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STABLE SPACE-LIKE HYPERSURFACES IN THE DE SITTER SPACE

LIU XIMIN AND DENG JUNLEI

Abstract. In this paper, we study the stability of space-like hypersurfaces with constant scalar curvature immersed in the de Sitter spaces.

1. Introduction

Hypersurfaces M^n with constant mean curvature in a Riemannian manifold $\bar{M}^{n+1}(c)$ of constant sectional curvature c are critical points of the area functional under variations that keep constant a certain volume function. In [3] a definition of stability for hypersurfaces of constant mean curvature in the Euclidean space R^{n+1} was given, and it was proved that the round spheres are the only compact hypersurfaces with constant mean curvature in R^{n+1} that are stable. Later in [4] Barbosa, do Carmo and Eschenburg extended this notion of stability to the case of immersions in Riemannian manifolds, and they proved that if M^n is compact and stable, and $\bar{M}^{n+1}(c)$ is complete and simply-connected, then M^n is a geodesic sphere.

Less widely known but equally true is that hypersurface M^n of $\bar{M}^{n+1}(c)$ with constant scalar curvature are solutions to a similar variational problem, namely, of extremizing the integral of the mean curvature for volume-preserving variations. In analogy with the case of constant mean curvature, questions of stability can be considered for hypersurfaces with constant scalar curvature. In [1], Alencar, do Carmo and Colaresthe extended to hypersurfaces with constant scalar curvature the above stability result on constant mean curvature. That is they proved that when the ambient space is Euclidean space R^{n+1}, or an open hemisphere of the sphere $S^{n+1}(1)$, geodesic spheres are the only stable immersed compact orientable hypersurfaces with constant scalar curvature.

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter

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space of index p, denote it by $S_p^{n+p}(c)$. The study of space-like hypersurfaces in de Sitter space has been recently of substantial interest from both physics and mathematical points of view. Akutagawa [2] and Ramanathan [8] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng [6] generalized this result to general submanifolds in a de Sitter space.

In the present paper, we would like to extend of stability to the case of immersions into the de Sitter spaces. We will define and discuss the stability of space-like hypersurfaces with constant scalar curvature in the de Sitter space.

2. The variational problem for constant scalar curvature

Let $S_1^{n+1}(c)$ be an $(n+1)$-dimensional de Sitter space of canstant curvature c and let $x : M^n \to S_1^{n+1}(c)$ be a space-like immersion of a compact, connected, orientable manifold M^n of constant scalar curvature with boundary ∂M (possibly, $\partial M = \emptyset$) into $S_1^{n+1}(c)$. By space-like we simply mean that the metric induced by x in M^n is Riemannian. Choose an orthonormal frame $e_1, \dots, e_{n+1}$ around $x(p)$, $p \in S_1^{n+1}(c)$, in $S_1^{n+1}(c)$ so that $e_1, \dots, e_n$ are tangent to $x(M)$ and $e_{n+1} = N$ is the time-like unit normal field globally defined on M^n and gives an orientation for M^n.

A variation of x is a differentiable map $X : (-\varepsilon, \varepsilon) \times M \to S_1^{n+1}(c)$, $\varepsilon > 0$, such that for each $t \in (-\varepsilon, \varepsilon)$, $X_t(p) = X(t,p)$, $p \in M^n$, is an immersion, $X_0 = x$, and $X_t|_{\partial M} = x|_{\partial M}$, for all t. We define the volume function: $V : (-\varepsilon, \varepsilon) \to R$ of X by

$$V(t) = \int_{[0,t]\times M} X^* dS_1^{n+1} \,.$$

In this paper, we will need the first three symmetric elementary functions of the principle curvatures $k_1, \dots, k_n$ of an immersion x, namely:

$$S_1 = \sum k_i \,, \quad S_2 = \sum_{i<j} k_i k_j \,, \quad S_3 = \sum_{i<j<l} k_i k_j k_l \,,$$

$i, j, l = 1, \dots, n$. We know that the mean curvature H and the scalar curvature R of x are given by:

$$H = \frac{1}{n} S_1 \,, \quad c - R = \frac{2}{n(n-1)} S_2 \,.$$

Let X be a variation of $x : M^n \to S_1^{n+1}(c)$ and $W(p) = \frac{\partial X}{\partial t}|_{t=0}$ be the variational vector of X. Let $f = \langle W, N \rangle$, where N is the unit normal vector along x. A variation is normal if W is parallel to N and volume-preserving if $V(t) = V(0)$ for all t.

Lemma 2.1. (i) $\frac{d}{dt} \int_M nH(t) dM_t|_{t=0} = \int_M (n(n-1)(R-c) + cn) f dM$,

(ii) $\frac{dV}{dt}|_{t=0} = \int_M f dM$.

Proof. (i) can be obtained from the formula for the first variation in p. 470 of [9] in our notation.

To prove (ii), fix a point $p \in M^n$ and choose a positive adapted orthonormal frame $e_1, \dots, e_n, e_{n+1} = N$ around $x(p)$, then we have

$$X^*(dS_1^{n+1}) = X^*(dS_1^{n+1})(\frac{\partial}{\partial t}, e_1, \dots, e_n) = (dS_1^{n+1})\Big(\frac{\partial X}{\partial t}, dX_t(e_1), \dots, dX_t(e_n)\Big)$$
$$= \mathrm{vol}\,\Big(\frac{\partial X}{\partial t}, dX_t(e_1), \dots, dX_t(e_n)\Big) = \Big\langle \frac{\partial X}{\partial t}, N_t \Big\rangle,$$

where N_t is a unit normal vector of the immersion X_t. It follows that

$$\frac{dV}{dt}(0) = \frac{d}{dt}\Big(\int_{[0,t]\times M} \Big\langle \frac{\partial X}{\partial t}, N_t \Big\rangle \wedge dM)_{t=0} = \int_M \Big\langle \frac{\partial X}{\partial t}(0), N \Big\rangle dM = \int_M f dM\,.$$

This completes the proof of Lemma 2.1. □

Now set

$$R_0 = A^{-1}\int_M R dM\,, \quad A = \int_M dM\,,$$

and define $J : (-\varepsilon, \varepsilon) \to R$ by

$$J(t) = n\int_M H(t)\, dM_t + \big(n(n-1)(c - R_0) - cn\big)V(t)\,.$$

Lemma 2.2. *Let $M^n \to S_1^{n+1}(c)$ be an immersion. Then the following statements are equivalent:*

(i) *x has constant scalar curvature R_0.*

(ii) *For all volume-preserving variations,*

$$\frac{d}{dt}\int_M nH(t)dM_t|_{t=0} = 0\,.$$

(iii) *For all variations, $J'(0) = 0$.*

Proof. The proof is essentially the same as in Proposition (2.7) of [4] using Lemma 2.1. We omit it here.

To compute the second variation of J we need to introduce the following operator. For each $p \in M^n$, consider the linear map $T:\ T_p \to T_p$

$$T = nHI - B\,,$$

where I is the identity map and B is the linear map associated to the second fundamental form of x along N. In the orthonormal frame $\{e_1, \dots, e_n\}$ around p, the matrix of T is

$$T_{ij} = nH\delta_{ij} - h_{ij}\,,$$

where h_{ij} is the matrix of B. Let f be a differentiable function on M^n and let f_{ij} be the matrix of the hessian of f. We define the operator $\Box$ acting on f by

$$\Box f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}\,.$$

This operator was first condidered by Cheng and Yau in [7]. From [7] we know that $\square$ is self-adjoint relative to the L^2 inner product of M^n, i.e.,

$$\int_M f\square g = \int_M g\square f.$$

Lemma 2.3. *Let $x : M^n \to S_1^{n+1}(c)$ be a hypersurface with constant scalar curvature R and let X be a variation of x. Then $J''(0)$ depends only on f and it is given by*

$$J''(0)(f) = 2\int_M \Big(f\square f - f^2\Big[\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H + 3S_3\Big]\Big)\, dM\,.$$

Proof. Note that

$$\frac{dJ}{dt} = \int_M \Big[(-n(n-1)(c-R_t)H + cn) + (n(n-1)(c-R_0) - cn)\Big] f_t\, dM_t\,.$$

Here R_t is the scalar curvature of X_t, dM_t is its volume element, and $f_t = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle$, where N_t is the unit normal vector of X_t. Set $n(n-1)(c-R_t) = -A_t$, we can write

$$\frac{DJ}{dt} = \int_M (A_t - A_0) f_t\, dM_t\,.$$

Then we have

$$\frac{d^2J}{dt^2} = \int_M A_t' f_t\, dM_t + \int_M A_t f_t'\, dM_t - \int_M A_0 f_t'\, dM_t + \int_M (A_t - A_0) f_t \frac{\partial}{\partial t}\, dM_t\,,$$

which, for $t = 0$, gives

$$\frac{d^2J}{dt^2}\Big|_{t=0} = \int_M A_0' f\, dM = -\int_M (n(n-1)\Big(\frac{\partial R_t}{\partial t}(0)\Big) f\, dM\,.$$

Using the formula (9c) in [9] we can obtain

$$\frac{1}{2}n(n-1)\frac{\partial R_t}{\partial t}(0) = f\Big\{\frac{1}{2}n^2(n-1)(c-R)H - 3S_3) + cn(n-1)H\Big\} + \square f$$

and this completes the proof of Lemma 2.3. $\square$

Definition 2.1. Let $x : M^n \to S_1^{n+1}(c)$ has constant scalar curvature. The immersion x is stable if

$$\frac{d^2}{dt^2}\int_M nH_t\, dM_t\big|_{t=0} \le 0\,,$$

for all volume-preserving variations of x. If M^n is non compact, x is stable if for every compact submanifold $M' \subset M^n$ with boundary, the restriction $x|_{M'}$ is stable.

Just as [5] we can prove the following criterion of stability. Let $\mathcal{G}$ be the set of differential functions $f : M \to R$ with $f|_{\partial M} = 0$ and $\int_M f\, dM = 0$. Then $x : M^n \to S_1^{n+1}(c)$ with constant scalar curvature is stable if and only if

$$J''(0)(f) \le 0\,,$$

for all $f \in \mathcal{G}$.

3. Stability of space-like hypersurfaces

Define a bilinear form $I : \mathcal{G} \to R$ by

$$I(f,g) = \int_M g\Big(\Box f - \Big[\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H - 3S_3\Big]\Big)f\,.$$

Definition 3.1. A normal vector field $V = fN$, $f \in \mathcal{G}$, to a space-like immersion $x : M^n \to S_1^{n+1}$ with constant scalar curvature is a Jacobi field if $f \in \mathrm{Ker}\ I$, that is, if $I(f,g) = 0$ for all $g \in \mathcal{G}$.

Proposition 3.1. *Let $f \in \mathcal{G}$. Then fN is a Jacobi field is and only if*

$$\Box f - \Big[\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H - 3S_3\Big]f = \mathrm{const}\,.$$

Proof. Clearly if the above formula holds, $f \in \mathrm{Ker}\ I$, since $g \in \mathcal{G}$. To show the converse, let F_0 be the mean value of

$$F = \Box f - \Big[\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H - 3S_3\Big]f$$

in M^n. Since $f \in \mathrm{Ker}\ I$, we have

$$\int_M g(F - F_0)\,dM = 0\,,$$

for all $g \in \mathcal{G}$. Now it is enough to prove that $F \equiv F_0$ which is similar to Proposition (2.7) in [4]. This completes the proof of Proposition 3.1. □

By direct computation, we can prove the following proposition.

Proposition 3.2. *Let W be a Killing vector field on $S_1^{n+1}(c)$, then $f = \langle W, N\rangle$ satisfies*

$$\Box f - \Big[\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H - 3S_3\Big]f = \mathrm{const}\,.$$

Now we can prove the following theorem.

Theorem 3.1. *Let $x : M^n \to S_1^{n+1}(c)$ be a space-like immersion with constant scalar curvature such that $\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H - 3S_3 = \lambda = \mathrm{const}$. If W is a Killing vector field on S_1^{n+1}, then x is stable if and only if $\lambda = \lambda_1$, the first eigenvalue of $\Box f$ on M^n.*

Proof. Since λ is an eigenvalue of $\Box$, we have either $\lambda = \lambda_1$ or $\lambda > \lambda_1$. In the first case, for any $f \in \mathcal{G}$,

$$I(f,f) = \int_M (f\Box f - \lambda f^2) \le (\lambda_1 - \lambda)\int_M f^2 = 0\,,$$

hence M is stable. In the latter case, choose f to be the first eigenfunction of the laplacian. Then $f \in \mathcal{G}$ and

$$I(f,f) = (\lambda_1 - \lambda)\int_M f^2 > 0$$

and therefore M is not stable. □

Theorem 3.2. *Let $\Sigma^n \subset S_1^{n+1}(c)$ be a geodesic sphere. Then Σ^n is stable.*

Proof. Choose $f : \Sigma \to R$ such that $\int_M f\,dM = 0$. Since Σ is umbilical, we have $\|B\|^2 = nH^2$ and

$$\Box f = (n-1)H\Delta f\,,$$

where Δf is the Laplacian of f in Σ. From the formula for the second variation of J, we have

$$\begin{aligned} J''(0)(f) = &-2(n-1)\int_\Sigma Hf\Delta f \\ &-2\int_\Sigma f^2\Big[\frac{1}{2}n^2(n-1)(c-R)H + cn(n-1)H - 3S_3\Big]\,. \end{aligned}$$

Since

$$\operatorname{tr} B^3 = nH\|B\|^2 - \frac{1}{2}n^2(n-1)H(c-R) + 3S_3\,,$$

by umbilicity, we have

$$-\frac{1}{2}n^2(n-1)H(c-R) + 3S_3 = \operatorname{tr} B^3 - nH\|B\|^2 = -n(n-1)H^3\,.$$

So by Stokes' theorem, we have

$$\begin{aligned} J''(0)(f) &= 2(n-1)H\int_\Sigma \big(\|\nabla f\|^2 - n(c+H^2)\big)f^2 \\ &\le 2(n-1)H\int_\Sigma \big(\lambda(\Sigma) - n(c+H^2)\big)f^2\,, \end{aligned}$$

where $\lambda(\Sigma)$ is the first eigenvalue of the Laplacian Δ in Σ. Since Σ is a sphere, $\lambda(\Sigma) = n(c+H^2)$. So $J''(0)(f) \le 0$, for all f such that $\int_\Sigma f\,dM = 0$, and Σ is stable. $\Box$

References

[1] Alencar, H., do Carmo, M. and Colares, A. G., *Stable hypersurfaces with constant scalar curvature*, Math. Z. **213** (1993), 117–131.

[2] Akutagawa, K., *On space-like hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), 13–19.

[3] Barbosa, J. L. and do Carmo, M., *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), 339–353.

[4] Barbosa, J. L., do Carmo, M. and Eschenburg, J., *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 123–138.

[5] Barbosa, J. L. and Oliker, V., *Stable spacelike hypersurfaces with constant mean curvature in Lorentz space*, Geometry and Global Analysis, Tohoku University, Sendai (1993), 161–164.

[6] Cheng, Q. M., *Complete space-like submanifolds in a de Sitter space with parallel mean curvature vector*, Math. Z. **206** (1991), 333–339.

[7] Cheng, S. Y. and Yau, S. T., *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195–204.

[8] Ramanathan, J., *Complete space-like hypersurfaces of constant mean curvature in the de Sitter space*, Indiana Univ. Math. J. **36** (1987), 349–359.

[9] Reilly, R. C., *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geom. **8** (1973), 465–477.

Department of Applied Mathematics, Dalian University of Technology
Dalian 116024, P. R. China
E-mail: xmliu@dlut.edu.cn