ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 345 – 353

# FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS IN MODULAR SPACES

POOM KUMAM

Abstract. In this paper, we extend several concepts from geometry of Banach spaces to modular spaces. With a careful generalization, we can cover all corresponding results in the former setting. Main result we prove says that if $\rho$ is a convex, $\rho$-complete modular space satisfying the Fatou property and $\rho_r$-uniformly convex for all $r > 0$, C a convex, $\rho$-closed, $\rho$-bounded subset of $X_\rho$, $T : C \to C$ a $\rho$-nonexpansive mapping, then $T$ has a fixed point.

## 1. Introduction

The theory of modular spaces was initiated by Nakano [15] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [14] in 1959. It is well known that one of the standard proof of Banach's fixed point theorem is based on Cantor's theorem in complete metric spaces [5, 6]. To this end, using some convenient constants in the contraction assumption, we present a generalization of Banach's fixed point theorem in some classes of modular spaces.

In this paper, we extend many concepts and results in normed spaces to modular spaces.

## 2. Preliminaries

We start by reviewing some basic facts about modular spaces as formulated by Musielak and Orlicz [14]. For more details the reader is refered to [7, 9, 10] and [13].

**Definition 2.1** (cf. [7])**.** Let $X$ be an arbitrary vector space.

(a) A function $\rho : X \to [0, \infty]$ is called a *modular* on $X$ if for arbitrary $x, y$ in $X$,

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $\mid \alpha \mid= 1$, and

---

2000 *Mathematics Subject Classification*: 46B20, 46E30, 47H10.

*Key words and phrases*: fixed point, modular spaces, $\rho$-nonexpansive mapping, $\rho$-normal structure, $\rho$-uniform normal structure, $\rho_r$-uniformly convex.

Received November 7, 2002.

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

(b) If (iii) is replaced by (iii)'$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, e say that $\rho$ is a *convex modular*.

(c) A modular $\rho$ defines a corresponding *modular space*, i.e. the vector space $X_\rho$ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

$X_\rho$ is a linear subspace of $X$.

In general the modular $\rho$ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an $F$-norm (see [13]).

The modular space $X_\rho$ can be equipped with an $F$-norm (see [13]) defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \leq \alpha\right\}.$$

Namely, if $\rho$ is convex, then the functional $\||x\||_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}$ is a norm in $X_\rho$ which is equivalent to the $F$-norm $\|.\|_\rho$.

**Definition 2.2** (cf. [7, 8])**.** Let $X_\rho$ be a modular space.

(a) A sequence $(x_n) \subset X_\rho$ is said to be *$\rho$-convergent* to $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) A sequence $(x_n)$ is called *$\rho$-Cauchy* whenever $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(c) The modular $\rho$ is called *$\rho$-complete* if any $\rho$-Cauchy sequence is $\rho$-convergent.

(d) A subset $B \subset X_\rho$ is called *$\rho$-closed* if for any sequence $(x_n) \subset B$ $\rho$-convergent to $x \in X_\rho$, we have $x \in B$.

(e) A $\rho$-closed subset $B \subset X_\rho$ is called *$\rho$-compact* if any sequence $(x_n) \subset B$ has a $\rho$-convergent subsequence.

(f) $\rho$ is said to satisfy the *$\Delta_2$-condition* if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

(g) We say that $\rho$ has the *Fatou property* if $\rho(x) \leq \liminf_n \rho(x_n)$ whenever $x_n \xrightarrow{\rho} x$.

(h) A subset $B \subset X_\rho$ is said to be *$\rho$-bounded* if

$$\operatorname{diam}_\rho(B) < \infty,$$

where $\operatorname{diam}_\rho(B) = \sup\{\rho(x - y); x, y \in B\}$ is called the *$\rho$-diameter* of $B$.

(i) Define the *$\rho$-distance* between $x \in X_\rho$ and $B \subset X_\rho$ as

$$\operatorname{dis}_\rho(x, B) = \inf\{\rho(x - y); y \in B\}.$$

(j) Define the *$\rho$-Ball*, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius $r$ as

$$B_\rho(x, r) = \{y \in X_\rho; \rho(x - y) \leq r\}.$$

Let $(X, \|.\|)$ be a normed space. Then $\rho(x) = \|x\|$ is a convex modular on $X$. One can check that $\rho$-balls are $\rho$-closed if and only if $\rho$ has the Fatou property (cf. [8]).

**Example 2.3.**
(1) The *Orlicz modular* is defined for every measurable real function $f$ by the formula

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|)\, dm(t)\,,$$

where $m$ denotes the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \to [0, \infty)$ is continuous. We also assume that $\varphi(u) = 0$ iff $u = 0$ and $\varphi(t) \to \infty$ as $n \to \infty$. The modular space induced by the Orlicz modular $\rho_\varphi$ is called the *Orlicz space* $L^\varphi$.
(2) The *Musielak-Orlicz modular spaces* (see. [17]). Let

$$\rho(f) = \int_{\Omega} \varphi(\omega, f(\omega)) d\mu(\omega)\,,$$

where $\mu$ is a $\sigma$-finite measure on $\Omega$, and $\varphi : \Omega \times \mathbb{R} \to [0, \infty)$ satisfy the following:
(i) $\varphi(\omega, u)$ is a continuous even function of $u$ which is nondecreasing for $u > 0$, such that $\varphi(\omega, 0) = 0, \varphi(\omega, u) > 0$ for $u \neq 0$, and $\varphi(\omega, u) \to \infty$ as $n \to \infty$.
(ii) $\varphi(\omega, u)$ is a measurable function of $\omega$ for each $u \in \mathbb{R}$.

The corresponding modular space is called the *Musielak-Orlicz spaces*, and is denoted by $L^\varphi$.

**Definition 2.4** (cf. [8])**.** A modular space $X_\rho$ is said to have *$\rho$-normal structure* if for any nonempty $\rho$-bounded $\rho$-closed convex subset $C$ of $X_\rho$ not reduced to a one point, there exists a point $x \in C$ such that

$$r_\rho(x, C) := \sup\{\rho(x - y); y \in C\} < \operatorname{diam}_\rho(C)\,.$$

A modular space $X_\rho$ is said to have *$\rho$-uniformly normal structure* if there exists a constant $c \in (0, 1)$ such that for any subset $C$ as above, there exists $x \in C$ such that

$$r_\rho(x, C) < c \operatorname{diam}_\rho(C)\,.$$

Clearly $\rho$-uniformly normal structure is $\rho$-normal structure.

Let $X_\rho$ be a modular space and let $C$ be a nonempty $\rho$-bounded and $\rho$-closed convex subset $C$ of $X_\rho$. We will say that $C$ has the *fixed point property (fpp)* if every $\rho$-nonexpansive selfmap defined on $C$ (i.e.,$T : C \to C, \rho(T(x) - T(y)) \leq \rho(x - y)$ for every $x, y \in C$) has a fixed point, that is, there eists $x \in C$ such that $T(x) = x$. Also, a modular space $X_\rho$ is said to have the *fixed point property (fpp)* if every nonempty $\rho$-bounded $\rho$-closed convex subset of $X_\rho$ has the fixed point property.

In Banach spaces, when we think about reflexivity automatically the dual space is present in our taught. But in modular spaces, it is very hard to conceive the dual space. To circumvent the problem, we use some characterization of reflexivity.

**Theorem 2.5** (Smulian 1939, cf. [12])**.** *A normed space $X$ is reflexive if and only if $\bigcap_n C_n \neq \emptyset$ whenever $(C_n)$ is a sequence of nonempty, closed bounded and convex subsets of $X$ such that $C_n \supseteq C_{n+1}$ for each $n \in \mathbb{N}$.*

**Definition 2.6** (cf. [8])**.** Let $X_\rho$ be a modular space. We will say that $X_\rho$ or $\rho$ satisfies the *property (R)* if every decreasing sequence of nonempty $\rho$-closed and $\rho$-bounded convex subsets of $X_\rho$, has a nonempty intersection.

The following theorem is known.

**Theorem 2.7** (cf. [8]). *Let $X_\rho$ be a $\rho$-complete modular space. Assume that $\rho$ is convex and satisfies the Fatou property. Moreover, assume that $X_\rho$ has the $\rho$-normal structure and has the property $(R)$ and $C$ is any $\rho$-closed $\rho$-bounded convex nonempty subset of $X_\rho$. Then any $\rho$-nonexpansive mapping $T : C \to C$ has a fixed point in $C$.*

## 3. Results

We start this chapter with generalizations as well as their corresponding results of uniform convexity and normal structure coefficients in modular spaces.

**Definition 3.1.** For $r > 0$, a modular space $X_\rho$ is said to be *$\rho_r$-uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X_\rho$, the conditions $\rho(x) \le r, \rho(y) \le r$ and $\rho(x - y) \ge r\varepsilon$ imply

$$\rho\Big(\frac{x+y}{2}\Big) \le (1-\delta)r\,.$$

**Definition 3.2.** Let $X_\rho$ be a Modular space. For any $\varepsilon \ge 0$ and $r > 0$, the *modulus of $\rho_r$-uniform convexity* of $X_\rho$ is defined by

$$\delta_\rho(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : \rho(x) \le r, \rho(y) \le r, \rho(x-y) \ge r\varepsilon\right\}.$$

**Definition 3.3.** The normal structure coefficient of $X_\rho$ is the number

$$N(X_\rho) = \inf\Big\{\frac{\operatorname{diam}_\rho(C)}{R_\rho(C)} : C \subset X_\rho\ C \text{ is } \rho\text{-closed convex, } \rho\text{-bounded and } \operatorname{diam}_\rho(C) > 0\Big\},$$

where $R_\rho(C) := \inf\{r_\rho(x, C) : x \in C\}$ which is called the *$\rho$-Chebyshev radius* of $C$ (cf. [7]).

**Remark 3.4.**

(1) It is not hard to show that $R_\rho(C) \ne 0$. Indeed, suppose $R_\rho(C) = 0$ and let, $x_0, y_0 \in C$ be such that $x_0 \ne y_0$. Since $R_\rho(C) = \inf_{y\in C} r_\rho(x, C) = 0$, so there exists a sequence $(x_n)$ in $C$ such that $\lim_{n\to\infty} r_\rho(x_n, C) = 0$. Thus

$$\rho\Big(\frac{x_0 - y_0}{2}\Big) = \rho\left(\frac{(x_0 - x_n) + (x_n - y_0)}{2}\right) \le \rho(x_0 - x_n) + \rho(x_n - y_0) \to 0$$

as $n \to \infty$. Therefore $x_0 = y_0$, a contradiction.

(2) For any $x \in C$ we have $R_\rho(C) \le r_\rho(x, C) \le \operatorname{diam}_\rho(C)$.

(3) It is obvious form the definition that $X_\rho$ has $\rho$-uniform normal structure if and only if $N(X_\rho) > 1$ (see [11]).

**Lemma 3.5.** *Let $r > 0$. A modular space $X_\rho$ is $\rho_r$-uniformly convex if and only if $\delta_\rho(r,\varepsilon) > 0$ for all $\varepsilon > 0$.*

**Proof.** Let $\varepsilon > 0$. If $X_\rho$ is $\rho_r$-uniformly convex, then there exists $\delta > 0$ such that for any $x, y \in X_\rho$ with $\rho(x) \le r, \rho(y) \le r$, and $\rho(x - y) \ge r\varepsilon$. we have $\rho\big(\frac{x+y}{2}\big) \le (1-\delta)r$. Thus, for these $x$ and $y$, $\delta \le 1 - \frac{1}{r}\rho\big(\frac{x+y}{2}\big)$. Hence $\delta_\rho(r,\varepsilon) \ge$

$\delta > 0$. Conversely, suppose $\delta_\rho(r,\varepsilon) \geq \delta > 0$ for some $\varepsilon > 0$ and $\delta > 0$. Take any $x, y \in X_\rho$ such that $\rho(x) \leq r, \rho(y) \leq r$ and $\rho(x-y) \geq r\varepsilon$. By definition of $\delta_\rho$, we get $\delta_\rho(r,\varepsilon) \leq 1 - \frac{1}{r}\rho\big(\frac{x+y}{2}\big)$. Hence

$$\frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) \leq 1 - \delta(r,\varepsilon) \leq 1 - \delta\,.$$

Therefore $X_\rho$ is $\rho_r$-uniformly convex. □

**Lemma 3.6.** *The modulus $\delta_\rho(r,.)$ of uniform convexity of $X_\rho$ is increasing on $[0,\infty)$.*

**Proof.** Let $r > 0$ and $\varepsilon_1 > \varepsilon_2 \geq 0$. Let $x, y \in X_\rho$ be such that $\rho(x) \leq r$ and $\rho(y) \leq r$. If $\rho(x-y) \geq \varepsilon_1 r$, then $\rho(x-y) \geq \varepsilon_2 r$. This show that

$$\left\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq r\varepsilon_1\right\}$$
$$\subseteq \left\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq r\varepsilon_2\right\}.$$

This implies that $\delta_\rho(r,\varepsilon_1) \geq \delta_\rho(r,\varepsilon_2)$. □

**Theorem 3.7.** *If the modulus $\delta_\rho$ of convexity of a modular space $X_\rho$ satisfies $\delta_\rho(d,\varepsilon) > 0$ for all $d, \varepsilon > 0$, then $X_\rho$ has $\rho$-normal structure.*

**Proof.** Let $C$ be a nonempty $\rho$-bounded $\rho$-closed convex subset of $X_\rho$ with $\operatorname{diam}_\rho(C) = d > 0$. Let $\varepsilon \in (0,1)$ there exist $x, y \in C$ such that

$$\rho(x-y) \geq d\varepsilon\,.$$

Let $z = \frac{x+y}{2}$ and $w \in C$. Thus, $z \in C$, $\rho(w-x) \leq d, \rho(w-y) \leq d$ and $\rho((w-x)-(w-y)) = \rho(x-y) \geq d\varepsilon$.
Consequently,

$$\rho\left(w - \Big(\frac{x+y}{2}\Big)\right) = \rho\left(\frac{(w-x)+(w-y)}{2}\right) \leq (1 - \delta_\rho(d,\varepsilon))d\,.$$

Hence

$$\sup_{w\in C} \rho(w-z) \leq \big(1 - \delta_\rho(d,\varepsilon)\big)d\,.$$

Since $\delta_\rho(d,\varepsilon) > 0$, we get

$$\sup_{w\in C} \rho(w-z) < d = \operatorname{diam}_\rho(C)\,.$$

Since this is true for any $C$, this proves that $X_\rho$ has $\rho$-normal structure.

Lemma 3.5 and Theorem 3.7 give us immediately

**Corollary 3.8.** *For a modular space $X_\rho$, if $X_\rho$ is $\rho_r$-uniformly convex for all $r > 0$, then $X_\rho$ has $\rho$-normally structure.*

**Corollary 3.9** (cf. [4])**.** *Closed bounded convex subsets of uniformly convex Banach spaces have normal structure.*

**Theorem 3.10.** *Let $X_\rho$ be a $\rho$-complete modular space. If $\rho$ is convex and satisfies the Fatou property and $X_\rho$ is $\rho_r$-uniformly convex for all $r > 0$, then $X_\rho$ has the property $(R)$.*

**Proof.** Let $(C_n)$ be a decreasing sequence of $\rho$-bounded, $\rho$-closed nonempty convex subsets of $X_\rho$, $z \in X_\rho$ which does not belong to $C_1$ and

$$r = \lim_{n\to\infty} \operatorname{dis}_\rho(z, C_n)\,.$$

Define $D_n = C_n \cap B_\rho(z, r)$ and let $d_n$ be the diameter of $D_n$. By the Fatou property of $\rho$, $(D_n)$ is a decreasing sequence of nonempty $\rho$-bounded, $\rho$-closed convex subsets of $X_\rho$ because $B_\rho(z, r)$ is then a $\rho$-closed set (see [8]).

Let $r_n$ be a sequence of positive number that decreases to zero and $d_n - r_n > 0$ for all $n$. There exist $x, y \in D_n$ such that $\rho(x - y) \geq d_n - r_n$. Thus, by the definition of $\delta_\rho(r, \frac{d_n - r_n}{r})$, we have

$$\rho(z - \frac{x+y}{2}) = \rho\left(\frac{(z-x)+(z-y)}{2}\right) \leq \left(1 - \delta_\rho\left(r, \frac{d_n - r_n}{r}\right)\right) r\,.$$

Hence

$$(*) \qquad \frac{1}{r}\operatorname{dis}_\rho(z, C_n) \leq \frac{1}{r}\rho\left(z - \frac{x+y}{2}\right) \leq 1 - \delta_\rho\left(r, \frac{d_n - r_n}{r}\right)\,.$$

Put $d = \lim_{n\to\infty} d_n$ and $a_n = d_n - \frac{1}{n}$, and consider two cases.

**Case 1** ( $a_n \geq d$, for all $n$ large enough). By $\delta_\rho$ being increasing and $(*)$, we have for all $n$ large enough,

$$\frac{1}{r}\operatorname{dis}_\rho(z, C_n) \leq 1 - \delta_\rho\left(r, \frac{a_n}{r}\right) \leq 1 - \delta_\rho\left(r, \frac{d}{r}\right)\,.$$

Letting $n \to \infty$, we get

$$1 \leq 1 - \delta_\rho\left(r, \frac{d}{r}\right)\,,$$

which implies that $\delta_\rho(r, \frac{d}{r}) = 0$. By $\rho_r$-uniform convexity of $X_\rho$ and Lemma 3.1.6 we have $\delta_\rho(r, \varepsilon) > 0$ for all $\varepsilon > 0$, whence $d = 0$.

**Case 2** ($0 < a_n < d$, for infinitely many $n$). There exists a subsequence $(a_{n'})$ such that $a_{n'} \nearrow d$, whence the limit $\lim_{n'\to\infty} \delta_\rho(r, \frac{a_{n'}}{r})$ exists and by $(*)$, we have

$$1 \leq 1 - \lim_{n'\to\infty} \delta_\rho\left(r, \frac{a_{n'}}{r}\right)\,.$$

Consequently, $\lim_{n'\to\infty} \delta_\rho(r, \frac{a_{n'}}{r}) = 0$. Since $a_{n'} \nearrow d$ and $\delta_\rho(r, \varepsilon) > 0$ for all $\varepsilon > 0$, we have $d = \lim_{n\to\infty} d_n = 0$ as well. Thus, there exists a $\rho$-Cauchy sequence $(x_n)$, where $x_n \in D_n$ for each $n$. Since $X_\rho$ is $\rho$-complete, $(x_n)\rho$-converges to some $x_0 \in X_\rho$. Using the $\rho$-closeness of $D_n$, we deduce that $x_0 \in D_n$ for all $n \geq 1$. This implies that $\cap_{n\in\mathbb{N}} D_n \neq \emptyset$ and so $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$ as well. The proof is therefore complete. $\square$

**Corollary 3.11** (cf. [4]). *Let $X_\rho$ be a $\rho$-complete modular space with $\rho$ convex and satisfying the Fatou property. If $X_\rho$ is $\rho_r$-uniformly convex for all $r > 0$, then $X_\rho$ has the fixed point property.*

**Proof.** By Corollary 3.8 and Theorem 3.10, $X_\rho$ has $\rho$-normal structure and property $(R)$. Consequently, Theorem 2.7 can be applied to conclude that $X_\rho$ has the fixed point property. □

**Corollary 3.12** (cf. [4]). *If $C$ is a nonempty closed bounded convex subset of a uniformly convex Banach space, then every nonexpansive mapping $T : C \to C$ has a fixed point in $C$.*

**Theorem 3.13.** *Let $X_\rho$ be a modular space with modulus of convexity $\delta_\rho(1, \varepsilon) \neq 1$ for some $\varepsilon \in (0, 1)$. If we assume that $\rho(\alpha x) = \alpha\rho(x)$ for all $\alpha > 0$, then*

$$N(X_\rho) \geq \frac{1}{1 - \delta_\rho(1, \varepsilon)}.$$

**Proof.** Let $C$ be a $\rho$-closed, $\rho$-bounded convex subset of $X_\rho$ with $\operatorname{diam}_\rho(C) = d > 0$. Since $\varepsilon \in (0, 1)$, there exist $x, y \in C$ such that

$$\rho(x - y) \geq d\varepsilon\,.$$

Let $z = \frac{x+y}{2} \in C$ and $w \in C$. Then $\rho(\frac{w-x}{d}) = \frac{1}{d}\rho(w - x) \leq 1, \rho(\frac{w-y}{d}) = \frac{1}{d}\rho(w - y) \leq 1$, and

$$\rho\left(\Big(\frac{w-x}{d}\Big) - \Big(\frac{w-y}{d}\Big)\right) = \frac{1}{d}\rho(x - y) \geq \varepsilon\,.$$

By the definition of $\delta_\rho(1, \varepsilon)$, we obtain

$$\frac{1}{d}\rho\Big(w - \frac{x+y}{2}\Big) = \frac{1}{d}\rho\left(\frac{(w - x) + (w - y)}{2}\right) \leq 1 - \delta_\rho(1, \varepsilon)\,.$$

Hence it follows that

$$R_\rho(C) \leq \sup_{w \in K} \rho(z - w) \leq d\big(1 - \delta_\rho(1, \varepsilon)\big)\,.$$

Consequently,

$$\frac{\operatorname{diam}_\rho(C)}{R_\rho(C)} \geq \frac{1}{1 - \delta_\rho(1, \varepsilon)}\,.$$

Therefore

$$N(X_\rho) \geq \frac{1}{1 - \delta_\rho(1, \varepsilon)}\,.$$

□

**Remark 3.14.** If we assume that in Colloray3.8 $\rho(\alpha x) = \alpha\rho(x)$ for all $\alpha > 0$, then $X_\rho$ will have $\rho$-uniformly normal structure.

**Corollary 3.15.** *If $X_\rho$ is a modular space with the modulus of convexity $\delta_\rho(1, \varepsilon) \in (0, 1)$ for some $\varepsilon \in (0, 1)$, then $X_\rho$ has $\rho$-uniformly normal structure.*

**Proof.** By Theorem 3.13 we have $N(X_\rho) > 1$. Thus, by Remarks 3.4 (3), $X_\rho$ has $\rho$-uniformly normal structure.

**Corollary 3.16.** *If $X$ is a Banach space space with modulus of convexity $\delta_X(\varepsilon) \in (0,1)$ for some $\varepsilon \in (0,1)$ and we put $\rho(x) = \|x\|$, then we get that $X$ has uniformly normal structure.*

Corollary 3.16 strongly improves [1] which states that any uniformly convex Banach space has uniformly normal structure.

Note that a Banach space $X$ is uniformly convex if and only if its modulus of convexity satisfies $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$ (see [5]).

**Acknowlegement.** The author is grateful to Prof. Sompong Dhompongsa for many helpful comments and suggestions. The author also wishes thanks to an anonymous referee for his/her suggestions which led to substantial improvements of this paper. Last but not least, I would like to thank Prof. Henryk Hudzik to his English suggestions and correction.

## References

[1] Aksoy, A. G. and Khamsi, M. A., *Nonstandard methods in fixed point theory*, Spinger-Verlag, Heidelberg, New York 1990.

[2] Ayerbe Toledano, J. M., Dominguez Benavides, T., and López Acedo, G., *Measures of noncompactness in metric fixed point theory: Advances and Applications Topics in metric fixed point theory*, Birkhäuser-Verlag, Basel, **99** (1997).

[3] Chen, S., Khamsi, M. A. and Kozlowski, W. M., *Some geometrical properties and fixed point theorems in Orlicz modular spaces*, J. Math. Anal. Appl. **155** No. 2 (1991), 393–412.

[4] Dominguez Benavides, T., Khamsi, M. A. and Samadi, S., *Uniformly Lipschitzian mappings in modular function spaces*, Nonlinear Analysis **40** No. 2 (2001), 267–278.

[5] Goebel, K. and Kirk, W. A., *Topic in metric fixed point theorem*, Cambridge University Press, Cambridge 1990.

[6] Goebel, K. and Reich, S., *Uniform convexity, Hyperbolic geometry, and nonexpansive mappings*, Monographs textbooks in pure and applied mathematics, New York and Basel, **83** 1984.

[7] Khamsi, M. A., *Fixed point theory in modular function spacesm*, Recent Advances on Metric Fixed Point Theorem, Universidad de Sivilla, Sivilla **No. 8** (1996), 31–58.

[8] Khamsi, M. A., *Uniform noncompact convexity, fixed point property in modular spaces*, Math. Japonica **41** (1) (1994), 1–6.

[9] Khamsi, M. A., *A convexity property in modular function spaces*, Math. Japonica **44**, No. 2 (1990).

[10] Khamsi, M. A., Kozlowski, W. M. and Reich, S., *Fixed point property in modular function spaces*, Nonlinear Analysis, **14**, No. 11 (1990), 935–953.

[11] Kumam, P., *Fixed Point Property in Modular Spaces*, Master Thesis, Chiang Mai University (2002), Thailand.

[12] Megginson, R. E., *An introduction to Banach space theory*, Graduate Text in Math. Springer-Verlag, New York **183** (1998).

[13] Musielak, J., *Orlicz spaces and Modular spaces*, Lecture Notes in Math., Springer-Verlag, Berlin, Heidelberg, New York **1034** (1983).

[14] Musielak, J. and Orlicz, W., *On Modular spaces*, Studia Math. **18** (1959), 591–597.

[15] Nakano, H., *Modular semi-ordered spaces*, Tokyo, (1950).

Department of Mathematics, Faculty of Science
King Mongkut's University of Technology Thonburi
Bangkok 10140, Thailand
*E-mail*: pooom.kum@kmutt.ac.th