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ON CERTAIN SINGULAR THIRD ORDER EIGENVALUE PROBLEM

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Dedicated to Professor's Šeda 70th birthday

Abstract. In this paper a singular third order eigenvalue problem is studied. The results of the paper complete the results given in the papers [3], [5].

1. We consider the third order linear differential equation in the normal form

$$y''' + 2A(t)y' + [A'(t) + \lambda b(t)]y = 0\,, \tag{a}$$

where $A(t) \geq 0$, $A'(t)$ and $b(t) > 0$ are continuous functions of $t \in [a, \infty)$, $a > -\infty$ and λ is a positive parameter. It is assumed that (a) is strongly nonoscillatory on $[a, \infty)$, that is (a) is nonoscillatory there for each real positive λ. By nonoscillation of (a) we mean that all of its nontrivial solutions are nonoscillatory on $[a, \infty)$.

A nontrivial solution of (a) is called oscillatory on $[a, \infty)$ if ∞ is a limit point of zeros of that solution. In the contrary case the solution is called nonoscillatory on $[a, \infty)$.

The equation (a) is said to be oscillatory on $[a, \infty)$ if it has at least one oscillatory solution on $[a, \infty)$.

The problem studied in Section 2, is to find a nontrivial (nonoscilatory) solution $y(t, \lambda)$ of (a) which satisfies either of the boundary conditions at finite points

$$y(a, \lambda) = y'(a, \lambda) = y(b, \lambda) = 0\,, \qquad a < b, \tag{1}$$

$$y(a, \lambda) = y(b, \lambda) = y(c, \lambda) = 0\,, \qquad a < b < c \tag{2}$$

a, b and c being any given constants, as well as the boundary condition at infinity

$$y(t, \lambda) = o(t[k_1 u_1(t) u_2(t) + k_2 u_2^2(t)]) \quad \text{for} \quad t \to \infty\,, \tag{3}$$

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together with the requirement that

$$y(t,\lambda) \neq 0$$

in a certain neighbourhood of infinity (t_0,∞), where $b \le t_0 < \infty$ (or $c \le t_0 < \infty$ in the conditions (2)), and $u_1,\ u_2$ form a fundamental set of solutions of the second order differential equation

$$u'' + \frac{1}{2}A(t)u = 0 \tag{4}$$

with initial conditions $u_1(t_0) = 1,\ u_1'(t_0) = 0,\ u_2(t_0) = 0, u_2'(t_0) = 1,\ k_1,\ k_2$ are suitable constants. The motivation for this paper was given by the paper [1] of A. Elbert, T. Kusano and M. Naito for linear second order nonoscillatory differential equations.

In the paper [3] the case $A(t) < 0$ on $[a,\infty)$ was studied.

2. At the beginning of this section we introduce certain auxiliary statements on the linear third order differential equation, given in monograph [4].

Consider equation (a) and the third order differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)] = 0\,, \tag{a_1}$$

Lemma 1. [4, Theorem 2.41] *Let the differential equation*

$$y'' + 2A(t)y = 0 \tag{5}$$

be disconjugate in (a,∞) and let the functions $A(t),\ A'(t)+b(t),\ b(t)-A'(t)$ be positive in $[a,\infty)$. If

$$\int_a^\infty t^2[b(t) - A'(t)]\,dt < \infty\,, \tag{6}$$

then the differential equation (a_1) is non-oscillatory in $[a,\infty)$.

Remark 1. If $A(t) \equiv 0$ on $[a,\infty)$ and (6) holds then [2, Theorem 4] equation (a_1) is non-oscillatory in $[a,\infty)$.

By Lemma 1 and Remark 1 the following lemma can be proved.

Lemma 2. *Let the differential equation (5) be disconjugated in $[a,\infty)$ and let $A(t) \ge 0,\ A'(t) \le 0,\ b(t) > 0$ and $A'(t) + b(t) > 0$ in $[a,\infty)$. Let further $\bar\lambda$ be any fixed positive value of the parameter λ. If (6) holds then*

$$\int_a^\infty t^2[\bar\lambda b(t) - A'(t)]\,dt < \infty\,, \tag{7}$$

and the differential equation (a) is non-oscillatory for $\lambda = \bar\lambda$ in $[a,\infty)$.

Lemma 3. [4, Theorem 2.31] *Let $b(t) > 0$, $A'(t)$ be continuous in $[a, \infty)$. The equation (a_1) is oscillatory in $[a, \infty)$ if and only if its adjoint equation*

$$(b_1) \qquad z''' + 2A(t) + [A'(t) - b(t)]z = 0\,,$$

is oscillatory in $[a, \infty)$.

If we apply Lemma 3 we can formulate Theorem 2.51 [4] as follows.

Lemma 4. *Let $A(t) \geq 0$, $A'(t) - b(t) \leq 0$, $b(t) > 0$ in $[a, \infty)$ and let $\int\limits_a^\infty b(t)\,dt = \infty$, then the differential equation (a_1) is oscillatory in $[a, \infty)$.*

Corollary 1. *Let the suppositions of Lemma 4 be fulfilled and $A'(t) \leq 0$ in $[a, \infty)$. Then the differential equation (a) for $\lambda = \bar{\lambda} > 0$ is oscillatory in $[a, \infty)$.*

Lemma 5. *Let $A(t) \geq 0$ in $[a, \infty)$ and let the differential equation (4) be disconjugated in $[a, \infty)$. Let $u_1,\ u_2$ be independent solutions of (4) and let $u_1(t_0) = 1$, $u_1'(t_0) = 0$, $u_2(t_0)$, $u_2(t_0) = 1$, $a < t_0 < \infty$. Then there is $u_2(t) > 0$ for $t > t_0$. And $u_1(t)$ has at most one zero to the right of t_0.*

Remark 2 [6, Lemma 2.23]**.** Let the suppositions of Lemma 5 be fulfilled. If u is a solution of (4) and $u(t) \neq 0$ for $t \geq t_1$, then

$$0 < (t + d)v(t) \leq 1,\ t \geq t_1$$

where $v(t) = \frac{u'(t)}{u(t)},\ d = -t_1 + 1/v(t_1)$.

Lemma 6. *Let the suppositions of Lemma 5 be fulfilled and let $b(t) > 0$ for $t \in [a, \infty)$ and $\lambda > 0$. Let further y be a solution of (a) and let for $\lambda = \bar{\lambda}$ be $y(t_0, \bar{\lambda}) = 0$, $y'(t_0, \bar{\lambda}) \neq 0$, $y''(t_0, \bar{\lambda}) \neq 0$ for $a \leq t_0 < \infty$. Let, moreover $y(t, \bar{\lambda}) \neq 0$ for $t > t_0$. Then*

$$\begin{aligned} y(t, \bar{\lambda}) = {} & u_2(t)\Big[\frac{y''(t_0, \bar{\lambda})}{2} u_2(t) + y'(t_0, \bar{\lambda}) u_1(t)\Big] \\ & - \frac{1}{2}\bar{\lambda} \int\limits_{t_0}^{t} b(\tau) \begin{vmatrix} u_1(t) & u_2(t) \\ u_1(\tau) & u_2(\tau) \end{vmatrix}^2 y(\tau, \bar{\lambda}) d\tau\,, \end{aligned} \tag{8}$$

where $u_1,\ u_2$ form a fundamental set of solutions of (4) with the properties given in Lemma 5.

The proof of Lemma 6 is given in [4] at the beginning of Section 3, Chap. I, §3.

Corollary 2. *If $y(t, \bar{\lambda}) > 0$ $[y(t, \bar{\lambda}) < 0]$ for $t > t_0$ in (8) then $y'(t_0, \bar{\lambda}) > 0$ $[y'(t_0, \bar{\lambda}) < 0]$, $u_2(t) > 0$ and $u(t) = y'(t_0, \bar{\lambda}) u_1(t) + \frac{y''(t_0, \bar{\lambda})}{2} u_2(t) > 0$ $[u(t) < 0]$ for $t > t_0$.*

Corollary 3. *Let the suppositions of Lemma 6 be fulfilled then there exist constants $k_1 = y'(t_0, \bar{\lambda})$, $k_2 = \frac{y''(t_0, \bar{\lambda})}{2}$ such that $|y(t, \bar{\lambda})| \leq u_2(t)|k_1 u_1(t) + k_2 u_2(t)|$ for $t > t_0$, or $y(t, \bar{\lambda}) = o(t u_2(t)[k_1 u_1(t) + k_2 u_2(t)])$ for $t \to \infty$.*

Adaptation of Oscillation theorem [4, Theorem 4.5] to (a) in our case yields the following lemma.

Lemma 7. *Suppose that $A(t) \geq 0, A'(t) \leq 0$ and $b(t) \geq k > 0$ for $t \in [a, \infty)$. Let $\lambda \in (0, \infty)$ and let $y(t, \lambda)$ be a nontrivial solution of (a) with $y(a, \lambda) = 0$. Then for any fixed $b > a$, the number of zeros of y on $[a, b]$ increases to infinity as $\lambda \to \infty$ and the distance between any consecutive zeros of y converges to zero.*

The continuous dependence of zeros of solutions of (a) upon the parameter λ is given in following lemma.

Lemma 8. [4, Lemma 4.2] *Let $A'(t)$, $b(t) > 0$ be continuous functions in $[a, \infty)$. Let y be a nontrivial solution of (a) on $[a, \infty)$ such that $y(\alpha, \lambda) = 0$, $a \leq \alpha < \infty$, for all $\lambda \in (0, \infty)$. Then the zeros of y on (α, ∞) (if they exist) are continuous functions of the parameter $\lambda \in (0, \infty)$.*

With the help of results given in the preceding lemmas, remarks and corollaries one can prove the following theorem regarding the singular eigenvalue problem problem (a), (1), (3) or (a), (2), (3).

Theorem 1. *Let $A(t) \geq 0$, $A'(t) \leq 0$, $b(t) > 0$ be continuous functions in $[a, \infty)$ and let $A'(t) + b(t) > 0$ for $t \in [a, \infty)$. Let $\int\limits_a^\infty t^2[b(t) - A'(t)]dt < \infty$ and let the second order differential equation (5) be disconjugate in $[a, \infty)$. Let further $a \leq b < c$ be fixed arbitrarily. Then there exists a natural number ν and a sequence of parameters $\lambda\Big\{\lambda_{\nu+p}\Big\}_{p=0}^{\infty}$ (eigenvalues) such that $\lambda_{\nu+p} < \lambda_{\nu+p+1}$, $p = 0, 1, 2, \ldots$ and $\lim\limits_{p\to\infty} \lambda_{\nu+p} = \infty$ and a corresponding sequence of functions $\Big\{y_{\nu+p}\Big\}_{p=0}^{\infty}$ (eigenfunctions) such that $y_{\nu+p} = y(t, \lambda_{\nu+p})$ is a solution of (a) for $\lambda = \lambda_{\nu+p}$, has a finite number of zeros on (a, ∞) with the last zero at $t_0^{\nu+p}$. This solution $y_{\nu+p}$ fulfills the boundary conditions (1), (3) or (2), (3) and has exactly $\nu + p$ zeros in (b, c).*

Proof. We prove the case $a < b < c$. In the case $a = b$, i.e. (1), (3) the proof is similar.

Let $a < b < c < \infty$. Let $y = y(t, \lambda), \lambda > 0$, be a nontrivial solution of (a) such that $y(a, \lambda) = y(b, \lambda) = 0$ for all $\lambda > 0$. By Lemma 2 solution y is nonoscillatory for each $\lambda = \bar{\lambda} > 0$. Now, construct the differential equation

$$Y''' + 2A(t)Y' + [A'(t) + \lambda B(t)]Y = 0 \quad \text{on} \quad [a, \infty) \tag{A}$$

where

$$B(t) = \begin{cases} b(t) & \text{for} \quad t \in [b, c] \\ b(c) & \text{for} \quad t \geq c\,. \end{cases}$$

Let $Y = Y(t, \lambda)$ be a solution of (A) on $[a, \infty)$ such that $Y(a, \lambda) = Y(b, \lambda) = 0$ and $Y(t, \lambda) = y(t, \lambda)$ for $t \in [a, c]$ and $\lambda \in (0, \infty)$.

By Lemma 4 and Corollary 1 the differential equation (A) is oscillatory on $[a, \infty)$ for each $\bar{\lambda} \in (0, \infty)$ and therefore the solution Y is oscillatory on $[a, \infty)$ for each $\bar{\lambda} > 0$.

Let $\lambda = \bar{\lambda}$ be fixed. Let $Y(t, \bar{\lambda})$ have exactly ν zeros in (b, c). Let $t_\nu(\lambda)$ be ν-th zero of $Y(t, \lambda)$. Then there is $t_\nu(\bar{\lambda}) < c \le t_{\nu+1}(\bar{\lambda})$. By Lemma 7 there exist λ^* such that $t_{\nu+1}(\lambda^*) < c$ and by Lemma 8 (continuous dependence of zeros) there exists $\lambda_\nu,\ \bar{\lambda} \le \lambda_\nu < \lambda^*$ such that $t_{\nu+1}(\lambda_\nu) = c$ and $y(t, \lambda_\nu)$ has exactly ν zeros in (b, c). But, we know that $Y(t, \lambda_\nu) = y(t, \lambda_\nu)$ on $[a, c]$. By Lemma 2 applied to λ_ν there exists $t_0^\nu \ge c$ such that $y(t, \lambda_\nu)$ has finite number of zeros to the right of c and t_0^ν is its last zero on $[c, \infty)$. Then by Corollary 3, when $t_0 = t_0^\nu$, the inequality (3) holds.

Continuing in the same manner we can find a sequence of values

$$\lambda_\nu,\ \lambda_{\nu+1},\ \cdots, \lambda_{\nu+p},\ \cdots$$

and the corresponding sequence of functions $\left\{ y_{\nu+p} \right\}_{p=0}^{\infty}$ (eigenfunctions) with the prescribed properties and the theorem is proved. □

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