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A counterexample to a conjecture on linear systems on $\mathbb{P}^3$

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Abstract. In his paper [1] Ciliberto proposes a conjecture in order to characterize special linear systems of $\mathbb{P}^n$ through multiple base points. In this note we give a counterexample to this conjecture by showing that there is a substantial difference between the speciality of linear systems on $\mathbb{P}^2$ and those of $\mathbb{P}^3$.

Let us take the projective space $\mathbb{P}^n$ and let us consider the linear system of hypersurfaces of degree d having some points of fixed multiplicity. The virtual dimension of such systems is the dimension of the space of degree d polynomials minus the conditions imposed by the multiple points and the expected dimension is the maximum between the virtual one and -1. The systems whose dimension is bigger than the expected one are called *special systems*.

There exists a conjecture due to Hirschowitz (see [5]), characterizing special linear systems on $\mathbb{P}^2$, which has been proved in some special cases [2], [3], [7], [6].

Concerning linear systems on $\mathbb{P}^n$, in [1] Ciliberto gives a conjecture based on the classification of special linear systems through double points. In this note we describe a linear system on $\mathbb{P}^3$ that we found in a list of special systems generated with the help of Singular [4] and which turns out to be a counterexample to that conjecture.

The paper is organized as follows: in Section 1 we fix some notation and state Ciliberto's conjecture, while Section 2 is devoted to the counterexample. In Section 3 we try to explain speciality of some systems by the Riemann–Roch formula, and we conclude the note with an appendix containing some computations.

1 Preliminaries

We start by fixing some notation.

Notation 1.1. Let us denote by $\mathbb{L}_n(d, m_1^{a_1}, \ldots, m_r^{a_r})$ the linear system of hypersurfaces of $\mathbb{P}^n$ of degree d, passing through a_i points with multiplicity m_i, for $i = 1, \ldots, r$. Let $\mathscr{I}_Z$ be the ideal of the zero-dimensional scheme of multiple points. We denote by $\mathscr{L}_n(d, m_1^{a_1}, \ldots, m_r^{a_r})$ the sheaf $\mathscr{O}_{\mathbb{P}^n}(d) \otimes \mathscr{I}_Z$. Given the system $\mathbb{L} = \mathbb{L}_n(d, m_1^{a_1}, \ldots, m_r^{a_r})$, its *virtual dimension* is

$$v(\mathbb{L}) = \binom{d+n}{n} - \sum_{i=1}^{r} a_i \binom{m_i+n-1}{n} - 1,$$

and the *expected dimension* is

$$e(\mathbb{L}) = \max(v(\mathbb{L}), -1).$$

A linear system will be called *special* if its expected dimension is strictly smaller than the effective one.

Remark 1.2. Throughout the paper, if no confusion arises, we will use sometimes the same letter to denote a linear system and the general divisor in the system.

We recall the following definition, see [1].

Definition 1.3. Let X be a smooth, projective variety of dimension n, let C be a smooth, irreducible curve on X and let $\mathcal{N}_{C|X}$ be the normal bundle of C in X. We will say that C is a *negative curve* if there is a line bundle $\mathcal{N}$ of negative degree and a surjective map $\mathcal{N}_{C|X} \to \mathcal{N}$. The curve C is called a (-1)-*curve* of size a, with $1 \leqslant a \leqslant n-1$, on X if $C \cong \mathbb{P}^1$ and $\mathcal{N}_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus a} \oplus \mathcal{N}$, where $\mathcal{N}$ has no summands of negative degree.

The main conjecture stated in [1] is the following.

Conjecture 1.4. *Let X be the blow-up of $\mathbb{P}^n$ at general points $p_1, \dots, p_r$ and let $\mathbb{L} = \mathbb{L}_n(d, m_1, \dots, m_r)$ be a linear system with multiple base points at $p_1, \dots, p_r$. Then*:

(i) *the only negative curves on X are (-1)-curves*;

(ii) $\mathbb{L}$ *is special if and only if there is a (-1)-curve C on X corresponding to a curve Γ on $\mathbb{P}^n$ containing $p_1, \dots, p_r$ such that the general member $D \in \mathbb{L}$ is singular along Γ*;

(iii) *if $\mathbb{L}$ is special, let B be the component of the base locus of $\mathbb{L}$ containing Γ according to Bertini's theorem. Then the codimension of B in $\mathbb{P}^n$ is equal to the size of C and B appears multiply in the base locus scheme of $\mathbb{L}$.*

In this note we give a counterexample to points (ii) and (iii) of this conjecture.

2 Counterexample

Let us consider the linear system of surfaces of degree nine with one point of degree six and eight points of degree four in $\mathbb{P}^3$, i.e. the system $\mathbb{L} = \mathbb{L}_3(9, 6, 4^8)$. In this section we are going to study this system, showing in particular that it is special but its general member is not singular along a rational curve. If we denote by $Q = \mathbb{L}_3(2, 1, 1^8)$ the quadric through the nine simple points, we have the following:

Claim 1. $\mathbb{L}_3(9, 6, 4^8) = Q + \mathbb{L}_3(7, 5, 3^8)$.

If we denote by H_1, H_2 two generators of Pic(Q), considering the restriction $\mathbb{L}_{|Q}$ we get the system of curves in $|9H_1 + 9H_2|$, with one point of multiplicity 6 and eight points of multiplicity 4. We denote for short this system by $|9H_1 + 9H_2| - 6p_0 - \sum 4p_i$.

Looking at Appendix 4.1, we can see that $|9H_1 + 9H_2| - 6p_0 - \sum 4p_i$ corresponds to the planar system $\mathbb{L}_2(12, 3^2, 4^8)$. This last system cannot be (-1)-special (see the Appendix 4.2) and $v(\mathbb{L}_2(12, 3^2, 4^8)) = -2$. Therefore, by [7] we may conclude that it is empty.

In particular, also $\mathbb{L}_{|Q} = \varnothing$, and hence $\mathbb{L}$ must contain Q as a fixed component. By subtracting Q from $\mathbb{L}$ we get $\mathbb{L}_3(7, 5, 3^8)$, which proves our claim.

This means that the free part of $\mathbb{L}$ is contained in $\mathbb{L}_3(7, 5, 3^8)$ which has virtual dimension 4. So $\mathbb{L}$ is a special system.

In order to show that $\mathbb{L}$ gives a counterexample to Conjecture 1.4 we are now going to prove that the general member of $\mathbb{L}$ is singular only along the curve C, intersection of Q and $\mathbb{L}_3(7, 5, 3^8)$, and that C does not contain rational components.

We can consider C as the restriction $\mathbb{L}_3(7, 5, 3^8)_{|Q}$. This is equal to $|7H_1 + 7H_2| - 5p_0 - \sum 3p_i$ on the quadric Q, which corresponds to $\mathbb{L}_2(9, 2^2, 3^8)$ on $\mathbb{P}^2$. This system is not special of dimension 0 and it does not contain rational components (see Appendix 4.3).

Clearly the curve C is contained in $\mathbb{L}_{\text{sing}}$ (i.e. the singular locus of $\mathbb{L}$). We are going to show that in fact $C = \mathbb{L}_{\text{sing}}$. First of all, let us denote by $\mathbb{L}_3(7, 5, 3^8, 1_Q)$ the subsystem of $\mathbb{L}_3(7, 5, 3^8)$ obtained by imposing one general simple point on the quadric. Since $\mathbb{L}_3(7, 5, 3^8, 1_Q)_{|Q} = \varnothing$, Q is a fixed component of this system and the residual part is given by $\mathbb{L}_3(5, 4, 2^8)$. Now $\mathbb{L}_3(5, 4, 2^8)_{|Q}$ is the system $|5H_1 + 5H_2| - 4p_0 - \sum 2p_i$ which corresponds to the non-special system $\mathbb{L}_2(6, 1^2, 2^8)$, of dimension 1. Therefore, imposing two general simple points on Q and restricting we get that the system $\mathbb{L}_3(5, 4, 2^8, 1_Q^2)_{|Q}$ is empty, which implies that $\mathbb{L}_3(5, 4, 2^8, 1_Q^2)$ has Q as a fixed component. The residual system $\mathbb{L}_3(3, 3, 1^8)$ is non-special of dimension 1 (because each surface of this system is a cone over a plane cubic through eight fixed points). This implies that the effective dimension of $\mathbb{L}_3(5, 4, 2^8)$ cannot be greater than 3. Therefore it must be 3 since the virtual dimension is 3. By the same argument one shows that the effective dimension of $\mathbb{L}_3(7, 5, 3^8)$ is 4.

Observe that $\text{Bs}(\mathbb{L}_3(7, 5, 3^8)) \subseteq \text{Bs}(2Q + \mathbb{L}_3(3, 3, 1^8))$ since $2Q + \mathbb{L}_3(3, 3, 1^8) \subseteq \mathbb{L}_3(7, 5, 3^8)$. So $\text{Bs}(\mathbb{L}_3(7, 5, 3^8))$ could have only Q as fixed component, but this is not the case since $\dim \mathbb{L}_3(7, 5, 3^8) = \dim \mathbb{L}_3(5, 4, 2^8) + 1$. The only curves that may belong to $\text{Bs}(\mathbb{L}_3(7, 5, 3^8))$ are the genus 2 curve $C = \mathbb{L}_3(7, 5, 3^8)_{|Q}$ and the nine lines of $\text{Bs}(\mathbb{L}_3(3, 3, 1^8))$ trough the vertex of the cone and each one of the nine base points of the pencil of plane cubics.

We can then conclude that the singular locus $\mathbb{L}_{\text{sing}}$ consists only of the curve C, since the subsystem $3Q + \mathbb{L}_3(3, 3, 1^8)$ is not singular along the nine fixed lines.

3 Speciality and Riemann–Roch theorem

Let Z be a zero-dimensional scheme of $\mathbb{P}^3$ and $\mathscr{I}_Z$ be its ideal sheaf. We put $\mathscr{L} = \mathscr{O}_{\mathbb{P}^3}(d) \otimes \mathscr{I}_Z$ and consider the exact sequence

$$\begin{aligned} 0 &\to H^0(\mathscr{L}) \to H^0(\mathscr{O}_{\mathbb{P}^3}(d)) \to H^0(\mathscr{O}_Z) \to H^1(\mathscr{L}) \to H^1(\mathscr{O}_{\mathbb{P}^3}(d)) \to H^1(\mathscr{O}_Z) \\ &\to H^2(\mathscr{L}) \to H^2(\mathscr{O}_{\mathbb{P}^3}(d)) \to H^2(\mathscr{O}_Z) \to H^3(\mathscr{L}) \to H^3(\mathscr{O}_{\mathbb{P}^3}(d)) \to H^3(\mathscr{O}_Z), \end{aligned}$$

obtained tensoring by $\mathscr{O}_{\mathbb{P}^3}(d)$ the sequence defining Z and taking cohomology. From this sequence we obtain that $h^i(\mathscr{L}) = h^i(\mathscr{O}_{\mathbb{P}^3}(d)) = 0$ for $i = 2, 3$ since $h^i(\mathscr{O}_Z) = 0$ for $i = 1, 2, 3$. We also obtain that the virtual dimension of $\mathbb{L}$, $h^0(\mathscr{O}_{\mathbb{P}^3}(d)) - h^0(\mathscr{O}_z) - 1$ is equal to $h^0(\mathscr{L}) - h^1(\mathscr{L}) - 1$ and hence to $\chi(\mathscr{L}) - 1$.

If $Z = \sum m_i p_i$ is a scheme of fat points, then on the blow-up $X \xrightarrow{\pi} \mathbb{P}^3$ along these points we may consider the divisor $\tilde{L} = \pi^* \mathscr{O}_{\mathbb{P}^3}(d) - \sum m_i E_i$ and the associated sheaf $\tilde{\mathscr{L}} = \mathscr{O}_X(\tilde{L})$. Since $h^i(X, \tilde{\mathscr{L}}) = h^i(\mathbb{P}^3, \mathscr{L})$, the virtual dimension of $\mathbb{L}$ is equal to $\chi(\tilde{\mathscr{L}}) - 1$. By the Riemann–Roch formula (see [4]) for a divisor $\tilde{L}$ on the threefold X,

$$\chi(\tilde{L}) = \frac{\tilde{L}(\tilde{L} - K_X)(2\tilde{L} - K_X) + c_2(X) \cdot \tilde{L}}{12} + \chi(\mathscr{O}_X),$$

we obtain the following formula for the virtual dimension of $\mathbb{L}$:

$$v(\mathbb{L}) = \frac{\tilde{L}(\tilde{L} - K_X)(2\tilde{L} - K_X) + c_2(X) \cdot \tilde{L}}{12}$$

since $\chi(\mathscr{O}_X) = 1$.

If the linear system $\mathbb{L}$ can be written as $F + \mathbb{M}$, where F is the fixed divisor and $\mathbb{M}$ is a free part, then on X we have $|\tilde{L}| = \tilde{F} + |\tilde{M}|$. Therefore the above formula says that

$$v(\tilde{L}) = v(\tilde{F}) + v(\tilde{M}) + \frac{\tilde{F}\tilde{M}(\tilde{L} - K_X)}{2}.$$

Let us suppose that the residual system $\mathbb{M}$ is non-special. The system $\mathbb{L}$ has the same effective dimension as $\mathbb{M}$, while their virtual dimensions differ by $v(\tilde{F}) + \tilde{F}\tilde{M}(\tilde{L} - K_X)/2$. Therefore we can conclude that $\mathbb{L}$ is special if $v(\tilde{F}) + \tilde{F}\tilde{M}(\tilde{L} - K_X)/2$ is smaller than zero.

Example 3.1. For instance, let us consider the system $\mathbb{L} := \mathbb{L}_3(4, 2^9)$. It is special because its virtual dimension is -2 while it is not empty since it is equal to $2Q$, where Q is the quadric through the nine simple points. In this case $F = 2Q$ and $\mathbb{M} = \mathbb{C}$, so $v(F) = -2$ and $\tilde{F}\tilde{M}(\tilde{L} - K_X)/2 = 0$.

Example 3.2. Let us consider now the example we described in the previous section, i.e. the system $\mathbb{L}_3(9,6,4^8)$. We have seen that it can be written as $Q+\mathbb{M}$, where Q is the quadric through the nine points, while $\mathbb{M}=\mathbb{L}_3(7,5,3^8)$ is the residual free part. The Chow ring $A^*(X)$ (where X is the blow-up of $\mathbb{P}^3$ along the nine simple points) is generated by $\langle H, E_0, E_1, \dots, E_8\rangle$, where H is the pull-back of the hyperplane divisor of $\mathbb{P}^3$ and the E_i's are the exceptional divisors. The second Chow group $A^2(X)$ is generated by $\langle h, e_0, e_1, \dots, e_8\rangle$, where $h=H^2$ is the pull-back of a line, while $e_i=-E_i^2$ is the class of a line inside E_i, for $i=0,1,\dots,8$. Clearly $H\cdot E_i=E_i\cdot E_j=0$ for $i\neq j$. With this notation we can write:

$$\mathbb{L}=\left|9H-6E_0-\sum 4E_i\right|$$

$$\mathbb{M}=\left|7H-5E_0-\sum 3E_i\right|$$

$$Q=2H-E_0-\sum E_i$$

$$K_X=-4H+2E_0+\sum 2E_i.$$

Therefore $Q\cdot\mathbb{M}=14h-5e_0-\sum 3e_i$, $\mathscr{L}-K_X=13H-8E_0-\sum 6E_i$ and hence $Q\tilde{M}(\tilde{L}-K_X)/2=-1$ (while $v(Q)=0$), which implies the speciality of $\mathbb{L}$.

4 Appendix

4.1 Linear systems on a quadric. In order to study linear systems on a quadric Q it may be helpful to transform them into planar systems by means of a birational transformation $Q\to\mathbb{P}^2$ obtained by blowing up a point and contracting the strict transforms of the two lines through it. Such a transformation gives rise to a $1:1$ correspondence between linear systems with one multiple point on the quadric and linear systems with two multiple points on $\mathbb{P}^2$.

In fact, let us consider a linear system $|aH_1+bH_2|-mp$ (i.e. a system of curves of kind (a,b) through one point p of multiplicity m). Blowing up at p, one obtains the complete system $|a\pi^*H_1+b\pi^*H_2-mE|$ which may be written as $|(a+b-m)\cdot(\pi^*H_1+\pi^*H_2-E)-(b-m)(\pi^*H_1-E)-(a-m)(\pi^*H_2-E)|$. Since the divisors π^*H_i-E $(i=1,2)$ are (-1)-curves, they may be contracted giving a linear system on $\mathbb{P}^2$ of degree $a+b-m$ through two points of multiplicity $b-m$ and $a-m$ and hence

$$|aH_1+bH_2|-mp\to\mathbb{L}_2(a+b-m,b-m,a-m).$$

4.2 (−1)-curves. In order to study the speciality of the systems $\mathbb{L}_2(12,3^2,4^8)$, $\mathbb{L}_2(9,2^2,3^8)$ and $\mathbb{L}_2(6,1^2,2^8)$, we need to produce a complete list of all the (-1)-curves of $\mathbb{P}^2$ of kind $\mathbb{L}_2(d,m_1,m_2,m_3,\dots,m_{10})$ which may have an intersection less than -1 with some of these systems. Clearly it is enough to consider the system $\mathbb{L}_2(12,3^2,4^8)$, whose degree and multiplicities are the biggest. From the condition of being contained twice in this system we deduce the following inequalities: $d \leqslant 6$, $0 \leqslant m_1, m_2 \leqslant 1$ and $0 \leqslant m_3, \dots, m_{10} \leqslant 2$. Moreover let us see that $m_3 = \cdots = m_{10} = m$. Otherwise the system would contain twice the compound (-1)-curve given by the union of all the simple (-1)-curves obtained by permuting the points $p_3, \dots, p_{10}$. In this case the multiplicities of the compound curve at these points would be too big. An explicit calculation shows that the only (-1)-curve of the form $\mathbb{L}_2(d, m_1, m_2, m^8)$ satisfying the preceding conditions is $\mathbb{L}_2(1, 1, 1, 0^8)$, but this has non-negative intersection with any of these systems.

4.3 $\mathbb{L}_2(9, 2^2, 3^8)$ does not contain rational components. Let S be the blow up of $\mathbb{P}^2$ along the ten points and let C be the strict transform of the curve given by $\mathbb{L} = \mathbb{L}_2(9, 2^2, 3^8)$. Suppose that there exists an irreducible rational component C_1 of C. Observe that $v(C_1) = 0$ since the system $|C_1|$ has dimension 0 and it is non-special by [7]. Therefore, from $g(C_1) = v(C_1) = 0$, we get that C_1 is a (-1)-curve.

We are going to see that if this is the case, then $C \cdot C_1 = -1$. Let us take the following exact sequence:

$$0 \to \mathcal{O}_S(C - C_1) \to \mathcal{O}_S(C) \to \mathcal{O}_{\mathbb{P}^1}(C \cdot C_1) \to 0.$$

By the subsection above, $h^1(\mathcal{O}_S(C)) = 0$. Let us see that also $h^1(\mathcal{O}_S(C - C_1)) = 0$. Otherwise the system $|C - C_1|$ would be special and in particular, by [7] there would exist a (-1)-curve C_2 such that $C_2 \cdot (C - C_1) \leqslant -2$. Since $\mathbb{L}$ is non-special, $C \cdot C_2 \geqslant -1$ and hence $C_1 \cdot C_2 \geqslant 1$. This implies that $|C_1 + C_2|$ has dimension at least 1, which is impossible since $C_1 + C_2$ is contained in the fixed locus of $\mathbb{L}$. Since $h^0(\mathcal{O}_S(C - C_1)) = h^0(\mathcal{O}_S(C)) = 1$, the cohomology of the preceding sequence gives $h^0(\mathcal{O}_{\mathbb{P}^1}(C \cdot C_1)) = h^1(\mathcal{O}_{\mathbb{P}^1}(C \cdot C_1)) = 0$, which means that $C \cdot C_1 = -1$ as claimed before.

Arguing as in the previous subsection, we get $|C_1| = \mathbb{L}_2(d, m_1, m_2, m^8)$ with $d \leqslant 9$, $0 \leqslant m_1, m_2 \leqslant 2$, $0 \leqslant m \leqslant 3$. An easy computation shows that the only (-1)-curve of this form is $\mathbb{L}_2(1, 1, 1, 0^8)$ and in this case $C \cdot C_1 = 5$.

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