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MAXIMAL COMPLETION OF A PSEUDO MV–ALGEBRA

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Abstract. In the present paper we investigate the relations between maximal completions of lattice ordered groups and maximal completions of pseudo MV-algebras.

1. Introduction

The notion of a pseudo MV-algebra (denoted also as a noncommutative MV-algebra) has been introduced independently by Georgescu and Iorgulescu [8], [9] and by Rachůnek [16]. It is defined to be an algebraic structure $\mathcal{A} = (A; \oplus, ^-, ^\sim, 0, 1)$ of type (2,1,1,0,0) satisfying certain axioms (cf. Section 2 below).

Dvurečenskij [5] proved that each pseudo MV-algebra $\mathcal{A}$ can be constructed by means of a lattice ordered group G with a strong unit u. This generalized the well-known result concerning MV-algebras (cf., e.g., the monograph Cignoli, D'Ottaviano and Mundici [2]).

In the method of Dvurečenskij a partial binary operation $+$ on the underlying set A of the pseudo MV-algebra $\mathcal{A}$ was applied in an essential way.

The maximal completion $M(\mathcal{A})$ of an MV-algebra $\mathcal{A}$ has been investigated in [12].

In the present paper we use Dvurečenskij's result for dealing with the maximal completion of a pseudo MV-algebra.

We prove that if $\mathcal{A}$ is constructed by means of a lattice ordered group G with a strong unit u (i.e., if $\mathcal{A} = \Gamma(G, u)$, in the notation of [5]), then the maximal completion $M(\mathcal{A})$ of $\mathcal{A}$ can be constructed by means of the maximal completion of the lattice ordered group G.

If the pseudo MV-algebra $\mathcal{A}$ is archimedean, then according to [5], $\mathcal{A}$ is an MV-algebra. In this case $M(\mathcal{A})$ coincides with the Dedekind completion $D(\mathcal{A})$ of $\mathcal{A}$.

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2. Preliminaries

For pseudo MV-algebras we apply the terminology and notation from [8], [9]; cf. also Dvurečenskij and Pulmannová [4]. For the sake of completeness, we recall the basic definition.

2.1. Definition. Assume that A is a nonempty set. Let $\mathcal{A} = (A; \oplus, ^-, ^\sim, 0, 1,)$ be an algebraic structure of type (2,1,1,0,0). For $x, y \in A$ we put

$$y \odot x = (x^- \oplus y^-)^\sim .$$

The structure $\mathcal{A}$ is a *pseudo MV-algebra* if the following axioms (A1) - (A8) are satisfied for each $x, y, z \in A$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

Suppose that $\mathcal{A}$ is a pseudo MV-algebra. For $x, y \in A$ we put $x \leqq y$ if $x^- \oplus y = 1$. Then $(A; \leqq)$ turns out to be a distributive lattice with the least element 0 and with the greatest element 1. We denote $(A; \leqq) = \ell(\mathcal{A})$.

In [4], a partial binary operation $+$ on the set A has been defined as follows: for $x, y \in A$ the partial operation $x + y$ is defined if and only if $x \leqq y^-$; in this case $x + y = x \oplus y$.

For lattice ordered groups we apply the notation as in Conrad [3]. Let G be a lattice ordered group with a strong unit u. For $x, y \in G$ we put

$$x \oplus y = (x + y) \wedge y ,$$
$$x^- = u - x, \quad x^\sim = -x, +u , \quad 1 = u .$$

Further, let A be the interval $[0, u]$ of G. Then the structure $(A; \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra which will be denoted by $\Gamma(G, u)$. (Cf. [9].)

2.2. Theorem (Cf. [5]). *For each pseudo MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$.*

Let us also remark that for $x, y \in [0, u]$ the above mentiond partial operation $+$ coincides with the operation $+$ as defined in G. Also, the partial order $\leqq$ on A is that induced from the partial order in G.

3. Maximal completion of a lattice ordered group

The maximal completion G_D of a lattice ordered group G has been constructed by Everett [6] in the case of an abelian lattice ordered group G and by Černák [1] in the general case; cf. also the monograph Fuchs [7], Chapter V, §10.

We will apply G_D by constructing the maximal completion $\mathcal{A}_D$ of a pseudo MV-algebra $\mathcal{A}$ (cf. Section 5 below).

In the present section we recall the corresponding definitions concerning G_D. For establishing a deeper insight into the structure of G_D, we present also some steps of its construction; analogous steps will be used by constructing $\mathcal{A}_D$.

We remark that in [6] and [7] a different terminology and notation have been applied.

Let G be a lattice ordered group. For each subset X of G we denote by X^u (and X^ℓ) the set of all upper bounds (or lower bounds, respectively) of the set X.

We denote by $d(G)$ the system of all sets

$$X^\# = X^{u\ell},$$

where X runs over the system of all nonempty upper bounded subsets of G. If $X = \{x\}$, then we write $x^\#$ instead of $\{x\}^\#$.

The system $d(G)$ is partially ordered by the set-theoretical inclusion. Then $d(G)$ is a conditionally complete lattice. Namely, if $\emptyset \neq \{C_i\}_{i\in I} \subseteq d(G)$, then

$$\bigwedge_{i\in I} C_i = \bigcap_{i\in I} C_i$$

holds in $d(G)$. Moreover, if $C_0 \in d(G)$ and $C_0 \geqq C_i$ for each $i \in I$, then the relation

$$\bigvee_{i\in I} C_i = (\bigcup_{i\in I} C_i)^\#$$

is valid in $d(G)$.

Let $\ell(G)$ be the underlying lattice of G. The mapping

$$x \to x^\# \quad (x \in G)$$

is an embedding of $\ell(G)$ into $d(G)$ preserving all suprema and infima existing in $\ell(G)$.

For $X, Y \subseteq G$ we put, as usual,

$$X + Y = \{x + y : x \in X, y \in Y\}, \quad -X = \{-x : x \in X\}.$$

Further, for $X, Y \in d(G)$ we set

$$X +_0 Y = (X + Y)^\#.$$

3.1. Lemma (Cf. [7])**.** *The system $d(G)$ with the relation $\leqq$ and with the operation $+_0$ is a partially ordered semigroup. If $x, y \in G$, then*

$$(x + y)^\# = x^\# +_0 y^\#.$$

Further, $0^\#$ is the neutral element of this semigroup.

3.2. Lemma (Cf. [7]). *The set G_D of all elements of $d(G)$ which have an inverse is a group.*

Let P be the positive cone of G.

3.3. Lemma (Cf. [7]). *Let $C \in d(G)$. Then C has a left inverse in $d(G)$ if and only if some of the following equivalent conditions is satisfied:*

(i) $((-C)^{\ell} + C)^u = P$.
(ii) *If $x \in G$ and $C^u + x \subseteq C^u$, then $x \in P$.*

An analogous result holds for the right inverses in $d(G)$.

Let $C \in d(G)$. Consider the following conditions for C:

(ii_1) The relation

$$\bigwedge_{c \in C, b \in C^u} (-c + b) = 0$$

is valid in G.

(ii_2) The relation

$$\bigwedge_{c \in C, b \in C^u} (b - c) = 0$$

is valid in G.

3.4. Proposition. *Let $C \in d(G)$. Then C has an inverse in $d(G)$ if and only if the conditions* (ii_1) *and* (ii_2) *are satisfied in G.*

Proof. This is a consequence of 1.3 in [1]. □

3.5. Lemma. *Let $C \in G_D$. Then $C \vee 0^{\#} \in G_D$.*

Proof. Denote

$$C^u = B, \quad C \vee 0^{\#} = C_1\,, \quad C_1^u = B_1\,.$$

We have

$$(C \cup 0^{\#})^u = (C \cup \{0\})^u\,,$$

whence

$$C_1 = (C \cup \{0\}^{\#})^{u\ell} = (C \cup \{0\})^{u\ell}\,,$$
$$B_1 = (C \cup \{0\})^{u\ell u} = (C \cup \{0\})^u\,.$$

Thus

$$B_1 = \{b_1 \in G : b_1 \geqq c \vee 0 \quad \text{for each } c \in C\}\,.$$

Since $C_1 \in d(G)$ we get $C_1 = B_1^{\ell}$ and hence C_1 is the set of all $c_1 \in G$ such that

(+) $c_1 \leqq b_1$ whenever $b_1 \in G$ and $b_1 \geqq c \vee 0$ for each $c \in C$.

Let $c_0 \in C$ and $b_0 \in B$. Then

$$b_0 \vee 0 \geqq c \vee 0 \quad \text{for each } c \in C\,,$$

whence $b_0 \vee 0 \in B_1$.

Let $b_1 \in G$, $b_1 \geqq c \vee 0$ for each $c \in C$. Then, in particular, $c_0 \vee 0 \leqq b_1$. Thus in view of (+) we get $c_0 \vee 0 \in C_1$.

Denote $x = b_0 \wedge (c_0 \vee 0)$. Since

$$b_0 \vee (c_0 \vee 0) = b_0 \vee 0\,,$$

we get

$$0 \leqq (b_0 \vee 0) - (c_0 \vee 0) = b_0 - x \leqq b_0 - c_0\,.$$

Therefore from the relation $C \in G_D$ and from 3.4 we obtain

$$\bigwedge_{b_0 \in B, c_0 \in C} ((b_0 \vee 0) - (c_0 \vee 0)) = 0\,.$$

We verified that $b_0 \vee 0 \in B_1$, $c_0 \vee 0 \in C_1$. Hence we conclude that

$$\bigwedge_{b_1 \in B_1, c_1 \in C_1} (b_1 - c_1) = 0\,.$$

Similarly we obtain

$$\bigwedge_{b_1 \in B_1, c_1 \in C_1} (-c_1 + b_1) = 0\,.$$

By applying 3.4 again we get $C_1 \in G_D$, completing the proof. □

The system G_D is partially ordered by the set-theoretical inclusion (i.e., by the relation of partial order induced from $d(G)$. Then G_D is a partially ordered group.

Since $C, 0^{\#}$ and $C \vee 0^{\#} = C_1$ belong to G_D, we conclude

3.6. Lemma. *$C \vee 0^{\#}$ is the least upper bound of the set $\{C, 0^{\#}\}$ in G_D.*

3.7. Lemma (Cf. [3]). *Let H be a partially ordered group such that for each $h \in H$, the element* $\sup\{0, h\}$ *exists in H. Then H is a lattice ordered group.*

From 3.6 and 3.7 we obtain

3.8. Proposition (Cf. Černák [1]). *G_D is a lattice ordered group.*

Since the mapping $x \to x^{\#}$ $(x \in G)$ is an embedding of G into $d(G)$ preserving the partial order, in view of 3.1 and of the fact that $x^{\#} \in G_D$ for each $x \in G$, we conclude that the mentioned mapping is an embedding of the lattice ordered group G into the lattice ordered group G_D.

We often identify the element x of G with the element $x^{\#}$ of G_D. Then G turns out to be an ℓ-subgroup of G_D.

We call G_D the maximal completion of G (the terms maximal Dedekind completion or Dedekind completion have also been used in the literature). We use the term 'Dedekind completion' for G_D in the case when G is archimedean. It is well-known that in such case we have $G_D = d(G)$; otherwise, $G_D \neq d(G)$.

4. Further results on G_D and $d(G)$

Assume that G, $d(G)$ and G_D are as above. In this section we denote the suprema and infima in $d(G)$ (or in G_D) by the symbols $\vee^1$, $\wedge^1$ (and by $\vee^2$, $\wedge^2$, respectively).

If $X \in d(G)$ and if X has an inverse element in $d(G)$, then this element will be denoted by $-_0 X$. In view of the definition of G_D, this element is also the inverse of X in G_D.

4.1. Lemma (Cf. [7])**.** *Let $A, B, C \in d(G)$. Then*

$$(A \vee^1 B) +_0 C = (A +_0 C) \vee^1 (B +_0 C)\,,$$
$$C +_0 (A \vee^1 B) = (C +_0 A) \vee^1 (C +_0 B)\,.$$

4.2. Lemma. *The lattice G_D is a sublattice of the lattice $d(G)$.*

Proof. Let $X, Y \in G_D$. Since G_D is a lattice ordered group, we have

$$(-_0 X) +_0 (X \vee^2 Y) = 0^{\#} \vee^2 (-_0 X +_0 Y).$$

Put $-_0 X +_0 Y = Z$. According to 3.6,

$$0^{\#} \vee^2 Z = 0^{\#} \vee^1 Z. \tag{1}$$

Further, 4.1 yields

$$(-_0 X) +_0 (X \vee^1 Y) = 0^{\#} \vee^1 (-_0 X +_0 Y).$$

By applying (1) we obtain $X \vee^1 Y = X \vee^2 Y$.

If $X \in d(G)$ and $y \in G$, then, obviously,

$$y \in X \Leftrightarrow y^{\#} \leqq X. \tag{$*$}$$

Further, in view of Section 3 we have

$$X \wedge^1 Y = X \cap Y.$$

Put $X \wedge^2 Y = Z$. Let $g \in G$. By $(*)$,

$$g \in Z \Leftrightarrow g^{\#} \leqq Z.$$

Further, $g^{\#} \leqq Z$ if and only if $g^{\#} \leqq X$ and $g^{\#} \leqq Y$; by using $(*)$ again we get that this is satisfied if and only if $g \in X$ and $g \in Y$. Hence $Z = X \cap Y$ and therefore $X \wedge^1 Y = X \wedge^2 Y$. □

4.3. Lemma. *Let $\{X_i\}_{i\in I}$ be a nonempty system of elements of G_D and $Z \in G_D$. Suppose that*

$$\bigvee_{i\in I}^{2} X_i = Z.$$

Then $\bigvee_{i\in I}^{1} X_i = Z$.

Proof. It suffices to verify that the element

$$T = \bigvee_{i\in I}^{1} X_i$$

of $d(G)$ belongs to G_D. By way of contradiction, assume that T fails to be an element of G_D. Then we must have

$$Z > T. \tag{1}$$

Denote $Y = \bigcup_{i\in I} X_i$. For $i \in I$ let

$$X_i = \{x_{ij}\}_{j\in J(i)}.$$

Then $T = Y^{\#}$.

Since T does not belong to G_D, in view of 3.4, some of the conditions (ii$_1$) or (ii$_2$) from Section 3 is not satisfied. Assume that (ii$_2$) is not valid (the case of (ii$_1$) is analogous).

Thus there exists $0 < a \in G$ such that for every $y \in T$ and every $p \in Y^u$ the relation

$$a + y \leqq p$$

is satisfied. In particular, the relation

$$a + x_{ij} \leqq p \tag{2}$$

is valid for each $i \in I$, $j \in J(i)$ and $p \in Y^u$.

In view of $(*)$, for each $C \in G_D$ we have

$$C = \bigvee_{c\in C}^{1} c^{\#}.$$

Thus we get

$$X_i = \bigvee_{j\in J(i)}^{2} x_{ij}^{\#},$$

whence

$$Z = \bigvee_{i\in I, j\in J(i)}^{2} x_{ij}^{\#},$$

$$Z < a^{\#} +_0 Z = \bigvee_{i\in I, j\in J(i)}^{2} (a^{\#} +_0 x_{ij}^{\#}) = \bigvee_{i\in I, j\in J(i)}^{2} (a + x_{ij})^{\#}.$$

From (2) we obtain

$$(a + x_{ij})^{\#} \leqq p^{\#}$$

for each $i \in I$, $j \in J(i)$ and $p \in Y^u$. Thus $Z \leqq p^{\#}$, hence $z \leqq p$ for each $z \in Z$. Therefore

$$Z \subseteq Y^{u\ell} = Y^{\#} = T.$$

In view of (1), we arrived at a contradiction. □

4.4. Corollary. *G_D is a conditionally complete sublattice of $d(G)$.*

Consider the mapping $\varphi(x) = x^{\#}$ of G into G_D.

4.5. Lemma. *The mapping φ preserves all suprema and infima existing in G.*

Proof. Let $\{x_i\}_{i\in I} \subseteq G$, $x \in G$ and suppose that $x = \bigvee_{i\in I} x_i$ in G. Then we have

$$x^{\#} = \bigvee_{i\in I}^{1} x_i^{\#}.$$

Since $x^{\#} \in G_D$, the above relation holds also in G_D, i.e.,

$$x^{\#} = \bigvee_{i\in I}^{2} x_i^{\#}.$$

Analogously we can verify the dual assertion. □

Now let us identify the element x of G with the element $x^{\#}$ of $d(G)$. We introduce the following definition.

4.6. Definition. Let G be as above and let H be a lattice ordered group such that

(a) G is an ℓ-subgroup of H;
(b) the underlying lattice $\ell(H)$ of H is a sublattice of the lattice $d(G)$;
(c) for $h_1, h_2 \in H$ we have $h_1 + h_2 = h_1 +_0 h_2$.

Then H is said to be a c-extension of G.

Let $\mathcal{C}(G)$ be the system of all c-extensions of G. This system is partially ordered by the set-theoretical inclusion.

From the definition of G_D and from 4.5 we obtain

4.7. Proposition. *G_D is the greatest element of the system $\mathcal{C}(G)$.*

For each $C \in G_D$ there exists $g \in G$ such that $C \leqq g^{\#}$. From this we conclude

4.8. Lemma. *Assume that G has a strong unit u. Then $u^{\#}$ is a strong unit of the lattice ordered group G_D.*

5. A construction for pseudo MV-algebras

In this section we define the notion of a maximal completion of a pseudo MV-algebra.

Let $\mathcal{A}$ be a pseudo MV-algebra with the underlying set A. The corresponding lattice is denoted by $\ell(\mathcal{A})$.

In view of 2.2, there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$.

For $T \subseteq A$ we denote by $T^{u(1)}$ (and $T^{\ell(1)}$) the set of all upper bounds (or the set of all lower bounds, respectively) of the set T in $\ell(\mathcal{A})$. We put

$$T^{u(1)\ell(1)} = T^{\#(1)}.$$

The system

$$d(A) = \{T^{\#(1)} : T \subseteq A\}$$

is partially ordered by the set-theoretical inclusion. Thus $d(A)$ is the Dedekind completion of the lattice $\ell(\mathcal{A})$. The mapping $x \to x^{\#(1)}$ is an embedding of $\ell(\mathcal{A})$ into $d(A)$ preserving all suprema and infima existing in $\ell(\mathcal{A})$.

Let A^* be the interval with the endpoints $0^\#$ and $u^\#$ of the lattice $d(G)$. For each $P \in A^*$ we put

$$\varphi_1(P) = P \cap A.$$

From Lemma 3.1 in [12] we obtain (since the proof of this lemmas remains valid in the non-commutative case as well)

5.1. Lemma. *φ_1 is an isomorphism of the lattice A^* onto the lattice $d(A)$.*

5.1.1. Lemma. *Let $\emptyset \neq C \subseteq G$. Assume that C is upper bounded. Then*

$$C^\# = \bigvee_{c \in C}^{1} c^\#.$$

Proof. Let $c \in C$. Hence $\{c\} \subseteq C$, thus $c^\# = \{c\}^\# \leqq C^\#$. Let $Z \in d(G)$ and $c^\# \leqq Z$ for each $c \in C$. Then $c \in Z$ for each $c \in C$, whence $C \subseteq Z$ and then $C^\# \leqq Z^\# = Z$. Thus the assertion of the lemma is valid. □

By a similar method as in the proof of 5.1.1 we can verify

5.1.2. Lemma. *Let $\emptyset \neq C \subseteq A$. Then the relation*

$$C^{\#(1)} = \bigvee_{c \in C} c^{\#(1)}$$

is valid in the lattice $d(A)$.

Let $g \in A$. Then

$$g^\# = \{g_1 \in G : g_1 \leqq g\}, \quad g^{\#(1)} = \{g_1 \in A : g_1 \leqq g\}.$$

Thus we have

5.1.3. Lemma. *For each $g \in A$, $\varphi_1(g^\#) = g^{\#(1)}$.*

Let T_1 and T_2 be elements of $d(A)$. We put

$$T_1 \oplus T_2 = \{t_1 \oplus t_2 : t_1 \in T_1, t_2 \in T_2\}^{\#(1)}.$$

5.2. Lemma. *Let $T_1, T_2 \in d(A)$. Then*

$$T_1 \oplus T_2 = \sup\{t_1 \oplus t_2 : t_1 \in T_1, t_2 \in T_2\}.$$

Proof. This is a consequence of 5.1.2. □

Since the operation $\oplus$ on A is associative, from 5.2 we conclude

5.3. Lemma. *The set $d(A)$ with the operation $\oplus$ is a semigroup.*

The following definition is analogous to 4.6; cf. also [12], 3.6.

5.4. Definition. Let $\mathcal{A}$ be as above and let $\mathcal{B}$ be a pseudo MV-algebra with the underlying set B such that

(a) $\mathcal{A}$ is a subalgebra of $\mathcal{B}$;
(b) $\ell(\mathcal{B})$ is a sublattice of $d(A)$;
(c) $(B, \oplus)$ is a subsemigroup of the semigroup $(d(A), \oplus)$.

Then $\mathcal{B}$ is called a c-extension of $\mathcal{A}$.

5.5. Definition. Let $\mathcal{B}_1$ be a c-extension of $\mathcal{A}$ such that, whenever $\mathcal{B}$ is a c-extension of $\mathcal{A}$, then $\mathcal{B}$ is a subalgebra of $\mathcal{B}_1$. We call $\mathcal{B}_1$ a maximal completion of $\mathcal{A}$. We denote $\mathcal{B}_1 = M(\mathcal{A})$.

Consider the lattice ordered group G_D from Section 3. In view of 4.8, G_D has the strong unit $u^\#$. Hence we can construct the pseudo MV-algebra $\mathcal{M}_0 = \Gamma(G_D, u^\#)$. Let M_0 be the underlying set of $\mathcal{M}_0$. We have

$$G_D \subseteq d(G), \quad M_0 \subseteq A^*.$$

5.6.1. Lemma. *Let $Z_1, Z_2 \in M_0$. Further, let $\oplus$ be the corresponding operation from $\mathcal{M}_0$. Then*

$$Z_1 \oplus Z_2 = \sup_{z_1 \in Z_1, z_2 \in Z_2} \{((z_1 + z_2) \wedge u)^\#\}$$

where sup *is taken with respect to the underlying lattice of $\mathcal{M}_0$.*

Proof. In view of the definition of $\Gamma(G_D, u^\#)$ we have

$$Z_1 \oplus Z_2 = (Z_1 +_0 Z_2) \wedge u^\#.$$

Further,

$$Z_1 +_0 Z_2 = \{z_1 + z_2 : z_1 \in Z_1, z_2 \in Z_2\}^\#.$$

Thus according to 5.1.1,

$$Z_1 +_0 Z_2 = \bigvee_{z_1 \in Z_1, z_2 \in Z_2}^{2} (z_1 + z_2)^{\#}.$$

Because each lattice ordered group is infinitely distributive, we get

$$\begin{aligned}(Z_1 +_0 Z_2) \wedge u^{\#} &= \Big(\bigvee_{z_1 \in Z_1, z_2 \in Z_2}^{2} (z_1 + z_2)^{\#}\Big) \wedge u^{\#} \\ &= \bigvee_{z_1 \in Z_1, z_2 \in Z_2}^{2} ((z_1 + z_2)^{\#} \wedge u^{\#}).\end{aligned}$$

Since the mapping $x \to x^{\#}$ of G into G_D preserves the operations $+$ and $\wedge$ we obtain

$$(Z_1 +_0 Z_2) \wedge u^{\#} = \bigvee_{z_1 \in Z_1, z_2 \in Z_2}^{2} ((z_1 + z_2) \wedge u)^{\#}.$$

The underlying lattice of $\mathcal{M}_0$ is an interval of the lattice $\ell(G)$, thus

$$\bigvee_{z_1 \in Z_1, z_2 \in Z_2}^{2} ((z_1 + z_2) \wedge u)^{\#} = \sup_{z_1 \in Z_1, z_2 \in Z_2} \{((z_1 + z_2) \wedge u)^{\#}\}.$$

□

Let $Z_1 \in M_0$ and $T_1 = \varphi_1(Z_1)$. Hence $z_1 \leqq u$ for each $z_1 \in Z_1$. Further, $0 \leqq z_1 \leqq z_1 \vee 0 \leqq u$, thus $z_1 \vee 0 \in T_1$ and analogously for Z_2. Therefore we have

$$Z_1 +_0 Z_2 = \bigvee_{z_1 \in T_1, z_2 \in T_2}^{2} (z_1 + z_2)^{\#}.$$

From this we conclude

5.6.2. Lemma. *In the formula for $Z_1 \oplus Z_2$ in* 5.6.1, *the relations $z_1 \in Z_1$, $z_2 \in Z_2$ can be replaced by $z_1 \in T_1$, $z_2 \in T_2$.*

Let Z_i and T_i $(i = 1, 2)$ be as above. According to 5.2 we have

$$T_1 \oplus T_2 \sup_{t_1 \in T_1, t_2 \in T_2} \{(t_1 + t_2) \wedge u\}^{\#(1)}.$$

Therefore according to 5.6.1, 5.6.2 and 5.1.2 we conclude

5.7. Lemma. *φ_1 is an isomorphism of the semigroup $(M_0, \oplus)$ onto the semigroup $(\varphi_1(M_0), \oplus)$.*

(In fact, we use the symbol φ_1 also for the partial mapping $\varphi_1 | M_0$.)

In view of 4.5, the lattice $\ell(G_D)$ is a sublattice of $d(G)$. Since M_0 is an interval of $\ell(G_D)$, we infer that M_0 is also a sublattice of $d(G)$. Thus in view of 5.1 we obtain

5.8. Lemma. *φ_1 is an isomorphism of the lattice M_0 onto the lattice $\varphi_1(M_0)$.*

We define the unary operation $-$ on the set $\varphi_1(M_0)$ as follows. Let $T \in \varphi_1(M_0)$; there exists $X \in M_0$ with $\varphi_1(X) = T$. We put $T^- = \varphi_1(X^-)$. Analogously we define the unary operation $^\sim$ on the set $\varphi_1(M_0)$.

Since M_0 is the underlying set of the pseudo MV-algebra $\mathcal{M}_0$, in view of 5.7 and 5.8 we obtain

5.8.1. Lemma. *The structure $(\varphi_1(M_0); \oplus, ^-, ^\sim, 0^{\#(1)}, u^{\#(1)})$ is a pseudo MV-algebra and φ_1 is an isomorphism of $\mathcal{M}_0$ onto this structure.*

The structure considered in 5.8.1 will be denoted by $\mathcal{A}_D$.

Similarly as in the case of G_D, we can identify the element a of A with the element $a^{\#(1)}$ of $\varphi_1(M_0)$. Then according to 5.7, 5.8 and 5.8.1 we obtain

5.9. Proposition. *$\mathcal{A}_D$ is a c-extension of the pseudo MV-algebra $\mathcal{A}$.*

Let $a, b \in A$. We put $a + b = a \oplus b$ if $x \leqq y^-$; otherwise, $a + b$ is not defined in A. (Cf. [4].) From the results of [5] we get

5.9.1. Lemma. *Let $a, b, c \in A$. Then $a + b = c$ if and only if this relation is valid in G.*

Consider the following conditions for $\emptyset \neq X \subseteq A$:

(i) There exists $0 < a \in A$ such that the relation $a + b \leqq c$ is valid for each $b \in X^\#$ and each $c \in X^u$.

(ii) There exists $0 < a \in A$ such that for each $b \in X^{\#(1)}$ the operation $a + b$ is defined in A and $a + b \leqq c$ for each $c \in X^{u(1)}$.

(We remark that in (i), $a + b$ has the meaning as in G.)

5.10. Lemma. *Let $\emptyset \neq X \subseteq A$. Then the conditions* (i) *and* (ii) *are equivalent.*

Proof. Assume that (i) holds. Let $b \in X^{\#(1)}$ and $c \in X^{u(1)}$. Then $b \in X^\#$ and $c \in X^u$. Hence $a + b \leqq c$ and so $a + b$ belongs to the interval $[0, u]$ of G. Therefore $a + b \in A$ and thus $a + b$ is defined in A. This shows that (ii) is satisfied.

Conversely, assume that (ii) is valid. Let $b \in X^\#$ and $c \in X^u$. Denote $b_1 = b \vee 0$ and $c_1 = c \wedge u$. We have $b \leqq u$, thus $b_1 \in X^{\#(1)}$ and $c_1 \in X^{u(1)}$. Let a be as in (ii). Hence $a + b_1$ is defined and $a + b_1 \leqq c_1$. We have clearly

$$a + b \leqq a + b_1 \leqq c_1 \leqq c\,.$$

Therefore (i) holds. □

Let us denote by (i_1) the condition analogous to (i) such that we have $b + a$ instead of $a + b$. Further, let (ii_1) be defined similarly. By the same method as above we obtain

5.10.1. Lemma. *Let $\emptyset \neq X \subseteq A$. Then the conditions* ($\mathrm{i}_1$) *and* ($\mathrm{ii}_1$) *are equivalent.*

Now assume that $\mathcal{B}_1$ is a c-extenstion of $\mathcal{A}$. Suppose that X belongs to B_1, where B_1 is the underlying set of $\mathcal{B}_1$. Then we have

5.11. Lemma. *X does not satisfy the condition* (ii) *above.*

Proof. By way of contradiction, assume that X satisfies the condition (ii). There exists a lattice ordered group G_1 with the strong unit $u^{\#(1)}$ such that $\mathcal{B}_1 = \Gamma(G_1, u^{\#(1)})$. Let us denote by $+_1$ the group operation in G_1.

In view of 5.11, the relation

$$X = \bigvee_{x\in X} x^{\#(1)} \tag{1}$$

is valid in $d(A)$. Let B_1 be the underlying set of $\mathcal{B}_1$. Since X and $x^{\#(1)}$ $(x \in X)$ belong to B_1, the relation (1) holds in the lattice $\ell(\mathcal{B}_1)$ and hence also in G_1.

Further, since $a+x$ is defined in A, in view of 5.9.1 we infer that $a+x = a+_1 x$. Also $a^{\#(1)} > 0^{\#(1)}$ in $\ell(\mathcal{B}_1)$. Hence we have

$$\begin{aligned} X < a^{\#(1)} +_1 X &= \bigvee_{x\in X} (a^{\#(1)} +_1 x^{\#(1)}) = \bigvee_{x\in X} (a +_1 x)^{\#(1)} \\ &= \bigvee_{x\in X} (a + x)^{\#(1)} . \end{aligned}$$

In view of (ii), $a + x \leqq c$ for each $c \in X^{u(1)}$, whence $a + x \in X^{u(1)\ell(1)} = X$ for each $x \in X$. Therefore

$$(a + x)^{\#(1)} \leqq X\,, \qquad \bigvee_{x\in X} (a + x)^{\#(1)} \leqq X\,, \qquad X < a^{\#(1)} +_1 X \leqq X\,,$$

which is a contradiction. □

5.12. Lemma. *Let $\mathcal{B}_1$ be a c-extension of $\mathcal{A}$. Then $B_1 \subseteq \varphi_1(A^*)$.*

Proof. Let $X \in B_1$. There exists $Y \in d(G)$ such that $X = \varphi_1(Y)$. In view of 5.11, X does not satisfy the condition (ii). Analogously, X does not satisfy (ii$_1$). Hence according to 5.10 and 5.10.1, Y satisfies neither (i) nor (i$_1$). Then 3.4 yields that Y belongs to A^*. Hence $X \in \varphi_1(A^*)$. □

The following assertion is easy to verify.

5.12.1. Lemma. *Let $a \in \mathcal{A}$. Then*

$$a^- = \max\{b \in A : b \oplus a = 1\},$$

$$a^\sim = \max\{b \in A : a \oplus b = 1\}.$$

5.13. Lemma. *Let $\mathcal{B}_1$ be a c-extension of $\mathcal{A}$. Then $\mathcal{B}_1$ is a subalgebra of $\mathcal{A}_D$.*

Proof. In view of 5.12, B_1 is a subset of the underlying set $A_D = \varphi_1(A^*)$ of $\mathcal{A}$. Then according to 5.4, $(B_1, \oplus)$ is a subsemigroup of $(A_D, \oplus)$, and $(B_1, \leqq)$ is a sublattice of $(A_D, \leqq)$. According to 5.12.1, in each pseudo MV-algebra, the operations $^-$ and $^\sim$ are uniquely determined by the operation $\oplus$ and the corresponding partial order. Hence $\mathcal{B}_1$ is a subalgebra of $\mathcal{A}_D$. □

From 5.9 and 5.13 we conclude

5.14. Theorem. *Let $\mathcal{A}$ be a pseudo MV-algebra. Then $\mathcal{A}_D$ is the maximal completion of $\mathcal{A}$.*

In view of the above results we also have

5.15. Proposition. *Let $\mathcal{A}$ be a pseudo MV-algebra. The underlying set of $\mathcal{A}_D$ consists of those elements X of $d(A)$ which satisfy neither* (ii) *nor* (ii$_1$). *If $\mathcal{A}=\Gamma(G,u)$, then the mapping φ_1 is an isomorphism of $\Gamma(G_D,u)$ onto $\mathcal{A}_D$.*

6. Another characterization of elements of $\mathcal{A}_D$

In the present section we apply the same notation as in Section 5.

For $X,Y\in d(G)$ the operation $X+_0 Y$ has been considered in Section 3.

We have $A^*\subseteq d(G)$. Let $X,Y\in A^*$. Put $X_1=\varphi_1(X)$, $Y_1=\varphi_1(Y)$. If $X+_0 Y\notin A^*$, then we say that the operation $X_1+_0 Y_1$ is not defined in $d(A)$; otherwise we put

$$X_1+_0 Y_1=\varphi_1(X+_0 Y).$$

Hence $+_0$ is a partial binary operation on the set $d(A)$. If $X_1\leqq Y_1$, $Z_1\in d(A)$, and if $X_1+_0 Z_1$, $Y_1+_0 Z_1$ exist in $d(A)$, then we have

$$X_1+_0 Z_1\leqq Y_1+_0 Z_1,$$

and analogously for $Z_1+_0 X_1$, $Z_1+_0 Y_1$. (Cf. 5.1.)

In Section 3, the set G_D has been characterized as the system of all $C\in d(G)$ having the property that there exists $X\in d(G)$ with

$$X+_0 C=C+_0 X=0^{\#}. \tag{$*$}$$

Hence G_D is characterized merely by $d(G)$ and the operation $+_0$ on $d(G)$.

The elements of $\mathcal{A}_D$ have been characterized in 5.15. The following result gives another characterization of these elements.

6.1. Theorem. *Let $C_1\in d(A)$. Then the following conditions are equivalent:*

(a) *C_1 is an element of $\mathcal{A}_D$.*

(b) *There exist $X_1,Y_1\in d(A)$ such that*

(1) $X_1+_0 C_1=C_1+_0 Y_1=u^{\#(1)}$,

(2) $Z+_0(-u)^{\#}=(-u)^{\#}+_0 T$,

where $T=\varphi_1^{-1}(X_1)$, $Z=\varphi_1^{-1}(Y_1)$.

Proof. Let (a) be valid. Put $C=\varphi_1^{-1}(C_1)$. Then according to 5.15 we have $C\in G_D$. Thus there exists $X\in d(G)$ such that the relation $(*)$ holds.

We have $0^{\#}\leqq C\leqq u^{\#}$. Since X is the inverse element of C in the lattice ordered group G_D we obtain

$$(-u)^{\#}\leqq X\leqq 0^{\#}.$$

Then

$$0^{\#} \leqq u^{\#} +_0 X \leqq u^{\#},$$
$$0^{\#} \leqq X +_0 u^{\#} \leqq u^{\#}.$$

Therefore the elements $u^{\#} +_0 X$, $X +_0 u^{\#}$ belong to the set A^*. We put

$$X_1 = \varphi_1(u^{\#} +_0 X), \quad Y_1 = \varphi_1(X +_0 u^{\#}).$$

Thus X_1 and Y_1 are elements of $d(A)$. From $(*)$ we obtain

$$(u^{\#} +_0 X) +_0 C = u^{\#},$$
$$C +_0 u^{\#}) = u^{\#},$$

whence applying the mapping φ_1 we get

$$X_1 +_0 C_1 = u^{\#(1)},$$
$$C_1 +_0 Y_1 = u^{\#(1)}.$$

Hence (1) holds. Also, (2) is obviously satisfied. Therebore (b) is valid.

Conversely, suppose that (b) holds. Again, let $C = \varphi_1^{-1}(C_1)$. Applying the mapping φ_1^{-1} for (1) we obtain

$$T +_0 C = u^{\#},$$
$$C +_0 Z = u^{\#},$$

whence

$$((-u)^{\#} +_0 T) +_0 C = 0^{\#},$$
$$(C +_0 (Z +_0 (-u)^{\#}) = 0^{\#}.$$

Then according to (2), the element $(-u)^{\#} +_0 T$ is the inverse element of C in $d(G)$. Hence C belongs to G_D. Thus in view of 5.15 we conclude that C_1 is an element of $\mathcal{A}_D$. □

6.1.1. Proposition. *Let $C_1 \in d(A)$ and $v \in A$ such that $c_1 \leqq v$ for each $c_1 \in C_1$. Then C_1 is an element of $d(A)$ if and only if the condition $(b(v))$ is satisfied, where $(b(v))$ is the modification of (b) from 6.1 consisting in replacing the element u by the element v.*

Proof. It suffices to replace the element u by the element v in the proof of 6.1.□

6.2. Proposition. *Assume that $\mathcal{A}$ is an MV-algebra. Let $C_1 \in d(A)$. The following conditions are equivalent:*

(a) *C_1 is an element of $\mathcal{A}_D$.*
(b_1) *There exists $X_1 \in d(A)$ such that $X_1 +_0 C_1 = u^{\#(1)}$.*

Proof. In view of 6.1 we have (a)⇒(b_1). Assume that (b_1) is valid. The operation $+_0$ is commutative. Put $Y_1 = X_1$. Then (b) holds, hence (a) is satisfied. □

Similarly as in 6.1.1 we have

6.2.1. Proposition. *Assume that $\mathcal{A}$ is an MV-algebra. Let $C_1 \in d(A)$, $v \in A$, $c_1 \leqq v$ for each $c_1 \in C_1$. Then the condition* (a) *from* 6.2 *is equivalent with the condition*

(b_2) *there exists $X_1 \in d(A)$ such that $X_1 +_0 C_1 = v^{\#(1)}$.*

6.2.2. Corollary. *Let $\mathcal{A}$ be an MV-algebra. Then the maximal completion of $\mathcal{A}$ is the set of all $T \in d(A)$ which satisfy the following condition*

(c) *either $T = u^{\#(1)}$, or there are $a \in A$ and $T_1 \in d(A)$ such that $a < u$ and $T +_0 T_1 = a^{\#(1)}$.*

We remark that there is a mistake in Proposition 3.19 of [12] (consisting in the fact that instead of the operation $\oplus$, the operation $+_0$ should be taken into account); the corrected version is Corollary 6.2.2 above. An analogous correction is to be performed in Lemma 3.15 of [12].

7. Strong subdirect products

In this section we assume that whenever $\mathcal{A}$ is a pseudo MV-algebra, then it is a subalgebra of its maximal completion (in view of the identification mentioned in Section 5). The same assumption is made for lattice ordered groups.

For the notion of the internal direct product decomposition of an MV-algebra cf. [11]; the same definition can be applied for pseudo MV-algebras.

Strong subdirect products of pseudo MV-algebras and of lattices have been investigated in [13]. In [15], strong subdirect products of MV-algebras have been dealt with.

For the sake of completeness, we recall the definition of the strong subdirect product decomposition of a pseudo MV-algebra.

Suppose that we are given a subdirect product decomposition

$$\varphi : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i = \mathcal{B} \tag{1}$$

of a pseudo MV-algebra $\mathcal{A}$. We apply the usual notation: A, A_i and B are the underlying subsets of $\mathcal{A}$, $\mathcal{A}_i$, or $\mathcal{B}$, respectively. Further, 0_i and 1_i is the least resp. the greatest element of A_i.

The elements of B are written in the form $(a_i)_{i\in I}$. Let $i \in I$, $a \in A$, $\varphi(a) = (a_i)_{i\in I}$. We put

$$\alpha_i(a) = a_i\,, \quad \beta_i(a) = (a_j)_{j\in J\setminus\{i\}}\,, \quad A'_i = \{\beta_i(a) : a \in A\}\,.$$

Then there exists a pseudo MV-algebra $\mathcal{A}'_i$ such that

(i) $\mathcal{A}'_i$ is a subalgebra of $\prod_{j\in I\setminus\{i\}} \mathcal{A}_j$,

(ii) the underlying subset of $\mathcal{A}'_i$ is equal to A'_i.

For each $a \in A$ we put

$$\varphi_i^0(a) = (\alpha_i(a), \beta_i(a))\,.$$

In view of (1), φ_i^0 is an injective mapping of $\mathcal{A}$ into the direct product $\mathcal{A}_i \times \mathcal{A}'_i$.

We say that (1) is a strong subdirect product decomposition of $\mathcal{A}$ if for each $i \in I$, φ_i^0 is an isomorphism of $\mathcal{A}$ onto $\mathcal{A}_i \times \mathcal{A}'_i$.

Let us suppose that this condition is satisfied.

Without loss of generality we can assume that for each $i \in I$, A_i is the set

$$\{a \in A : \varphi(a)_j = 0_j \quad \text{for each } j \in I \setminus \{i\}\}$$

and that for each $x \in A_i$ we have $\varphi(x)_i = x$, $\varphi(x)_j = 0_j$ whenever, $j \in I, j \neq i$.

Then (in view of the definition of the internal direct product) we have an internal direct product decomposition

$$\varphi_i^0 : \mathcal{A} \to \mathcal{A}_i \times \mathcal{A}'_i\,. \tag{2}$$

From Theorem 6.1 in [14] we obtain that (2) induces a direct product decomposition of the corresponding lattice

$$\varphi_i^0 : \ell(\mathcal{A}) \to \ell(\mathcal{A}_i) \times \ell(\mathcal{A}'_i)\,. \tag{2.1}$$

We remark that Proposition 2.4 and Proposition 2.5 proven in [11] for MV-algebras remain valid (with the same proofs) for pseudo MV-algebras as well; hence from 2.1 we conclude that there exists an internal direct product decomposition

$$\psi_i^0 : G \to G_i \times G'_i\,, \tag{3}$$

where (under the notation as above)

$$\mathcal{A} = \Gamma(G, u)\,, \quad \mathcal{A}_i = \Gamma(G_i, u_i)\,, \quad \mathcal{A}'_i = \Gamma(G'_i, u'_i)\,.$$

Now we recall that in [10] there have been investigated the relations between the direct product decompositions of a lattice ordered group G and the direct product decompositions of the maximal completion of G. It was proved that each internal direct product decomposition of G induces an internal direct product decomposition of G_D. It was assumed that the lattice ordered group under consideration is abelian, but the proof remains valid for the non-abelian case as well. Hence from (3) we infer that there exists an internal direct product decomposition

$$\chi_i^0 : G_D \to (G_i)_D \times (G_i')_D \tag{4}$$

such that χ_i^0 is an extension of ψ_i^0 in the sense that for each $g \in G$ we have

$$\psi_i^0(g) = \chi_i^0(g)\,.$$

Let us apply again Proposition 2.5 of [11] (we already remarked that it remains valid for pseudo MV-algebras as well); further we use Theorem 6.4 of [14]. Then in view of (4) there exists an internal direct product decomposition

$$\psi_i' : \mathcal{A}^0 \to \mathcal{A}_i^0 \times \mathcal{A}_i^{01}\,, \tag{5}$$

where

$$\mathcal{A}^0 = \Gamma(G_D, u)\,,$$
$$\mathcal{A}_i^0 = \Gamma((G_i)_D, u_i), \quad \mathcal{A}_i^{01} = \Gamma((G_i')_D, u_i')\,.$$

Then in view of 5.15 we have

$$\mathcal{A}_i^0 \simeq (\mathcal{A}_i)_D, \quad \mathcal{A}_i^{01} \simeq (\mathcal{A}_i')_D, \quad \mathcal{A}^0 \simeq \mathcal{A}_D\,, \tag{6}$$

where $\simeq$ denotes the relation of isomorphism between pseudo MV-algebras.

The above construction can be performed for each $i \in I$. Let $a_0 \in A^0$ (as usual, we denote by A^0 the underlying set of $\mathcal{A}^0$; the meaning of A_i^0 is analogous). Consider the mapping

$$\psi^0 : A^0 \to \prod_{i\in I} A_i^0$$

defined by

$$\psi^0(a_0) = (\psi_i^1(a_0))_{i\in I} \text{ for each } a \in A^0.$$

Then ψ^0 is a homomorphism of the pseudo MV-algebra $\mathcal{A}^0$ into the direct product

$$\prod_{i\in I} \mathcal{A}_i^0 = \mathcal{D}\,.$$

Suppose that

$$x = (x_i)_{i\in I} \in \prod_{i\in I} A_i^0.$$

Let $i \in I$. Thus $x_i \in A_i^0$. Hence $x_i \in (G_i)_D$. According to 4.3.2 there exists a subset $\{a_{ij}\}_{j\in J_i}$ of G_i such that

$$x_i = \bigvee_{j\in J_i} a_{ij} \tag{7}$$

is valid in $(G_i)_D$. Then, clearly, this relation is valid also in $\mathcal{A}_i^0$ and, consequently, also in $\mathcal{A}^0$. Put

$$B_1 = \{a_{ij}\} \qquad (i \in I, j \in J_i)\,,$$
$$C = B_1^u, \quad B = C^\ell\,,$$

where the symbols u and ℓ are taken with respect to the partially ordered set $\ell(\mathcal{A}^0)$. Hence B is an element of $d(\mathcal{A}_0)$.

7.1. Lemma. *B is an element of $(\mathcal{A}^0)_D$.*

Proof. By way of contradiction, assume that B does not belong to $(\mathcal{A}^0)_D$. Then in view of 3.2 and 3.3 there exists $0 < y \in A^0$ such that either

a) $y + b \leqq c$ for each $b \in B$, $c \in C$,

or

b) $b + y \leqq c$ for each $b \in B$, $c \in C$.

Let us consider the case a). Hence, in particular,

$$y + a_{ij} \leqq c \quad \text{for each } a_{ij} \in B_1 \text{ and each } c \in C\,. \tag{8}$$

Since $y > 0$, there exists $i \in I$ with $y_i > 0$. Denote

$$B_{1i} = B_1(\mathcal{A}_i)\,, \quad B_i = B(\mathcal{A}_i), \quad C_i = C(\mathcal{A}_i)\,.$$

Then C_i is the set of all upper bounds of B_{1i} in $\ell(\mathcal{A}_i)$ and B_i is the set of all lower bounds of C_i in $\ell(\mathcal{A}_i)$. Moreover, because of

$$(a_{ij})_i = a_{ij} \quad \text{if } j \in J_i \quad \text{and} \quad (a_{ij})_i = 0 \text{ otherwise,}$$

we have $B_{i1} = \{a_{ij}\}_{j \in J_i}$.

Let x_i be as in (7). Then

$$C_i = \{c \in A_i : c \geqq x_i\}\,,$$
$$B_i = \{c \in A_i : c \leqq x_i\}\,.$$

Therefore the condition a) yields $y + x_i \leqq x_i$, which is a contradiction. The case when b) is valid can be treated analogously. □

From 7.1 and 5.1.1 we conclude (in view of the indentification mentioned above)

$$B = \bigvee_{i \in I, j \in J_i} a_{ij}\,,$$

whence according to (7) we obtain

7.2. Lemma. *The relation* $B = \bigvee_{i\in I} x_i$ *is valid in* $(\mathcal{A}^0)_D$.

Let $i_0 \in I$ be fixed. Then 7.2 yields

$$u_{i_0} \wedge B = \bigvee_{i\in I} (u_{i_0} \wedge x_i) = x_{i_0}\,,$$

since $x_{i_0} \leqq u_{i_0}$ and $u_{i_0} \wedge x_i = 0$ if $i \neq i_0$. From this we easily obtain

$$(B)_{i_0} = x_{i_0} \quad \text{for each } i_0 \in I\,.$$

Thus we have

7.3. Lemma. *The mapping* ψ^0 *is surjective.*

7.4. Lemma. *The mapping* ψ^0 *is an isomorphism.*

Proof. Let $x, y \in A^0$ and suppose that $\psi^0(x) = \psi^0(y)$; in other words, $x_i = y_i$ for each $i \in I$. According to 7.2 we have

$$x = \bigvee_{i\in I} x_i, \quad y = \bigvee_{i\in I} y_i\,.$$

Hence $x = y$. Thus the mapping ψ^0 is injective. Since ψ^0 is a homomorphism, by 7.3 it is an isomorphism. □

7.5. Theorem. *Let* $\mathcal{A}$ *be a pseudo* MV*-algebra which can be expressed as a strong subdirect product of pseudo* MV*-algebras* $\mathcal{A}_i$ $(i \in I)$. *Then the maximal completion* $\mathcal{A}_D$ *of* $\mathcal{A}$ *is isomorphic to the direct product of pseudo* MV*-algebras* $(\mathcal{A}_i)_D$ $(i \in I)$.

Proof. Let us apply the notation as above. Then we have

$$\mathcal{A} \simeq \mathcal{A}^0, \quad \mathcal{A}_i \simeq \mathcal{A}_i^0 \quad \text{for } i \in I.$$

Now, it suffices to use 7.4. □

For the particular case when $\mathcal{A}$ is an MV-algebra, cf. [15].

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