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A FUZZY VERSION OF TARSKI'S FIXPOINT THEOREM

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Abstract. A fuzzy version of Tarski's fixpoint Theorem for fuzzy monotone maps on nonempty fuzzy compete lattice is given.

1. Introduction

Let X be a nonempty set. A fuzzy set in X is a function of X in $[0,1]$. Fuzzy set theory is a powerful tool for modelling uncertainty and for processing vague or subjective information in mathematical models. In [9], Zadeh introduced the notion of fuzzy order and similarity. Recently, several authors studied the existence of fixed point in fuzzy setting, Heilpern [7], Hadzic [6], Fang [5] and Beg [1, 2, 3]. In fuzzy ordered sets, I. Beg [1] proved the existence of maximal fixed point of fuzzy monotone maps. The aim of this note is to give the following fuzzy version of Tarski's fixpoint Theorem [8]: suppose that (X, r) is a nonempty r-fuzzy complete lattice and $f : X \to X$ is a r-fuzzy monotone map. Then the set $\mathrm{Fix}(f)$ of all fixed points of f is a nonempty r-fuzzy complete lattice.

2. Preliminaries

In this note we shall use the following definition of order due to Claude Ponsard (see [4]).

Definition 2.1. Let X be a crisp set. A fuzzy order relation on X is a fuzzy subset R of $X \times X$ satisfying the following three properties

(i) for all $x \in X$, $r(x,x) \in [0,1]$ (f-reflexivity);

(ii) for all $x, y \in X$, $r(x,y) + r(y,x) > 1$ implies $x = y$ (f-antisymmetry);

(iii) for all $(x,y,z) \in X^3$, $[r(x,y) \geq r(y,x)$ and $r(y,z) \geq r(z,y)]$ implies $r(x,z) \geq r(z,x)$ (f-transitivity).

A nonempty set X with fuzzy order r defined on it, is called r-fuzzy ordered set. We denote it by (X, r). A r-fuzzy order is said to be total if for all $x \neq y$ we

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have either $r(x, y) > r(y, x)$ or $r(y, x) > r(x, y)$. A r-fuzzy ordered set on which the r-fuzzy order is total is called r-fuzzy chain.

Let A be a nonempty subset of X. We say that $x \in X$ is a r-upper bound of A if $r(y, x) \geq r(x, y)$ for all $y \in A$. A r-upper bound x of A with $x \in A$ is called a greatest element of A. An $x \in A$ is called a maximal element of A if there is no $y \neq x$ in A for which $r(x, y) \geq r(y, x)$. Similarly, we can define r-lower bound, minimal and least element of A. As usual, $\sup_r(A)$= the unique least element of r-upper bound of A (if it exists),

$\max_r(A)$= the unique greatest element of A (if it exists),

$\inf_r(A)$= the unique greatest element of r-lower bound of A (if it exists),

$\min_r(A)$= the unique least element of A (if it exists).

Definition 2.2. Let (X, r) be a nonempty r-fuzzy ordered set. The inverse fuzzy relation s of r is defined by $s(x, y) = r(y, x)$, for all $x, y \in X$.

Definition 2.3. Let (X, r) be a nonempty r-fuzzy ordered set. We say that (X, r) is a r-fuzzy complete lattice if every nonempty subset of X has a r-infimum and a r-supremum.

Let X be a r-fuzzy ordered set and let $f : X \to X$ be a map. We say that f is r-fuzzy monotone if for all $x, y \in X$ with $r(x, y) \geq r(y, x)$, then $r(f(x), f(y)) \geq r(f(y), f(x))$.

We denote the set of all fixed points of f by $\mathrm{Fix}(f)$.

3. The results

In this section, we establish a fuzzy version of Tarski's fixpoint Theorem [8]. More precisely, we show the following:

Theorem 3.1. *Let (X, r) be a nonempty r-fuzzy complete lattice and let $f : X \to X$ be a r-fuzzy monotone map. Then the set $\mathrm{Fix}(f)$ of all fixed points of f is a nonempty r-fuzzy complete lattice.*

In this section, we shall we need the three following technical lemmas which their proofs will be given in the Appendix.

Lemma 3.2. *Let X be a nonempty r-fuzzy ordered set and let E be a nonempty fuzzy ordered subset of X. If $\sup_r(E) = s$, then we have*

$$\{x \in X : r(s, x) = r(x, s)\} = \{s\} .$$

Lemma 3.3. *Let (X, r) be a nonempty r-fuzzy ordered set and let s be the inverse fuzzy relation of r. Then,*

(i) *The fuzzy relation s is a fuzzy order on X.*

(ii) *Every r-fuzzy monotone map $f : X \to X$ is also s-fuzzy monotone.*

(iii) *If a nonempty subset A of X has a r-infimum, then A has a s-supremum and $\inf_r(A) = \sup_s(A)$.*

(iv) *If a nonempty subset A of X has a r-supremum, then A has a s-infimum and $\inf_s(A) = \sup_r(A)$.*

(v) *If* (X, r) *is a fuzzy complete lattice, then* (X, s) *is also a fuzzy complete lattice.*

For starting the third Lemma, we have to introduce the following subset E of X by $x \in E$ if and only if $r(x, f(x)) \geq r(f(x), x)$ and $r(f(x), y) \geq r(y, f(x))$ for all $y \in A$, where A is a subset of $\mathrm{Fix}(f)$.

Lemma 3.4. *Let* (X, r) *be a nonempty* r*-fuzzy complete lattice and let* $f : X \to X$ *be a* r*-fuzzy monotone map. Let us suppose that* E *is defined as above and* $t = \sup_r(E)$*. Then* t *is a fixed point of* f*.*

In order to prove Theorem 3.1, we need the following proposition:

Proposition 3.5. *Let* (X, r) *be a nonempty* r*-fuzzy complete lattice and let* $f : X \to X$ *be a* r*-fuzzy monotone map. Then* f *has a greatest and least fixed points. Furthermore,*

$$\max_r(\mathrm{Fix}(f)) = \sup_r \{x \in X : r(x, f(x)) \geq r(f(x), x)\} ,$$

and

$$\min_r(\mathrm{Fix}(f)) = \inf_r \{x \in X : r(f(x), x) \geq r(x, f(x))\} .$$

Proof of Proposition 3.2. Let D be the fuzzy ordered subset defined by

$$D = \{x \in X : r(x, f(x)) \geq r(f(x), x)\} .$$

Since $\min_r(X) \in D$, so D is nonempty. Let d be the r-supremum of D.
Claim 1. The element d is the greatest fixed point of f. Indeed, as $d = \sup_r(D)$, then $r(x, d) \geq r(d, x)$ for all $x \in D$. Since f is r-fuzzy monotone, so $r(f(x), f(d)) \geq r(f(d), f(x))$, for all $x \in D$. We know that $r(x, f(x)) \geq r(f(x), x)$, for every $x \in D$. Then by fuzzy transitivity, we obtain $r(x, f(d)) \geq r(f(d), x)$, for all $x \in D$. Thus, $f(d)$ is a r-upper bound of D. On the other hand, d is the least r-upper bound of D. So,

$$r(d, f(d)) \geq r(f(d), d) . \tag{3.1}$$

From this and fuzzy monotonicity of f, we get

$$r(f(d), f(f(d))) \geq r(f(f(d)), f(d)) . \tag{3.2}$$

Hence, we get $f(d) \in D$. From this and as $d = \sup_r(D)$, then

$$r(f(d), d) \geq r(d, f(d)) . \tag{3.3}$$

By combining (3.1) and (3.3), we get $r(d, f(d)) = r(f(d), d)$. From Lemma 3.2, we conclude that we have $f(d) = d$. Now let $x \in \mathrm{Fix}(f)$. Then $x \in D$. So $\mathrm{Fix}(f) \subset D$. From this and as d is the r-supremum of D, then we deduce that d is a r-upper bound of $\mathrm{Fix}(f)$. Since $d \in \mathrm{Fix}(f)$. Therefore d is the greatest element of $\mathrm{Fix}(f)$.
Claim 2. The map f has a least fixed point. Let s be the fuzzy inverse order relation of r and let B be the following ordered subset of X defined by

$$B = \{x \in X : r(f(x), x) \geq r(x, f(x))\} .$$

Since $\min_r(X) \in B$, then $B \neq \emptyset$. On the other hand, by the definition of inverse fuzzy relation, we have

$$B = \{x \in X : s(x, f(x)) \geq s(f(x), x)\} \ .$$

By hypothesis, (X, r) is a nonempty fuzzy complete lattice, then from Lemma 3.3, (X, s) ia also a nonempty fuzzy complete lattice. Furthermore, f is s-fuzzy monotone. Then by Claim 1, f has a greatest fixed point l in (X, s) with

$$l = \sup_s \{x \in X : s(x, f(x)) \geq s(f(x), x)\} \ .$$

Thus l is a least fixed point of f in (X, r). By Lemma 3.3, we get

$$l = \inf_r \{x \in X : r(f(x), x) \geq r(x, f(x))\} \ . \qquad \square$$

Now we are able to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let X be a nonempty r-fuzzy complete lattice and $f : X \to X$ be a r-fuzzy monotone map.

First Step. We shall prove that every nonempty subset A of $\mathrm{Fix}(f)$ has a r-infimum in $(\mathrm{Fix}(f), r)$. Let E and F be the two following subsets of X defined by $x \in E$ if and only if

$$r(x, f(x)) \geq r(f(x), x) \quad \text{and} \quad r(f(x), y) \geq r(y, f(x))$$

for all $y \in A$, and

$$F = \{x \in \mathrm{Fix}(f) : r(x, y) \geq r(y, x) \quad \text{for all} \quad y \in A\} \ .$$

By Proposition 2.5, $\min_r(\mathrm{Fix}(f))$ exists in (X, r). Since $\min_r(Fix(f)) \in F$, then $F \neq \emptyset$. Let $m = \sup_r(F)$ and $t = \sup_r(E)$. We claim that the element m is the r-infimum of A in $(\mathrm{Fix}(f), r)$. Indeed, Since $F \subset E$, then $r(\sup_r(F), \sup_r(E)) \geq r(\sup_r(E), \sup_r(F))$. Thus $r(m, t) \geq r(t, m)$. On the other hand $t \in F$, hence $r(t, m) \geq r(t, m)$. It follows that we have $r(t, m) = r(m, t)$. From Lemma 3.2, we get $m = t$. By Lemma 3.4, t is a fixed point of f. Therefore A has a r-infimum in $\mathrm{Fix}(f)$.

Second Step. We shall prove that every nonempty subset A of $\mathrm{Fix}(f)$ has a r-supremum in $(\mathrm{Fix}(f), r)$. Let G be the following ordered subset of X defined by $x \in G$ if and only if

$$r(y, f(x)) \geq r(f(x), y)$$

for all $y \in A$, and

$$r(f(x), x) \geq r(x, f(x)) \, .$$

By Proposition 3.5, $\max_r(\mathrm{Fix}(f))$ exists in (X, r). As $\max_r(\mathrm{Fix}(f)) \in G$, then $G \neq \emptyset$ and $p = \inf_r(G)$ exists in (X, r). Let s be the fuzzy inverse order relation of r. Then we get, $x \in G$ if and only if

$$s(f(x), y) \geq s(y, f(x))$$

for all $y \in A$ and

$$s(x, f(x)) \geq s(f(x), x) .$$

We know by Lemma 3.3 that (X, s) is a nonempty fuzzy complete lattice. Moreover, f is s-fuzzy monotone and $p = \sup_s(G)$. From Lemma 3.4, we get $f(p) = p$. On the other hand, by the first step above, p is the s-supremum of A. Therefore, we deduce by Lemma 3.3 that the element p is the r-infimum of A in $(\mathrm{Fix}(f), r)$. □

4. Appendix

In this section, we give the proofs of Lemmas 3.2, 3.3 and 3.4.

Proof of Lemma 3.2. Let $s = \sup_r(E)$ and let $x \in X$ such that $r(s, x) = r(x, s)$.

Claim 1. The element x is a r-upper bound of E. Indeed, if $a \in E$, then $r(a, s) \geq r(s, a)$. Since $r(s, x) = r(x, s)$, then by fuzzy transitivity we get $r(a, s) \geq r(s, a)$ for all $a \in E$ and our claim is proved.

Claim 2. The element x is a least r-upper bound of E. Indeed, if b is a r-upper bound of E, then $r(s, b) \geq r(b, s)$. As $r(s, x) = r(x, s)$, then $r(x, b) \geq r(b, x)$. It follows that x is a least r-upper bound of E. Hence x is a r-supremum of E.

By Claims 1 and 2, we deduce that the element x is a r-supremum of A. From hypothesis, the r-supremum of A is unique, therefore $x = s$. □

Proof of Lemma 3.3. (i) For all $x \in X$, we have $s(x, x) = r(x, x) \in [0, 1]$. Let $x, y \in X$ such that $s(x, y) + s(y, x) > 1$. Since $r(x, y) + r(y, x) = s(x, y) + s(y, x) > 1$, so $r(x, y) + r(y, x) > 1$. By r-fuzzy antisymmetry, we deduce that we have $x = y$. Let $x, y, z \in X$ with $s(x, y) \geq s(y, x)$ and $s(y, z) \geq s(z, y)$. Then we have $r(z, y) \geq r(y, z)$ and $r(y, x) \geq r(x, y)$. By r-fuzzy transitivity, we obtain $r(z, x) \geq r(x, z)$. Therefore we get $s(x, z) \geq s(z, x)$. Thus the fuzzy relation s is a fuzzy order on X.

(ii) Let $x, y \in X$ with $s(x, y) \geq s(y, x)$. Then we get $r(y, x) \geq r(x, y)$. Since f is r-fuzzy monotone, hence $r(f(y), f(x)) \geq r(f(x), f(y))$. Therefore $s(f(x), f(y)) \geq s(f(y), f(x))$. Thus the map f is s-fuzzy monotone.

(iii) Let $m = \sup_r(A)$. Then $r(x, m) \geq r(m, x)$, for all $x \in A$. So $s(m, x) \geq s(x, m)$, for all $x \in A$. Thus m is a s-lower bound of A. Now let t be another s-lower bound of A. Hence $s(t, x) \geq s(x, t)$, for all $x \in A$. Then $r(x, t) \geq r(t, x)$. Thus t is a r-upper bound of A. From this and as $m = \sup_r(A)$, we deduce that we have $r(m, t) \geq r(t, m)$. So $s(t, m) \geq s(m, t)$. Thus m is a greatest s-lower bound of A. Suppose that p is another greatest s-lower bound of A. By using a simlar proof as above we deduce that p is a least r-upper bound of A. By hypothesis, the r-supremum of A is unique. Therefore, we conclude that $p = m$. Thus $m = \inf_s(A)$.

(iv) Since s is the inverse fuzzy relation of r, then r is the inverse fuzzy relation of s. By (iii), we get $\inf_s(A) = \sup_r(A)$.

(v) Let A be a nonempty set in X. Then A has a r-infimum and a r-supremum. From (iii) and (iv), we deduce that A has a s-infimum and a s-supremum. Thus (X, s) is a nonempty fuzzy complete lattice. □

Proof of Lemma 3.4. Let E be the subset of X defined by $x \in E$ if and only if

$$r(x, f(x)) \geq r(f(x), x) \quad \text{and} \quad r(f(x), y) \geq r(y, f(x))$$

for all $y \in A$.

By Proposition 2.5, $\min_r(\mathrm{Fix}(f))$ exists in (X, r). As $\min_r(\mathrm{Fix}(f)) \in E$, then $E \neq \emptyset$ and $t = \sup_r(E)$ exists in X. We claim that we have: $t = f(t)$. Indeed, since for all $x \in E$, we have $r(x, t) \geq r(t, x)$ and as f is r-fuzzy monotone, then

$$r(f(x), f(t)) \geq r(f(t), f(x))\,, \quad \text{for all} \quad x \in E\,. \tag{4.1}$$

By definition, we have

$$r(x, f(x)) \geq r(f(x), x)\,, \quad \text{for all} \quad x \in E\,. \tag{4.2}$$

From (4.1) and (4.2) and fuzzy-transitivity, we get $r(x, f(t)) \geq r(f(t), x)$ for all $x \in E$. Thus $f(t)$ is a r-upper bound of E. From this and as $t = \sup_r(E)$ so

$$r(t, f(t)) \geq r(t, f(t))\,. \tag{4.3}$$

From (4.3) and fuzzy monotonicity of f, we obtain

$$r(f(t), f(f(t))) \geq r(f(f(t)), f(t))\,. \tag{4.4}$$

Now let $y \in A$. Then for all $x \in E$, we have $r(f(x), y) \geq r(y, f(x))$. By using (4.2) and r-fuzzy transitivity, we obtain $r(x, y) \geq r(y, x)$ for all $x \in E$. Thus every element of A is a r-upper bound of E. Since t is the least r-upper bound of E, then we get $r(t, y) \geq r(y, t)$, for all $y \in A$. Then by fuzzy monotonicity of f, we deduce that we have

$$r(f(t), y) \geq r(y, t)\,, \quad \text{for all} \quad y \in A\,. \tag{4.5}$$

Combining (4.4) and (4.5) we get $f(t) \in E$. On the other hand the element t is the r-supremum of E, then we deduce that we have

$$r(f(t), t) \geq r(t, f(t))\,. \tag{4.6}$$

By using (4.3) and (4.6) we deduce that we have $r(f(t), t) = r(t, f(t))$. Therefore by Lemma 3.2, we conclude that we have $f(t) = t$. □

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