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GENERALIZATIONS OF THE FAN-BROWDER FIXED POINT THEOREM AND MINIMAX INEQUALITIES

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Abstract. In this paper fixed point theorems for maps with nonempty convex values and having the local intersection property are given. As applications several minimax inequalities are obtained.

1. Introduction

A *map* (or a *multifunction*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is a function with the values $T(x) \subset Y$. For $y \in Y$, $T^{-1}(y)$ is called the *fiber* of T on y.

Using an infinite dimensional version of the Knaster-Kuratowski-Mazurkiewicz theorem, Fan [10] proved in 1961 the following:

Theorem 0. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and M be a closed subset of $X \times X$ such that*:
(i) $(x, x) \in M$ *for all* $x \in X$;
(ii) *for each* $y \in X$ *the set* $\{x \in X : (x, y) \notin X\}$ *is convex* (*or empty*).
Then $X \times \{y_0\} \subset M$ *for some* $y_0 \in X$.

Subsequently, Browder [4] obtained in 1968 the following fixed point theorem:

Theorem 1. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $T : X \multimap X$ be a map with nonempty convex values and open fibers. Then T has a fixed point.*

Browder's proof for his theorem was based on the existence of a partition of unity for open coverings of compact sets and on the Brouwer fixed point theorem. Let us observe that Browder's theorem is just Theorem 0 reformulated in a more convenient form (to see this, take $T(x) = \{y \in X : (x, y) \notin M\}$). For this reason Theorem 1 is known in the literature as the Fan-Browder fixed point theorem.

The existence of many significant applications in nonlinear functional analysis, game theory and economic theory gave rise to a number of generalizations or

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versions of Theorem 1 (see [1], [2], [3], [6], [7], [16], [17], [19]). In Section 2 we give new generalizations of Theorem 1 involving maps with the local intersection property. Two well-known applications of the Fan-Browder fixed point theorem will be considered in this paper. The first one is the following Fan's minimax inequality [12]

Theorem 2. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$ be a function quasiconvex in y and upper semicontinuous in x. Then*

$$\inf_{x\in X} f(x,x) \leq \max_{x\in X} \inf_{y\in X} f(x,y) .$$

The second application is a two-function minimax inequality due also to Fan [11] which generalizes the celebrate Sion's minimax theorem [18]. We state this result as follows

Theorem 3. *Let X, Y be nonempty compact convex subsets of topological vector spaces and $f, g : X \times Y \to \mathbb{R}$. Suppose that f is lower semicontinuous in y and quasiconcave in x, g is upper semicontinuous in x and quasiconvex in y, and $f \leq g$ on $X \times Y$. Then*

$$\min_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} g(x,y) .$$

Note that "quasiconvex" and further notions will be explained in the last section of the paper. In the same section, from each fixed point theorem established in Section 2 we derive a Fan type minimax inequality and a Fan-Sion type minimax theorem. Throughout this paper we assume that the topological vector spaces are separated.

2. Local intersection property and fixed point theorems

Let X be a topological space and Y be a set. A map $T : X \multimap Y$ is said to have the *local intersection property* (see[20]) if for each $x \in X$ with $T(x) \neq \emptyset$ there exists an open neighbourhood $V(x)$ of x such that $\bigcap_{z\in V(x)} T(z) \neq \emptyset$. It is not hard to see that each map with open fibers has the local intersection property but the example given in [20, p.63], shows that the converse is not true.

The following lemma is useful in what follows and can be found in [9].

Lemma 4. *Let X be a topological space, Y be a set and $T : X \multimap Y$ be a map with nonempty values. Then the following assertions are equivalent*

(i) *T has the local intersection property;*

(ii) *There exists a map $F : X \multimap Y$ such that $F(x) \subset T(x)$ for each $x \in X$, $F^{-1}(y)$ is open for each $y \in Y$ and $X = \bigcup_{y\in Y} F^{-1}(y)$.*

Theorem 5. *Let X be a topological space, Y be a convex subset of a topological vector space and $T : X \multimap Y$ be a map with nonempty convex values and having the local intersection property. Then T admits a selection G (i.e. $G(x) \subset T(x)$ for all $x \in X$) with nonempty convex values and open fibers.*

Proof. By Lemma 4, T admits a selection F with open fibers such that

$$X = \bigcup_{y \in Y} F^{-1}(y) \,. \tag{1}$$

From (1) we infer that $F(x) \neq \emptyset$ for all $x \in X$. Define the map $G : X \multimap Y$, by $G(x) = coF(x)$. Since T has convex values, $G(x) \subset T(x)$ and $G(x)$ is convex for each $x \in X$. Since F has open fibers, by Lemma 5.1 in [21], it follows that G has also open fibers. □

The first generalization of the Fan-Browder fixed point theorem is the following

Theorem 6. *Let X be a compact convex subset of a topological vector space and $T : X \multimap X$ be a map with nonempty convex values having the local intersection property. Then T has a fixed point.*

Proof. By Theorem 5, T has a selection G with nonempty convex values and open fibers, and Theorem 1 guarantees the existence of a point $x_0 \in X$ such that $x_0 \in G(x_0) \subset T(x_0)$. □

Theorem 7. *Let X be a compact convex subset of a topological vector space and Y a nonempty set. Suppose that $F : X \multimap Y, T : X \multimap X$ are two maps satisfying the following conditions*

(i) *T takes convex values;*
(ii) *F has nonempty values and open fibers;*
(iii) *for each $y \in Y$ there exists $z \in X$ such that $F^{-1}(y) \subset T^{-1}(z)$.*

Then T has a fixed point.

Proof. Since F has nonempty values, $\bigcup_{y \in Y} F^{-1}(y) = X$, and from (iii) we get $\bigcup_{z \in X} T^{-1}(z) = X$, hence T has also nonempty values. According to Theorem 6 it suffices to show that T has the local intersection property. Let $x \in X$. Since $F(x) \neq \emptyset$ there exist $y \in Y$ and $z \in X$ such that

$$x \in F^{-1}(y) \subset T^{-1}(z) \,. \tag{2}$$

Then $F^{-1}(y)$ is an open neighbourhood of x and, by (2), it follows that $z \in \bigcap_{x' \in F^{-1}(y)} T(x')$. Thus the proof is complete. □

The following result extends the Fan-Browder fixed point theorem to the case when the convex set X is not compact.

Theorem 8. *Let X be a convex subset of a topological vector space and $T : X \multimap X$ be a map with nonempty convex values, having the local intersection property. Suppose that there exist a nonempty compact convex subset X_0 of X and a compact subset K of X satisfying the following condition*

for each $x \in X \backslash K$ there exists an open neighbourhood $V(x)$ of x such that

$$\bigcap_{z \in V(x)} T(z) \cap X_0 \neq \emptyset \,. \tag{3}$$

Then T has a fixed point.

Proof. Define the maps $H, G : X \multimap X$ by

$$H(y) = \operatorname{int}\left(T^{-1}(y)\right) \quad \text{for} \quad y \in X$$

and

$$G(x) = \operatorname{co} H^{-1}(x) \quad \text{for} \quad x \in X\,.$$

We see that H takes open values and $H(y) \subset T^{-1}(y)$ for each $y \in X$. Since the values of T are convex, $G(x) \subset T(x)$ for all $x \in X$. Using once again Lemma 5.1 in [21] we infer that G has open fibers. For an arbitrary $x \in X$, since T has the local intersection property, there exist a neighbourhood $V(x)$ of x and a point y such that

$$x \in V(x) \subset T^{-1}(y) \quad \text{whence} \quad x \in H(y) \subset G^{-1}(y)\,.$$

Consequently, G has nonempty values and

$$X = G^{-1}(X)\,. \tag{4}$$

For each $x \in X \backslash K$, by (3), there exists $y \in X_0$ such that $x \in H(y) \subset G^{-1}(y)$, hence

$$X \backslash K = G^{-1}(X_0)\,. \tag{5}$$

On the other hand, by (4), $K \subset G^{-1}(X)$ and, since K is compact, there exists a finite set $A \subset X$ such that

$$K \subset G^{-1}(A)\,. \tag{6}$$

Thus, by (5) and (6), we have $X = G^{-1}(X_0 \cup A)$.
Let $C = \operatorname{co}(X_0 \cup A)$. Then C is a compact, convex subset of X and

$$C \subset G^{-1}(X_0 \cup A) \subset G^{-1}(C)\,. \tag{7}$$

Define the map $\widetilde{G} : C \to C$ by $\widetilde{G}(x) = G(x) \bigcap C$. Then the values of $\widetilde{G}$ are nonempty (by (7)) and convex. Since $\widetilde{G}^{-1}(y) = G^{-1}(y) \bigcap C$ for each $y \in C$, the fibers of $\widetilde{G}$ are open in C. Applying Theorem 1 to the map $\widetilde{G}$ we find a point $x_0 \in C$ such that $x_0 \in \widetilde{G}(x_0) \subset T(x_0)$. □

Remark. The local intersection property imposed on T and condition (3) can be unified in the following condition

$$\text{the map } \widetilde{T} : X \multimap X, \text{ defined by } \widetilde{T}(x) = \begin{cases} T(x) & \text{for} \quad x \in K \\ T(x) \cap X_0 & \text{for} \quad x \in X \backslash K \end{cases}$$

has the local intersection property.

In our opinion it is worth comparing Theorem 8 with other noncompact generalizations of the Fan-Browder fixed point theorem due to Browder [4], Lassonde [15], Mehta [16] and Park [17].

3. Minimax inequalities

Let X, Y nonempty convex subsets of topological vector spaces. Recall that a function $f : X \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be:

(i) *quasiconcave* (resp. *upper semicontinuous*) *in* x if for each $y \in Y$ and $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x, y) \geq \lambda\}$ is convex (resp. closed);

(ii) *quasiconvex* (resp. *lower semicontinuous*) *in* y if for each $x \in X$ and $\lambda \in \mathbb{R}$ the set $\{y \in Y : f(x, y) \leq \lambda\}$ is convex (resp. closed).

A function $f : X \times Y \to \overline{\mathbb{R}}$ (X, Y topological spaces) is said to be:

(iii) *transfer upper semicontinuous* in x (see [8]) if, for each $\lambda \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x, y) < \lambda$, there exist a neighbourhood $V(x)$ of x and a point $y' \in Y$ such that $f(z, y') < \lambda$, for all $z \in V(x)$;

(iv) *transfer lower semicontinous* in y (see [8]) if, for each $\lambda \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x, y) > \lambda$, there exist a neighbourhood $V(y)$ of y and a point $x' \in X$ such that $f(x', u) > \lambda$, for all $u \in V(y)$.

It is clear that every function which is upper semicontinuous in x (resp. lower semicontinuous in y) is transfer upper semicontinous in x (resp. transfer lower semicontinuous in y) but the converse is not true (see [8]).
From each fixed point theorem obtained in the previous section we shall derive a Fan type minimax inequality and a Fan-Sion type minimax theorem.

Theorem 9. *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \overline{\mathbb{R}}$ be a function quasiconvex in y and transfer upper semicontinuous in x. Then*

$$\inf_{x \in X} f(x, x) \leq \sup_{x \in X} \inf_{y \in X} f(x, y) .$$

Proof. We may assume that $\sup_{x \in X} \inf_{y \in X} f(x, y) < \infty$. Let $\lambda > \sup_{x \in X} \inf_{y \in X} f(x, y)$ be arbitrarily fixed; we define the map $T : X \multimap X$ by

$$T(x) = \{y \in X : f(x, y) < \lambda\} .$$

From $\lambda > \sup_{x \in X} \inf_{y \in X} f(x, y)$ it follows that $T(x)$ is nonempty for each $x \in X$. Since f is quasiconvex in y, the values of T are convex; since f is transfer upper semicontinuous in x, T has the local intersection property. By Theorem 6 there exists a point $x_0 \in X$ such that $x_0 \in T(x_0)$. Hence $\inf_{x \in X} f(x, x) \leq f(x_0, x_0) \leq \lambda$, which proves the theorem. □

Theorem 10. *Let X and Y be nonempty compact convex subsets of topological vector spaces and $f, g : X \times Y \to \overline{\mathbb{R}}$ be two functions satisfying the following conditions:*

(i) *$f \leq g$;*

(ii) *f is quasiconcave in x;*

(iii) *f is transfer lower semicontinuous in y*

(iv) *g is quasiconvex in y;*

(v) *g is transfer upper semicontinuous in x.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\inf_{y\in Y} g(x,y) .$$

Proof. Suppose that there exists a real λ such that

$$\sup_{x\in X}\inf_{y\in Y} g(x,y) < \lambda < \inf_{y\in Y}\sup_{x\in X} f(x,y) .$$

Define the map $T : X \times Y \to X \times Y$ by

$$T(x,y) = \{x' \in X : f(x',y) > \lambda\} \times \{y' \in Y : g(x,y') < \lambda\} .$$

Then $T(x,y)$ is nonempty and convex (by (ii) and (iv)) for each $(x,y) \in X \times Y$. By (iii) and (v) one can easily prove that T has the local intersection property. Applying Theorem 6 we get a fixed point $(x_0,y_0) \in T(x_0,y_0)$. Therefore $\lambda < f(x_0,y_0) \le g(x_0,y_0) < \lambda$, a contradiction. □

Theorem 11. *Let X be a compact convex subset of a topological vector space and Y be a nonempty set. Suppose that $f : X \times X \to \overline{\mathbb{R}}$, $g : X \times Y \to \overline{\mathbb{R}}$ are two functions satisfying the following conditions:*

(i) *f is quasiconvex in the second variable;*

(ii) *g is upper semicontinuous in x;*

(iii) *for each $y \in Y$ there exists $z \in X$ such that $f(\cdot,z) \le g(\cdot,y)$.*

Then

$$\inf_{x\in X} f(x,x) \le \sup_{x\in X}\inf_{y\in Y} g(x,y) .$$

Proof. We may assume that $\sup_{x\in X}\inf_{y\in Y} g(x,y) < \infty$. Let $\lambda > \sup_{x\in X}\inf_{y\in Y} g(x,y)$ be arbitrarily fixed; we define the maps $T : X \multimap X$, $F : X \multimap Y$, by

$$T(x) = \{z \in X : f(x,z) < \lambda\}$$

and

$$F(x) = \{y \in Y : g(x,y) < \lambda\} .$$

Since $\lambda > \sup_{x\in X}\inf_{y\in Y} g(x,y)$, $F(x)$ is nonempty for each $x \in X$. It is easy to prove that conditions (i), (ii), (iii) in our theorem imply the conditions similarly denoted in Theorem 7. By Theorem 7, T has a fixed point x_0. It follows that $\inf_{x\in X} f(x,x) \le f(x_0,x_0) < \lambda$ and the proof is complete. □

Theorem 12. *Let X_1, Y_1 be nonempty compact convex subsets of topological vector spaces and X_2, Y_2 be nonempty sets. Let $f : X_2 \times Y_1 \to \overline{\mathbb{R}}$, $g : X_1 \times Y_2 \to \overline{\mathbb{R}}$, $h,k : X_1 \times Y_1 \to \overline{\mathbb{R}}$ be four functions satisfying:*

(i) *$h \le k$;*

(ii) *f is lower semicontinuous on Y_1;*

(iii) *g is upper semicontinous on X_1;*

(iv) *h is quasiconcave on X_1;*

(v) *k is quasiconvex on Y_1;*

(vi) *for each* $x_2 \in X_2$ *there exists* $x_1 \in X_1$ *such that* $f(x_2, \cdot) \leq h(x_1, \cdot)$;

(vii) *for each* $y_2 \in Y_2$ *there exists* $y_1 \in Y_1$ *such that* $k(\cdot, y_1) \leq g(\cdot, y_2)$.

Then

$$\inf_{y_1 \in Y_1} \sup_{x_2 \in X_2} f(x_2, y_1) \leq \sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} g(x_1, y_2) .$$

Proof. Suppose that there exists a real λ such that

$$\sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} g(x_1, y_2) < \lambda < \inf_{y_1 \in Y_1} \sup_{x_2 \in X_2} f(x_2, y_1) . \tag{8}$$

Define the maps $T : X_1 \times Y_1 \to X_1 \times Y_1$, $F : X_1 \times Y_1 \to X_2 \times Y_2$ by

$$T(x_1, y_1) = \{x_1' \in X_1 : h(x_1', y_1) > \lambda\} \times \{y_1' \in Y_1 : k(x_1, y_1') < \lambda\}$$

and

$$F(x_1, y_1) = \{x_2' \in X_2 : f(x_2', y_1) > \lambda\} \times \{y_2' \in Y_2 : g(x_1, y_2') < \lambda\} .$$

By (8), F has nonempty values. In view of conditions (iv) and (v) the values of T are convex and by (ii) and (iii), F has open fibers. From (vi) and (vii) it follows readily that for each $(x_2, y_2) \in X_2 \times Y_2$ there exists $(x_1, y_1) \in X_1 \times Y_1$ such that $F^{-1}(x_2, y_2) \subset T^{-1}(x_1, y_1)$. Therefore all hypotheses of Theorem 7 are verified. Applying Theorem 7 we get a point $(\overline{x_1}, \overline{y_1}) \in X_1 \times Y_1$ such that $(\overline{x_1}, \overline{y_1}) \in T(\overline{x_1}, \overline{y_1})$. Taking into account condition (i) we obtain the following contradiction

$$\lambda < h(\overline{x_1}, \overline{y_1}) \leq k(\overline{x_1}, \overline{y_1}) < \lambda .$$

□

When $X_1 = Y_1$, $X_2 = Y_2$ and conditions (vi), (vii) are replaced by a unique stronger condition one can get at once the following known result (see [3]).

Corollary 13. *Let* X *and* Y *be nonempty compact convex subsets of topological vector spaces and* f, g, h, $k : X \times Y \to \overline{\mathbb{R}}$, *be four functions satisfying:*

(i) $f \leq h \leq k \leq g$;

(ii) f *is lower semicontinuous in* y;

(iii) g *is upper semicontinous in* x;

(iv) h *is quasiconcave in* x;

(v) k *is quasiconvex in* y.

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y) .$$

Theorem 14. *Let* X *be a nonempty convex subset of a topological vector space and* $f : X \times X \to \overline{\mathbb{R}}$ *be a function quasiconvex in* y *and transfer upper semicontinuous in* x. *Suppose that there exists a nonempty compact convex subset* X_0 *of* X *and a compact subset* K *of* X *satisfying the following condition*

$$\begin{gathered} \text{for each } x \in X \setminus K \text{ and any } y' \in X \text{ there exists a neighbourhood } V(x) \text{ of} \\ x \text{ and a point } y_0 \in X_0 \text{ such that } f(z, y_0) \leq f(z, y') \text{ for all } z \in V(x) . \end{gathered} \tag{9}$$

Then

$$\inf_{x\in X} f(x,x) \leq \sup_{x\in X}\inf_{y\in Y} f(x,y) .$$

Proof. As in previous proof we assume $\sup_{x\in X}\inf_{y\in Y} f(x,y) < \infty$ and fix a real $\lambda > \sup_{x\in X}\inf_{y\in Y} f(x,y)$. The map $T : X \multimap X$ defined by

$$T(x) = \{y \in X : f(x,y) < \lambda\}$$

takes nonempty convex values and has the local intersection property. We show that it satisfies condition (3) from Theorem 8. Let $x \in X\backslash K$. Since $T(x) \neq \emptyset$ and f is transfer upper semicontinous in x, there exists a neighbourhood $V'(x)$ of x and a point $y' \in X$ such that $f(z,y') < \lambda$ for each $z \in V'(x)$. By (9) there exist a neighbourhood $V''(x)$ of x and a point $y_0 \in K$ such that $f(z,y_0) \leq f(z,y')$ for all $z \in V''(x)$. Then for each $z \in V(x) = V'(x) \cap V''(x)$ we have $f(z,y_0) \leq f(z,y') < \lambda$, hence

$$y_0 \in \bigcap_{z\in V(x)} T(z) \cap X_0 .$$

Theorem 8 implies that $x_0 \in T(x_0)$ for some $x_0 \in X$. Hence

$$\inf_{x\in X} f(x,x) \leq f(x_0,x_0) < \lambda$$

and the proof is complete. □

Combining the lines of the proofs of Theorems 10 and 14 one can easily prove the following result

Theorem 15. *Let X and Y be nonempty compact convex subsets of topological vector spaces and $f,g : X \times Y \to \overline{\mathbb{R}}$ be two functions satisfying the following conditions*

(i) *$f \leq g$;*
(ii) *f is quasiconcave in x;*
(iii) *f is transfer lower semicontinuous in y;*
(iv) *there exist a nonempty compact convex subset Y_0 of Y and a compact subset K of X satisfying the following condition:*

for each $x \in X\backslash K$ and any $y' \in Y$ there exists a neighbourhood $V(x)$ of x and a point $y_0 \in Y_0$ such that $f(z,y_0) \leq f(z,y')$ for all $z \in V(x)$;

(v) *g is quasiconvex in y;*
(vi) *g is transfer upper semicontinuous in x;*
(vii) *there exist a nonempty compact convex subset X_0 of X and a compact subset L of Y satisfying the following condition:*

for each $y \in Y\backslash L$ and any $x' \in X$ there exists a neighbourhood $V(y)$ of y and a point $x_0 \in X_0$ such that $g(x_0,u) \geq g(x',u)$ for all $u \in V(y)$.

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y) .$$

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