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PAIRWISE BOREL AND BAIRE MEASURES IN BISPACES

PRATULANANDA DAS AND AMAR KUMAR BANERJEE

Abstract. In this paper we continue the study of the concepts of pairwise Borel and Baire measures in a bispace, recently introduced in [10]. We investigate some of its consequences including the problem of a pairwise regular Borel extension of a pairwise Baire measure.

1. Introduction

One of the important generalizations of the notion of a topological space is that of σ-space (further, "space"; A. D. Alexandroff [1]). In this paper we will consider a related concept of a bispace as introduced in [10].

Integration in a locally compact space depend essentially on the regularity of measures. If one wishes to study the theory of integration in bitopological spaces [7] or more generally in bispaces [10], the foremost necessity is the considerations of (pairwise) regularity of Borel and Baire measures. Following Polexe [12], Lahiri and Das ([8], [9]) have recently developed the theory of Borel and Baire measures in a bitopological space [7] where many of the results have been proved under two very strong additional assumptions of the bitopological space to be pairwise compact and pairwise Hausdorff. In this paper we modify the methods and do the same in a more general structure of a bispace [10] without these additional assumptions. As a result our nature of study does not appear to be analogous, particularly in respect of the problem of pairwise regular Borel extension of a pairwise Baire measure.

2. Preliminaries

Definition 1 ([1])**.** An Alexandroff space (or a σ-space, briefly space) is a set X together with a system F of subsets of X satisfying the following axioms:

(1) The intersection of a countable number of sets from F is a set in F.
(2) The union of a finite number of sets from F is a set in F.
(3) φ and X are in F.

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Sets of F are called closed sets. Their complementary sets are called open. The collection of all such open set will sometimes be denoted by τ and the space by (X, τ).

Note 1. In general τ is not a topology as can be easily seen by taking $X = R$ and τ as the collection of all F_σ-sets in R.

Throughout the paper by a space we shall mean an Alexandroff space.

Definition 2 ([1])**.** With every set $\bar{M}$ of (X, τ), we associate its closure $\bar{M}$, the intersection of all closed sets containing M.

Note that $\bar{M}$ is not necessarily closed.

Definition 3 ([1])**.** A space (or a set) is called bicompact if every open cover of it has a finite subcover.

Definition 4 ([10])**.** Let X be a non empty set. If τ_1 and τ_2 be two collections of subsets of X such that (X, τ_1) and (X, τ_2) are two spaces then X is called a bispace and is denoted by (X, τ_1, τ_2).

Many examples of bispaces can be seen in [10].

Definition 5 ([7])**.** A set X on which are defined two arbitrary topologies $\mathbf{P}$, $\mathbf{Q}$ is called a bitopological space and is denoted by $(X, \mathbf{P}, \mathbf{Q})$.

Note 2. If τ_1 and τ_2 are topologies then a bispace reduces to a bitopological space.

Definition 6 ([10])**.** (X, τ_1, τ_2) is said to be pairwise Hausdorff if for any two distinct points x and y of X, there exist $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \in V$, $U \cap V = \varphi$.

Definition 7 ([10])**.** A cover $\mathbf{B}$ of (X, τ_1, τ_2) is said to be pairwise open if $B \subset \tau_1 \cup \tau_2$ and $\mathbf{B}$ contains at least one non empty member from each of τ_1 and τ_2.

Definition 8 ([10])**.** (X, τ_1, τ_2) is said to be pairwise bicompact if every pairwise open cover of it has a finite subcover.

Definition 9 ([10])**.** Let (X, τ_1, τ_2) be a bispace. τ_1 is called locally bicompact with respect to τ_2 if for each $x \in X$, there is a τ_1-open set G containing x such that $\tau_2 - \mathrm{cl}\,(G)$ is pairwise bicompact. If both τ_1 and τ_2 are locally bicompact with respect to each other then (X, τ_1, τ_2) is called pairwise locally bicompact.

Definition 10 ([10])**.** In (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if for any $x \in X$ and a τ_1-closed set F not containing x, there exist $U \in \tau_1$, and $V \in \tau_2$ such that $x \in U$, $F \subset V$, $U \cap V = \varphi$. (X, τ_1, τ_2) is called pairwise regular if τ_1 and τ_2 both are regular with respect to each other.

We now introduce the following definition.

Definition 11. In (X, τ_1, τ_2), τ_1 is said to be completely regular with respect to τ_2 if for each τ_1-closed set C and each point $x \notin C$, there is a real valued function $f : X \to [0, 1]$, $f(x) = 0$, $f(C) = 1$ and f is τ_1-upper and τ_2-lower semicontinuous. (X, τ_1, τ_2) is called pairwise completely regular if both τ_1 and τ_2 are completely regular with respect to each other.

We shall make use of the following lemmas in Section 5.

Lemma 1. *τ_1 or τ_2-closed subset of a pairwise bicompact set is pairwise bicompact.*

Lemma 2. *Finite union of pairwise bicompact sets is pairwise bicompact.*

Lemma 3. *The intersection of two bicompact sets which are τ_1-closed and τ_2-closed respectively is pairwise bicompact.*

The proofs are straightforward and so are omitted.

3. Assumption

Throughout our discussion (X, τ_1, τ_2) stands for a bispace, $(X, \mathbf{P}, \mathbf{Q})$ for a bitopological space. R for the set of real numbers unless otherwise stated. We also assume that (X, τ_1, τ_2) is pairwise locally bicompact and pairwise completely regular.

4. Pairwise Borel and pairwise Baire sets

The following definitions are first introduced.

Definition 12 (cf. [8])**.** A set A in (X, τ_1, τ_2) is said to be bounded if it is contained in a pairwise bicompact set. A is called σ-bounded if it is contained in the union of a sequence of pairwise bicompact sets.

Definition 13 (cf. [9])**.** A set in (X, τ_1, τ_2) is called pairwise G_δ if it can be expressed as the intersection of countable number of sets of the form $P \cup Q$, $P \in \tau_1$, $Q \in \tau_2$.

Definition 14 (cf. [8])**.** The σ-ring generated by the class of all pairwise bicompact (pairwise bicompact pairwise G_δ-sets) in (X, τ_1, τ_2) which are either τ_1-closed or τ_2-closed is called the class of pairwise Borel (Baire) sets.

Remark 1. The reason behind taking the additional assumption on the set being τ_1-closed or τ_2-closed in Definitions 14 and 15 is the same as in [8], [9] which is throughly illustrated in [8].

Definition 15 (cf. [8], [9])**.** A measure μ defined on the class of pairwise Borel (Baire) sets such that $\mu(D) < \infty$ for pairwise bicompact (bicompact G_δ) members D is called a pairwise Borel (Baire) measure.

Evidently pairwise Baire sets are pairwise Borel sets and pairwise Borel measure is a pairwise Baire measure.

We require the following classes of sets in order to define the pairwise regularity of pairwise Borel and Baire measures. Let

M^* The class of all subsets of X which can be expressed as the union of countable number of sets of the form $P \cap Q$, $P \in \tau_1$, $Q \in \tau_2$.

$L(L_1)$ The class of pairwise Borel (Baire) sets.

$M(M_1)$ The subfamily of $L(L_1)$ whose members are also the members of M^*.

$N(N_1)$ The subfamily of $L(L_1)$ whose members can be expressed as the intersection of a countable number of sets of the form $C_1 \cup C_2$ where C_1 and C_2 are pairwise bicompact (bicompact G_δ) sets which are either τ_1 or τ_2-closed.

We now introduce the definition of pairwise regularity.

Definition 16 (cf. [8], [9])**.** A set $A \in L$ is called pairwise outer (Borel) regular if

$$\mu(A) = \inf\{\mu(U); A \subset U \in M\}$$

and pairwise inner (Borel) regular if

$$\mu(A) = \sup\{\mu(C); A \supset C \in N\}\,.$$

$A \in L$ is called pairwise (Borel) regular if it is both pairwise outer (Borel) and pairwise inner (Borel) regular. If every member of L is pairwise (Borel) regular then μ is called a pairwise regular Borel measure.

Pairwise regularity of Baire measures is similarly defined.

As in a bitopological space ([8], [9]), the basic properties of pairwise Borel and Baire measures in respect of pairwise regularity can be established here also. We just state below the main results in this respect without giving their proofs.

Theorem 1 (cf. Theorem 9 [8])**.** *The necessary and sufficient condition for μ to be pairwise regular on L is that every bounded set in M is pairwise inner regular.*

Theorem 2 (cf. Theorem 10 [8])**.** *If L satisfies the condition*
($*$): *For each bounded $U \in M$ there is a $D \in N$ and a set C which is both τ_1 and τ_2-closed such that $U \subset C \subset D$, then the pairwise outer regularity of all sets of L of the form $A - B$, $A \in N$, $B \in M$, $B \subset A$ implies the pairwise regularity of μ on L.*

Theorem 3 (cf. Theorem 4 [9])**.** *The necessary and sufficient condition for μ to be pairwise* (*Baire*) *regular on L_1 is that every bounded set in M_1 is pairwise inner* (*Baire*) *regular.*

5. Pairwise content

In this section we define pairwise regular content and show that every pairwise Baire measure can be used to construct a pairwise regular content. For this however we need several results which are given below.

Definition 17. A real valued function defined on a bispace (X, τ_1, τ_2) is said to have a pairwise bicompact support if there exists a pairwise bicompact set $C \subset X$ such that $f = 0$ on $X - C$.

Lemma 4. *For any pairwise bicompact set C there is a bounded $U \in M^*$ such that $C \subset U$ where U is of the form $U = P \cup Q$, $P \in \tau_1$, $Q \in \tau_2$, $P \cap C \neq \varphi \neq Q \cap C$.*

Proof. Since (X, τ_1, τ_2) is pairwise locally bicompact, for each $x \in C$ there exist $P_x \in \tau_1$ and $Q_x \in \tau_2$ such that $x \in P_x$, $x \in Q_x$ and $\tau_1 - \mathrm{cl}\,(Q_x)$ and $\tau_2 - \mathrm{cl}\,(P_x)$ are pairwise bicompact. Now the collection $\{P_x, Q_x \text{ and } x \in C\}$ forms a pairwise open cover of C. Since C is pairwise bicompact, there is a finite subfamily $\{U_1, U_2, \ldots, U_n\}$ (say) of $\{P_x, Q_x \text{ and } x \in C\}$ such that $C \subset \bigcup_{i=1}^{n} U_i = U$ (say). Clearly U can be expressed as $U = P \cup Q$, $P \in \tau_1$, $Q \in \tau_2$ by taking P and Q as the union of τ_1 and τ_2-open sets from the collection $\{U_1, U_2, \ldots, U_n\}$. If the collection $\{U_1, U_2, \ldots, U_n\}$ does not contain any τ_1-open (or τ_2-open) set then we take an additional τ_1-open (or τ_2-open) set from the collection $\{P_x, Q_x \text{ and } x \in C\}$ to form the finite subcover of C. □

Again $U = \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (U_i \cap X) \in M^*$. Also $U \subset \bigcup_{i=1}^{n} \big(\tau - \mathrm{cl}\,(U_i)\big)$ where $\tau = \tau_2$ or τ_1 according as $U_i \in \tau_1$ or $U_i \in \tau_2$, which being a finite union of pairwise bicompact sets is also pairwise bicompact. Hence U is bounded.

Lemma 5. *If $x \in X$ and V is a τ_1-neighbourhood (or τ_2-neighbourhood) of x then there exists a function $f : X \to [0,1]$ which is τ_2-upper, τ_1-lower (τ_1-upper, τ_2-lower) semicontinuous with pairwise bicompact support such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X - V$.*

Proof. Suppose V is a τ_1-neighbourhood of x. Since X is pairwise locally bicompact and pairwise completely regular so there exist a τ_1-open set U such that $x \in U \subset \tau_2 - \mathrm{cl}\,(U) \subset V$ where $\tau_2 - \mathrm{cl}\,(U)$ is pairwise bicompact. Since $X - U$ is τ_1-closed and $x \notin X - U$, there exist a real valued function $f_1 : X \to [0,1]$ such that $f_1(x) = 0$ and $f_1(y) = 1\ \forall\ y \in X - U$ which is τ_1-upper and τ_2-lower semicontinuous. Then $f = 1 - f_1$ is a τ_1-lower and τ_2-upper semicontinuous function such that $f(x) = 1$, $f(y) = 0\ \forall\ y \in X - V$ (since $X - V \subset X - \tau_2 - \mathrm{cl}\,(U) \subset X - U$, $f(x) = 0\ \forall\ y \in X - \tau_2 - \mathrm{cl}\,(U)$). Since $\tau_2 - \mathrm{cl}\,(U)$ is pairwise bicompact, so f has pairwise bicompact support. This proves the lemma. □

From this stage onwards the bispace X is assumed to satisfy the following additional assumption (cf. [9]).

(I) Every cover of a pairwise bicompact set by the sets of the form $P \cap Q$, $P \in \tau_1$, $Q \in \tau_2$ has a finite subcover.

Remark 2. Since a bitopological space is always a bispace, from [9] it follows that in an arbitrary bispace X, the condition (I) need not hold but there also exist bispaces where the condition (I) hold.

Remark 3. Under the supposition (I) it is easy to verify that a pairwise bicompact set is also τ_1-bicompact and τ_2-bicompact. We call such a set s-bicompact because of its similarity with the notion of compactness introduced by Swart [13] for a bitopological space.

In contrast with [9] the following lemmas implicating members of M_1 are proved here without the additional assumption of (X, τ_1, τ_2) being pairwise bicompact or pairwise Hausdorff. As a result the methods of proofs of the following lemmas are also not analogous to [9].

Lemma 6. *If C is a pairwise bicompact set, there is $U \in M_1$ and a pairwise bicompact G_δ set $D \in L_1$ such that $C \subset U \subset D$.*

Proof. By Lemma 4, we can find $V \in M$ such that $V = P \cup Q$, $P \in \tau_1$, $Q \in \tau_2$, $C \subset V$ and $P \cap C \neq \varphi \neq Q \cap C$. Let $x \in C$. If $x \in P$, by Lemma 5, there is a function f_x on X which is τ_2-upper and τ_1-lower semicontinuous with pairwise bicompact support such that $f_x(x) = 1$, $f_x(y) = 0\ \forall\ y \in X - P$, $0 \leq f \leq 1$. If $x \in Q$, then we can similarly obtain a function $f_x : X \to [0, 1]$ such that $f_x(x) = 1$, $f_x(y) = 0$, $\forall\ y \in X - Q$ and f_x is τ_1-upper and τ_2-lower semicontinuous with pairwise bicompact support. Since $P \cap C \neq \varphi \neq Q \cap C$, the collection of sets $U_x = \{y \in X; f_x(y) > \frac{1}{2}\}$ when x varies over C consists of both τ_1 and τ_2 open sets and so forms a pairwise open cover of C. If x belongs to both P and Q then the modification is evident. Since C is pairwise bicompact there are $U_{x_1}, U_{x_2} \ldots, U_{x_k}$, such that $C \subset \bigcup_{i=1}^{k} U_{x_i}$. Let $g = \min\{f_{x_1}, f_{x_2}, \ldots, f_{x_k}\}$. Then $g(y) = 0\ \forall\ y \in X - (P \cup Q) = X - V$, $g(x) > \frac{1}{2}\ \forall\ x \in C$. Then

$$\begin{aligned} C &\subset \left\{x \in X; g(x) > \frac{1}{2}\right\} = U \quad \text{(say)} \\ &\subset \left\{x \in X; g(x) \geq \frac{1}{2}\right\} = D \quad \text{(say)} \end{aligned}$$

Now $U = \{x; g(x) > \frac{1}{2}\} = \bigcup_{i=1}^{k} \{x \in X; f_{x_i}(x) > \frac{1}{2}\} \in M^*$. Also $U = \bigcup_{n=2}^{\infty} \{x; g(x) \geq \frac{1}{2} + \frac{1}{2}^n\}$. For each n, $\{x \in X; g(x) \geq \frac{1}{2} + \frac{1}{2}^n\} = \bigcup_{i=1}^{k} \{x \in X; f_{x_i}(x) \geq \frac{1}{2} + \frac{1}{2}^n\}$. Again since each f_{x_i} has pairwise bicompact support, there is a pairwise bicompact set B such that $f_{x_i}(y) = 0\ \forall\ y \in X - B$. Now the set $\{x \in X; f_{x_i}(x) \geq \frac{1}{2} + \frac{1}{2}^n\}$ being a τ_1 or τ_2-closed subset of B is pairwise bicompact. Evidently $\{x \in X; f_{x_i}(x) \geq \frac{1}{2} + \frac{1}{2}^n\}$ is pairwise G_δ. Hence $\{x; g(x) \geq \frac{1}{2} + \frac{1}{2}^n\} \in L_1$ for each n and consequently $U \in L_1$. Thus $U \in M_1$. Again $D = \{x \in X; g(x) \geq \frac{1}{2}\} = \bigcup_{i=1}^{k} \{x \in X; f_{x_i}(x) \geq \frac{1}{2}\}$ being finite union of pairwise bicompact sets is pairwise bicompact. Further

$$\left\{x \in X; f_{x_i}(x) \geq \frac{1}{2}\right\} = \bigcap_{n=2}^{\infty} \left\{x \in X; f_{x_i}(x) > \frac{1}{2} - \frac{1}{2}^n\right\}$$

is a pairwise G_δ set and so is also D. This completes the proof. □

The following lemma gives a result which is analogous to the Baire Sandwitch theorem in the context of a bispace.

Lemma 7. *If C is pairwise bicompact and $U \in M_1$ be such that $C \subset U$ then there is a $V \in M_1$ and a pairwise bicompact set D such that*

$$C \subset V \subset D \subset U\,.$$

Proof. We note that $U \in M_1$ is of the form $\bigcup_{i=1}^{\infty}(P_i \cap Q_i)$ where $P_i \in \tau_1$, $Q_i \in \tau_2$. Let $x \in C$. Then $x \in (P_i \cap Q_i)$ for some i. So there is a function $f_x : X \to [0,1]$ with $f_x(x) = 1$, $f_x(y) = 0$, for all $y \in X - P_i$ and f_x is τ_2-upper and τ_1-lower semicontinuous with pairwise bicompact support. Similarly there is a function $f'_x : X \to [0,1]$ with $f'_x(x) = 1$, $f'_x(y) = 0$ for all $y \in X - Q_i$ and f'_x is τ_1-upper and τ_2-lower semicontinuous with pairwise bicompact support. Let $g_x = \min\{f_x, f'_x\}$. Then $g_x(x) = 1$, $g_x(y) = 0$ for all $y \in (X - P_i) \cup (X - Q_i)$. Now C is covered by the collection of the sets of the form $U_x = \{y; g_x(y) > \frac{1}{2}\}$ as varies over C. But $U_x = \{y; f_x(y) > \frac{1}{2}\} \cap \{y; f'_x(y) > \frac{1}{2}\}$ where the first set is τ_1-open and the second is τ_2-open. So by condition (I) there are $x_1, x_2, \ldots, x_n \in C$ such that $C \subset \bigcup_{i=1}^{n} U_{x_1}$. Thus

$$\begin{aligned} C &\subset \bigcup_{i=1}^{n} U_{x_i} = V \quad \text{(say)} \\ &\subset \bigcup_{i=1}^{n} \left\{ g_{x_i}(y) \geq \frac{1}{2} \right\} = D \quad \text{(say)} \\ &\subset \bigcup_{i=1}^{\infty} (P_i \cap Q_i) \subset U \end{aligned}$$

where P_i , Q_i correspond to the element x_i. Clearly $V = \bigcup_{i=1}^{n} U_{x_i} = \bigcup_{i=1}^{n} \big[\{y; f_{x_i}(y) > \frac{1}{2}\} \cap \{y; f'_{x_i}(y) > \frac{1}{2}\big]\big\} \in M^*$. Again $\{y; f_{x_i}(y) > \frac{1}{2}\} = \bigcup_{k=2}^{\infty} \{y; f_{x_i}(y) \geq \frac{1}{2} + \frac{1}{2}^k\}$ where for each k, $\{y; f_{x_i}(y) \geq \frac{1}{2} + \frac{1}{2}^k\}$ is pairwise G_δ and τ_2-closed. Also since f_{x_i} has pairwise bicompact support, there exists a pairwise bicompact set B such that $f_{x_i}(y) = 0$ for all $y \in X - B$. Hence $\{y; f_{x_i}(y) \geq \frac{1}{2} + \frac{1}{2}^k\}$ being a τ_2-closed subset of B is pairwise bicompact. So $\{y; f_{x_i}(y) > \frac{1}{2}\} \in L_1$. Similarly $\{y; f'_{x_i}(y) > \frac{1}{2}\} \in L_1$. Thus $V \in L_1$ and hence $V \in M_1$. Finally we see that

$$D = \bigcup_{i=1}^{n} \left[\left\{ y; f_{x_i}(y) \geq \frac{1}{2} \right\} \cap \left\{ y; f'_{x_i}(y) \geq \frac{1}{2} \right\} \right].$$

Where each set in the union is pairwise bicompact (by Lemma 3) and hence D is pairwise bicompact. This proves the lemma. □

Note 3. From the proof of Lemma 7 it is clear that the above result holds for the class M^* also which is larger than M_1.

From this stage onwards we assume that (X, τ_1) or (X, τ_2) is Hausdorff.

Lemma 8. *If C and D are two disjoint pairwise bicompact sets then there exist U, $V \in M_1$ such that $C \subset U$, $D \subset V$, $U \cap V = \varphi$.*

The proof is omitted.

We now intoduce the following definition.

Definition 18. By a pairwise content we mean a real valued function λ defined over the class of all pairwise bicompact sets such that
(i) $0 \leq \lambda(C) < \infty$,
(ii $\lambda(C \cup D) \leq \lambda(C) + \lambda(D)$,
(iii) $C \subset D \Rightarrow \lambda(C) \leq \lambda(D)$ and
(iv) $C \cap D = \varphi \Rightarrow \lambda(C \cup D) = \lambda(C) + \lambda(D)$.

Further λ is said to be regular if $\lambda(C) = \text{glb}\{\lambda(D); C < D\}$.
Where $C < D$ means that there is a $U \in M$ such that $C \subset U \subset D$.
Though the proof of the following theorem is analogous to Theorem 5 [9], we give its proof for the sake of completeness.

Theorem 4. *Let ν be a pairwise Baire measure. Then the set function λ defined for all pairwise bicompact sets C by the formula*

$$\lambda(C) = \text{glb}\{\nu(U); C \subset U, U \in M_1\}$$

is a pairwise regular content on X.

Proof. If C is a pairwise bicompact set then by Lemma 6 there is a $U \in M_1$ and a pairwise bicompact G_δ set $D \in L_1$ such that $C \subset U \subset D$. Then $\lambda(C) \leq \nu(U) \leq \nu(D) < \infty$.

The proof of λ being monotone and subadditive are straightforward. Let C, D be two pairwise bicompact sets such that $C \cap D = \varphi$. Then by Lemma 8 there are $U, V \in M_1$ such that $C \subset U$, $D \subset V$ and $U \cap V = \varphi$. Let W be an arbitrary member of M_1 containing $C \cup D$. Then $C \subset U \cap W$, $D \subset V \cap W$, where $U \cap W$, $V \cap W \in M_1$. Thus

$$\begin{aligned} \nu(W) &\geq \nu[(U \cap W) \cup (V \cap W)] = \nu(U \cap W) + \nu(V \cap W) \\ &\geq \lambda(C) + \lambda(D) \,. \end{aligned}$$

Taking lower bound, $\lambda(C \cup D) \geq \lambda(C) + \lambda(D)$.

To show that λ is regular, let C be a pairwise bicompact set and let $\varepsilon > 0$ be arbitrary. Then there is $U \in M_1$ such that $C \subset U$ and $\lambda(C)+\varepsilon \geq \nu(U)$. By Lemma 7 there is $V \in M_1$ and a pairwise bicompact set D such that $C \subset V \subset D \subset U$. Evidently then $C < D$ and $\lambda(D) \leq \nu(U) \leq \lambda(C) + \varepsilon$. This proves the theorem. □

6. Pairwise regular Borel extension of a pairwise Baire measure

Here we use the idea of pairwise content to generate a pairwise regular Borel measure which is used in the last theorem for the extension of pairwise regular Baire measure.

The proofs of the following results are parallel to [9] and so are omitted.

Lemma 9. *If $C \subset U \cup V$ where $U, V \in M$ and C is pairwise bicompact then there are two pairwise bicompact sets D, E, such that $C \subset D \cup E$ where $D \subset U$, $E \subset V$.*

Theorem 5. *Let for all $U \in M$,*

$$\lambda_*(U) = \text{lub}\,\{\lambda(C); C \subset U, C \ \textit{ is pairwise bicompact}\}\,.$$

The set function λ_* is called the pairwise inner content induced by λ and have the following properties:
(i) $\lambda_*(\varphi) = 0$,
(ii) $\lambda_*(U) < \infty$ *for every bounded member U of M,*
(iii) λ_* *is monotone nondecreasing,*
(iv) λ_* *is countable subadditive,*
(v) λ_* *is countably additive.*

Theorem 6. *Let $\lambda^*(A) = \text{glb}\,\{\lambda^*(U); A \subset U, U \in M\}\ \forall\ A \in H$ where H is the σ-bounded subsets of X. Then*
(i) λ^* *is an outer measure on H,*
(ii) $\lambda^*(A) < \infty$ *for every bounded set $A \in H$, and*
(iii) $\lambda_*(U) = \lambda^*(U)\,\forall\, U \in M$.

Lemma 10. *$B \in H$ is λ^*-measurable if and only if*

$$\lambda^*(U) = \lambda^*(B \cap U) + \lambda^*(B^c \cap U)\,\forall\, U \in M$$

where B^c denotes the complement of B.

We now prove our main result.

Theorem 7. *Let λ be a pairwise regular content on X and λ^* be the outer measure induced by λ. Then every pairwise Borel set is λ^*-measurable and the restriction μ of λ^* on L is a pairwise regular Borel measure such that $\lambda(C) = \mu(C)$ for all pairwise bicompact members of L.*

Proof. Since the class M of λ^*-measurable sets is a σ-ring and (x, τ_1) or (x, τ_2) is Hausdorff, to prove that $M \supset L$, it is sufficient to show that each pairwise bicompact set is λ^*-measurable. Let $U \in M$ and let C be any pairwise bicompact member of L. Then $U \cap C^c \in M$. Let D be a pairwise bicompact subset of $U \cap C^c$. Then clearly $U \cap D^c \in M$. Let E be any pairwise bicompact subset of $U \cap D^c$. Then E and D are mutually disjoint pairwise bicompact subset of U. Hence $\lambda^*(U) = \lambda_*(U) \geq \lambda(D \cup E) = \lambda(D) + \lambda(E)$. Varying E we get

$$\lambda^*(U) \geq \lambda(D) + \lambda_*(U \cap D^c) = \lambda(D) + \lambda^*(U \cap D^c)\,.$$

Since $D \subset C^c$, we have $D^c \supset C$. So by monotonicity of λ^*, $\lambda^*(U \cap D^c) \geq \lambda^*(U \cap C)$. Therefore $\lambda^*(U) \geq \lambda(D) + \lambda^*(U \cap C)$. Varying D, we get

$$\lambda^*(U) \geq \lambda_*(U \cap C^c) + \lambda^*(U \cap C) = \lambda^*(U \cap C^c) + \lambda^*(U \cap C)\,.$$

This shows that C is λ^*-measurable.

Let μ be the restriction of λ^* on L. Then μ is a pairwise Borel measure.

To show that μ is pairwise regular it will suffice to show that each bounded member of M is pairwise inner regular. Let $U \in M$ be bounded. Then

$$\mu(U) = \lambda^*(U) = \lambda_*(U) = \text{lub}\,\{\lambda(C); C \subset U, C \text{ pairwise bicompact}\}\,.$$

For each pairwise bicompact set $C \subset U$ there exists (by Lemma 7) a $V \in M_1 \subset M$ and a pairwise bicompact set D such that $C \subset V \subset D \subset U$. Then

$$\lambda(C) \leq \lambda_*(V) = \lambda^*(V) = \mu(V) \leq \mu(D) \leq \mu(U)\,.$$

We consider now the collection N_0 of all those members D of N such that there are $V \in M_1$ and a pairwise bicompact set C satisfying $C \subset V \subset D \subset U$. Then,

$$\begin{aligned}\mu(U) &\geq \text{lub}\,\{\mu(U); U \in N_0\}\\ &\geq \text{lub}\,\{\lambda(C); C \subset U, C \text{ pairwise bicompact}\}\\ &= \lambda_*(U) = \lambda^*(U) = \mu(U)\,.\end{aligned}$$

Hence, $\mu(U) = \text{lub}\,\{\mu(D); U \supset D, D \in N\}$ consequently U is pairwise inner regular.

We now show that $\lambda(C) = \mu(C)$ for all pairwise bicompact members C of L. Since λ is regular, given $\varepsilon > 0$ arbitrary, there is a pairwise bicompact set D such that $C < D$ and $\lambda(D) \leq \lambda(C) + \varepsilon$. Now $C < D$ implies $C \subset U \subset D$ where $U \in M$. Therefore,

$$\mu(C) = \lambda^*(C) \leq \lambda^*(U)\,.$$

If C_1 is pairwise bicompact and $C_1 \subset U$, then $C_1 \subset D$ and so $\lambda(C_1) \leq \lambda(D)$. Varying C_1, $\lambda^*(U) = \lambda_*(U) \leq \lambda(D)$.

Hence $\mu(C) \leq \lambda^*(U) \leq \lambda(D) \leq \lambda(C) + \varepsilon$, i.e. $\mu(C) \leq \lambda(C)$.

Again if $V \in M$ be such that $C \subset V$, then $\lambda(C) \leq \lambda_*(V)$. Taking lower bound over V,

$$\lambda(C) \leq \lambda^*(C) = \mu(C)\,.$$

Hence, $\lambda(C) = \mu(C)$. The proof is now complete. □

Theorem 8. *A pairwise regular Baire measure ν defined on (X, τ_1, τ_2) can be extended to a pairwise regular Borel measure μ provided the following condition holds:* (II) *For each $C \in N$ and $E \in L$ such that $C \subset E$, there is a pairwise bicompact member D of L_1 satisfying $C \subset D \subset E$.*

The proof is parallel to the proof of Theorem 9 [9] and so is omitted.

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Department of Mathematics, Jadavpur University
Kolkata - 700 032, West Bengal, India
E-mail: pratulananda@yahoo.co.in

Department of Mathematics, St. Paul's C.M. College
R.R.R Sarani, Kolkata - 700 009, West Bengal, India