0(\mathscr{E})$. The family of curves of type $C_3^0 \cup q$ ($C_3^0 \in \mathscr{C}_3^0[2]_q$) being 3-dimensional, C deforms inside $\mathbb{P}H^0(\mathscr{E})$ into a curve with only one node, hence $\mathscr{E}$ is in the closure of the union of $M_X(x)$, $x \in X$ generic. Hence $M_X(q) \subset M_X^1$ is an irreducible component, birational to Γ''.

It remains to see that M_X^1 is irreducible. Let $\mathscr{E} \in M_X(p)$ for generic p. Then $\mathbb{P}H^0(\mathscr{E})$ contains a 3-dimensional family of curves D with two nodes x, x' for which

each one of the points x, x' runs over a dense subset of X. As $p_a(D) = 1$, the generic D is reducible. There is no line through a generic point of X, so the only possible type of decomposition of D is a conic plus a cubic. The cubic is rational and nonsingular, because X is an intersection of quadrics. Hence $\mathscr{E} \in M_X(q)$ and $M_X^1 = M_X(q)$ for some q.   □

As Γ', Γ'' are curves of the same genus, Γ'' does not have a dominant rational map to Γ' of degree > 1, so we can deduce from the proof of the lemma the following corollary:

Corollary 6.10. *Let $p \in X$ be a generic point. Let $\mathscr{E}_C$ be the vector bundle on X obtained by Serre's construction from a generic good quintic $C \in \mathscr{C}_5^0[2]_p$ and $\ell = \Phi_p(C)$ the line in Y_5 meeting Γ' at a point u. Then there are at most three distinct lines ℓ_i in Y_5 meeting Γ' at u ($\ell_1 = \ell$), and the vector bundles obtained by Serre's construction from the curves $C_i := \Phi_p^{-1}(\ell_i)$ are isomorphic to $\mathscr{E}$. Equivalently, the family $\mathbb{P}^4 = \mathbb{P}H^0(\mathscr{E})$ of curves $(s)_0$ of zeros of all the sections s of $\mathscr{E}$ contains at most three distinct curves $C_i \in \mathscr{C}_5^0[2]_p$ and their images ℓ_i under Φ_p are precisely all the lines in Y_5 meeting Γ' at u.*

In Section 5 we constructed a map $\rho_X : \Gamma \to M_X$ and proved that $\rho_X(\Gamma) = M_X^0$ is an irreducible component of M_X.

Lemma 6.11. $M_X^0 = M_X^1$.

Proof. Let $\mathscr{E} \in M_X^0$ be generic. The sections of $\mathscr{E}$ embed X into the Grassmannian $G = G(2, 5)$ of lines in $\mathbb{P}^4$ and $\mathscr{E} \simeq \mathscr{Q}_G|_X$. The sections of $\mathscr{Q}_G$ correspond to linear forms on $\mathbb{P}^4$ and the restriction to X sends them isomorphically onto the sections of $\mathscr{E}$. Let $q \subset X$ be any smooth conic. Considered as a family of lines in $\mathbb{P}^4$, q is either a pencil of generators of a quadric in $\mathbb{P}^3 \subset \mathbb{P}^4$, or the curve of tangents to a conic in $\mathbb{P}^2$. In the first case, there is a unique, up to proportionality, section of $\mathscr{Q}_G$ vanishing on q, the one defined by the $\mathbb{P}^3$ which is the linear span of the quadric. In the second case there is a pencil of such sections in $\mathbb{P}H^0(\mathscr{Q}_G)$. The span of the lines parametrized by a reducible conic q is $\mathbb{P}^3$, hence the same is true for a generic q. So, for generic q, the projective space $\mathbb{P}H^0(\mathscr{E})$ of zero loci of sections of $\mathscr{E}$ contains a unique reducible curve D having q as one of its components, and D does not contain other conics or lines. Hence D is a union of the conic q and of a rational twisted cubic. Hence $\mathscr{E} \in M_X(q) = M_X^1$.   □

M_X^0 being birational to Γ, we obtain:

Corollary 6.12. $\Gamma \simeq \Gamma' \simeq \Gamma''$.

7   Smoothness and irreducibility of M_X

Let $X = X_{12}$ be a generic Fano threefold of degree 12 with Picard number 1, and Γ its dual curve of genus 7. Let M_X, ρ be defined as in Section 5 and let M_X^0 be the component of M_X that is the image of ρ. Propositions 7.1 and 7.8 prove Theorem 1.1.

Proposition 7.1. *For generic X, M_X^0 is in the smooth locus of M_X.*

Proof. Let $\mathscr{E} \in M_X^0$. By Lemma 4.4, to prove the smoothness of M_X at $\mathscr{E}$ it suffices to find a quintic C in $\mathbb{P}H^0(X, \mathscr{E})$ such that $\mathscr{N}_{C/X} \not\simeq \mathscr{O}_C \oplus \mathscr{O}_C(1)$. Lemmas 7.2, 7.3 and 7.4 prove, for generic $p \in X$, the existence of such a quintic having a node at p. $\square$

Lemma 7.2. *Let $\mathscr{E} \in M_X^0$, $p \in X$ a generic point. Let $Z_p = \{C_i\}_{1 \leqslant i \leqslant n}$ $(n = 1, 2 \text{ or } 3)$ be the set of curves with singularity at p which are zero loci of sections of $\mathscr{E}$, $q_1, \ldots, q_{24}$ all the conics on X passing through p and $\tilde{C}_i, q_j'$ the proper transforms of the above curves on the blowup of p. Then $\tilde{C}_i \cap q_j' = \varnothing$ for all $i = 1, \ldots, n$, $j = 1, \ldots, 24$.*

Proof. The curves C_i are rational quintics with only one node. The assertion of the lemma is not satisfied for $\mathscr{E}, p$ if and only if there is a pair i, j, for which C_i meets q_j in such a way that $\text{length}(C_i \cap q_j) \geqslant 3$.

We want to prove that for generic $p \in X$, there is no conic q through p meeting any one of the curves C_i in such a way that $\text{length}(C_i \cap q) \geqslant 3$. Assume the contrary. By Theorem 6.3, the number of conics through a generic point p is 24. Hence, if we assume that at most seven conics q_j can meet a curve C_i with length $\geqslant 3$, then such conics form a proper irreducible component of $\mathscr{F}(X)$. This contradicts the irreducibility of the family of conics on X (Lemma 6.2). But it is immediate to see that two different conics q_1, q_2 passing through p cannot meet C_i with length of intersection $\geqslant 3$. Indeed, if we assume that such conics q_1, q_2 exist, then $C_i \cup q_j$ $(j = 1, 2)$ are two distinct reducible fibers of the ruled surface M (notation from Theorem 6.3). The flop transforms the components q_j into secant lines l_j to the curve $\Gamma \subset Y_5$, and the proper transform of C_i is contracted to a point of Γ. This is absurd, because two distinct fibers of M are contracted to two distinct points of Γ. Hence each C_i meets at most one conic q_j and we are done. $\square$

Lemma 7.3. *Let $\mathscr{E} \in M_X^0$, $U \subset X$ the open set of points $p \in X$ for which the birational map Φ_p of Theorem 6.3 exists, and $p \in U$. Assume that there are at least two nodal rational quintic curves C_1, C_2 with only one node at p as singularity which are zero loci of sections of $\mathscr{E}$. Assume also that the proper transform $\tilde{C}_i$ of C_i $(i = 1, 2)$ on the blowup X' of X at p does not meet any of the flopping curves of κ introduced in Theorem 6.3. Then the normal bundle of C_i in X is indecomposable and M_X is nonsingular at the point representing the vector bundle $\mathscr{E}$.*

Proof. Assume the contrary, that is, $\mathscr{N}_{C/X} \simeq \mathscr{O}_C \oplus \mathscr{O}_C(1)$, where C is one of the two curves C_i, say, $C = C_1$. Then $\tilde{C} \simeq \mathbb{P}^1$ and $v^* \mathscr{N}_{C/X} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(5)$, where $v : \tilde{C} \to C$ is the normalization map; $v = \sigma|_{\tilde{C}}$. Let $\mathscr{N}_{\tilde{C}}$ denote the normal bundle to C as a parametrized curve in X: $\mathscr{N}_{\tilde{C}} = \text{coker}(dv : \mathscr{T}_{\tilde{C}} \to v^* \mathscr{N}_{C/X})$. Then $v^* \mathscr{N}_{C/X}$ is obtained from $\mathscr{N}_{\tilde{C}}$ by a positive elementary transformation described in [9]: $v^* \mathscr{N}_{C/X} = \text{elm}_T^+ \mathscr{N}_{\tilde{C}}$, where T is the set of two points in $\mathbb{P}(\mathscr{N}_{\tilde{C}})$ corresponding to the tangent directions of the two branches at p. Hence $\mathscr{N}_{\tilde{C}} \simeq \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(5)$, $\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(4)$ or $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(3)$. Hence $\mathscr{N}_{\tilde{C}/X'} \simeq \mathscr{N}_{\tilde{C}}(-T) \simeq \mathscr{O}_{\mathbb{P}^1}(-4) \oplus \mathscr{O}_{\mathbb{P}^1}(3)$, $\mathscr{O}_{\mathbb{P}^1}(-3) \oplus \mathscr{O}_{\mathbb{P}^1}(2)$ or $\mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$.

In the notation of Theorem 6.3, the image ℓ of $\tilde{C}$ under $\tau \circ \kappa$ is a line in Y meeting Γ transversely at one point $z = z(p, \mathscr{E})$, so we have for the normal bundles $\mathscr{N}_{\ell/Y} = \mathrm{elm}^+_{1\ \mathrm{point}}\ \mathscr{N}_{\tilde{C}/X'}$ Hence the only possible case is $\mathscr{N}_{\ell/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. According to the description of lines on Y, if $z \notin C_6^0$ (notation from the proof of Lemma 6.4), then there is another line on Y passing through z with normal bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ giving the other nodal quintic C_2 and, in replacing C by C_2 in the above argument, we come to the conclusion that $\mathscr{N}_{C_2/X} \not\simeq \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(1)$. This ends the proof. □

We have seen that $\mathscr{E}$ may be a singular point of M_X only if for generic $p \in X$ C is the unique nodal quintic from $\mathbb{P}H^0(X, \mathscr{E})$, or, equivalently, $z(p, \mathscr{E}) \in C_6^0$ and $\ell = T_z C_6^0$ is the unique line on Y passing through $z(p, \mathscr{E})$.

Lemma 7.4. *Let $\mathscr{E} \in M_X^0$, $p \in X$ a generic point. Then there are at least two different curves with a singular point at p which are zero loci of sections of $\mathscr{E}$.*

Proof. The assertion that we are to prove now is equivalent to the fact that for generic X and generic $p \in X$ the curve $\Gamma \subset Y = Y_5$ defined by Theorem 6.3 does not meet the closed orbit C_6^0 of SL(2). We prove this fact in three steps in Lemmas 7.5, 7.6 and Proposition 7.7: we prove, first, that the family of canonical curves Γ in Y is irreducible. Second, we present a canonical curve $\Gamma \in Y$ which does not meet C_6^0, and this implies that the generic Γ does not meet C_6^0 either. Third, we show that a generic canonical curve $\Gamma \in Y$ is obtained as in Theorem 6.3 via the map Φ_p applied to some X and some $p \in X$.

Lemma 7.5. *Let Y be a Del Pezzo threefold of degree 5, that is, a nonsingular threefold linear section $\mathbb{P}^6 \cap G(2, 5)$ of the Grassmannian $G(2, 5)$ in $\mathbb{P}^9$. Then the family of canonical curves Γ of genus 7 in Y is irreducible of dimension 24.*

Proof. When saying that Γ is *a canonical curve in Y*, we mean that the restriction of the class of the hyperplane section of Y to Γ is canonical and that the dimension of the linear span of Γ in $\mathbb{P}^9$ coincides with that of the canonical linear system of Γ, so $\langle \Gamma \rangle = \langle Y \rangle = \mathbb{P}^6$. By [18], Y is unique up to projective equivalence and $h^0(\mathbb{P}^6, \mathscr{I}_Y(2)) = 5$. Hence the hypotheses of Lemmas 5.1, 5.5 are verified for the embedding $\Gamma \hookrightarrow Y \hookrightarrow G(2, 5)$ and this embedding is determined uniquely, up to projective equivalence, by a point $p \in X$, where X is the dual Fano threefold of Γ. Let I be the variety parametrizing all the canonical embeddings $\Gamma \hookrightarrow G(2, 5)$ such that Γ is a generic genus-7 curve and $\langle \Gamma \rangle \cap G(2, 5)$ is a (smooth) Del Pezzo threefold. Then I is irreducible of dimension

$$\dim I = \dim \mathfrak{M}_7 + \dim X + \dim \mathrm{PGL}(5) = 18 + 3 + 24 = 45.$$

Let pr be the natural projection from I to the open subset $\mathscr{U}$ of $G(7, 10)$ parametrizing linear sections of $G(2, 6)$ which are (smooth) Del Pezzo threefolds. Then for generic $u \in \mathscr{U}$, the fiber $\mathrm{pr}^{-1}(u)$ is equidimensional and $\dim \mathrm{pr}^{-1}(u) = \dim I - \dim G(7, 10) = 24$. By the monodromy argument, to prove the assertion of the

lemma, it suffices to present a distinguished component of the family of canonical curves in Y. Then it has to be the unique one.

There is such an obvious component: the one containing all the curves Γ obtained from the generic pairs (p, X) via the map Φ_p of Theorem 6.3. This ends the proof of the lemma. □

Lemma 7.6. *Let $F \subset Y$ be any curve. Then there exists a reducible smoothable canonical curve $C_0 = E_1 \cup E_2 \cup q$ such that the following conditions are verified*:

(a) E_1, E_2 *are elliptic quintics meeting each other transversely at 3 distinct points*;

(b) q *is a conic meeting each one of the curves E_i transversely at 2 points*;

(c) $C_0 \cap F = \varnothing$.

Proof. Choose a conic q not meeting F; let $\mathbb{P}_q^2 = \langle q \rangle$. Choose a generic 3-secant $\mathbb{P}^2$ to Y meeting $\mathbb{P}_q^2$. Let $z = \mathbb{P}^2 \cap \mathbb{P}_q^2$, $\{p_1, p_2, p_3\} = \mathbb{P}_q^2 \cap Y$. Let l_1, l_2 be two generic lines in $\mathbb{P}_q^2$ passing through z, and $\{p_4^i, p_5^i\} = l_i \cap q$. Let $\mathbb{P}_i^3 = \langle \mathbb{P}^2, l_i \rangle$ and $\mathbb{P}_i^4$ generic 4-spaces containing $\mathbb{P}_i^3$ $(i = 1, 2)$. Then $E_i = \mathbb{P}_i^4 \cap Y$ are elliptic curves such that $\mathbb{P}_i^3 \cap E_i = \{p_1, p_2, p_3, p_4^i, p_5^i\}$, that is, the intersection of E_i with the residual curve $E_{3-i} \cup q$ defines the divisor from the linear system $\mathcal{O}(1)$. For q, the degree of the intersection with $E_1 \cup E_2$ is 4, which is $\deg K_q + \deg \mathcal{O}(1)|_q$. Hence $C_0 = E_1 \cup E_2 \cup q$ is embedded by a subsystem of the canonical system, and since $\langle C_0 \rangle = \mathbb{P}^6$, it is a canonical curve. It is obvious that for a generic choice of the above $\mathbb{P}_i^4$, we have $\mathbb{P}_i^4 \cap F = \varnothing$.

It remains to see that C_0 is smoothable, but this follows immediately by the technique of [9] and from the known normal bundles to the components of our curve: $\mathcal{N}_{E_i} = 2\mathcal{O}(1)|_{E_i}$ and $\mathcal{N}_q = 2\mathcal{O}_{\mathbb{P}^1}(1)$. □

Proposition 7.7. *Let Y be a Del Pezzo threefold of degree 5 and Γ a generic canonical curve of genus 7 in Y. Let X be the dual Fano threefold of Γ. Then there exists a point $p \in X$ such that the embedding $\Gamma \hookrightarrow Y$ is obtained via the birational map Φ_p of Theorem 6.3.*

Proof. We have to invert the construction of the map Φ_p for generic $\Gamma \subset Y$. Let $\Gamma_0 \hookrightarrow Y$ be an embedding corresponding to some pair (p_0, X_0). Then the anticanonical linear system $|-K_{\tilde{Y}_0}|$ on the blowup $\tilde{Y}_0$ of Γ_0 in Y defines a small contraction π_0 of 24 $(-1, -1)$-lines which are bisecant to Γ_0 whose image is a quartic threefold in $\mathbb{P}^4$ with 24 ordinary double points, and the flop over this small contraction gives rise to an exceptional $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$ which can be contracted to a nonsingular point. It is easy to verify that this situation is stable under small deformations. Let $\Gamma_t \hookrightarrow Y$ be a deformation of $\Gamma_0 \hookrightarrow Y$ with parameter t. By the semicontinuity of $h^i(\tilde{Y}_t, \mathcal{O}(-K_{\tilde{Y}_t})) = 0$ and because the h^i vanish for $t = 0$ and $i > 0$, the linear system $|-K_{\tilde{Y}_t}|$ defines also a map π_t to $\mathbb{P}^4$ for small t. The base-point-free condition is open, so π_t is a morphism. By the semicontinuity of the degree of a morphism, π_t is birational. The stability of the $(-1, -1)$-curves under deformations is

well-known. See for example [21], where the stability of nonsingular subvarieties with negative normal bundle is proved, and the type of the normal bundle is preserved in our case, because a vector bundle of type $\mathcal{O}(k) \oplus \mathcal{O}(k)$ has no nontrivial infinitesimal deformations on $\mathbb{P}^1$. So, $\tilde{Y}_t$ contains 24 $(-1, -1)$-curves, which have to be contracted by the anticanonical system. Their intersection indices with the exceptional divisor of $\tilde{Y}_t \to Y$ being constant, they descend to bisecant lines of Γ_t in Y.

Now we can blow up all the 24 $(-1, -1)$-curves and blow down the obtained quadrics $\mathbb{P}^1 \times \mathbb{P}^1$ along the second ruling simultaneously in all the varieties $\tilde{Y}_t$ for small t. We will obtain the family of nonsingular projective threefolds X'_t. The fiber X_0 possesses an exceptional $\mathbb{P}^2$. Again by [21], this $\mathbb{P}^2$ deforms in a unique way to a compact submanifold in the neighboring fibers; it is again $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$, because both $\mathbb{P}^2$ and the line bundle $\mathcal{O}(k)$ on it have no nontrivial infinitesimal deformations. Thus there is a relative contraction $X'_t \to X_t$ blowing down $\mathbb{P}^2$ to a nonsingular point in each fiber. The fiber X_0 is a Fano threefold of genus 7 with Picard group Z, and these properties are stable under small deformations. $\square$

Proposition 7.8. *For a generic Fano threefold X of degree* 12 *with Picard group $\mathbb{Z}$, the moduli space M_X is irreducible.*

Proof. Let $\mathcal{E} \in M_X$. By Proposition 5.9, the sections of $\mathcal{E}$ define a regular map from $j : X \to G(2, 5)$. Restricting to a generic hyperplane section H and using Proposition 5.7, we see that $i = j|_H$ embeds H linearly into $G(2, 5)$ and satisfies the hypotheses of Lemma 3.6. Hence the linear subspace U defined in the lemma is a maximal isotropic subspace in $V = H^0(\mathbb{P}^7, \mathcal{I}_H(2)) = H^0(\mathbb{P}^8, \mathcal{I}_X(2))$. Hence, by Lemma 3.5, j is a linear embedding associated to a point $w \in \check{X}$, and hence $\mathcal{E} \in M_X^0$. $\square$

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