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NEW ASPECTS ON CR-STRUCTURES OF CODIMENSION 2 ON HYPERSURFACES OF SASAKIAN MANIFOLDS

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Abstract. We introduce a torsion free linear connection on a hypersurface in a Sasakian manifold on which we have defined in natural way a CR-structure of CR-codimension 2. We study the curvature properties of this connection and we give some interesting examples.

1. Introduction

In 1995, P. Matzeu & V. Oproiu have introduced a torsion free linear connection adapted to an almost contact structure associated with a given pseudoconvex CR-manifold (of hypersurface type) (see [5]). The fundamental tensor field and the 1-form of the associated almost contact structure are no longer parallel with respect to this connection. Yet, this connection yields to the same Bochner type curvature tensor field for the CR-manifold as it was obtained by using the Tanaka Webster connection.

In this paper we consider CR-structures of codimension 2 on hypersurfaces in Sasakian manifolds. We use a natural f-structure with complemented frames in order to obtain a torsion free linear connection. This is a generalization of the Matzeu Oproiu connection for the CR-codimension 2. Then, we give a relation between the adapted connection and the Levi Civita connection on the hypersurface. Finally, we examine some symmetry properties of the curvature tensor field of this connection.

In the end of the paper we present some examples in the case when the ambient is $\mathbf{R}^5$ and S^5 endowed with the canonical Sasakian structures.

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2. The adapted torsion free canonical connection

Let $\widetilde{M}$ be a smooth Sasakian manifold of dimension $2n+3$ with the contact metric structure $(\widetilde{\phi},\ \widetilde{\xi},\ \widetilde{\eta},\ \widetilde{g})$ and let $\widetilde{\nabla}$ be its Levi-Civita connection. The following relation

$$(2.1) \qquad (\widetilde{\nabla}_X\widetilde{\phi})Y = -\widetilde{g}(X,Y)\xi + \widetilde{\eta}(Y)X, \quad \text{for} \quad X,Y \in \chi(\widetilde{M}),$$

holds on $\widetilde{M}$ and actually characterizes Sasakian manifolds among almost contact Riemannian manifolds. Let M be an oriented C^∞ hypersurface in $\widetilde{M}$ tangent [1] to the structure vector field $\widetilde{\xi}$ and let $\iota : M \hookrightarrow \widetilde{M}$ the immersion of M in $\widetilde{M}$. On $\widetilde{M}$ we have

$$(2.2) \qquad \widetilde{g}(X,Y) = d\widetilde{\eta}(X,\widetilde{\phi}Y) + \widetilde{\eta}(X)\widetilde{\eta}(Y), \quad \forall\ X,Y \in \chi(\widetilde{M})$$

(if ω is a 1-form, recall the formula: $2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$). On M we set the 1-form

$$(2.3) \qquad \eta = \iota^*\widetilde{\eta}.$$

Let ξ be the restriction of $\widetilde{\xi}$ to M (since the last one is tangent to the hypersurface ξ belongs to $\chi(M)$) and consequently we have

$$(2.4) \qquad \iota_*\xi = \widetilde{\xi}.$$

If we denote by $N \in \chi(\widetilde{M})$ the unit vector field normal to M we can define $U \in \chi(M)$ such that

$$(2.5) \qquad \iota_*U = \widetilde{\phi}N$$

(since $\widetilde{\phi}N$ is a vector field tangent to M). Let g be the induced metric

$$(2.6) \qquad g = \iota^*\widetilde{g}.$$

and define the 1-form u on M by

$$(2.7) \qquad u(X) = g(U,X), \quad \forall\ X \in \chi(M).$$

Consider the distribution

$$H(M) = \{X \in \chi(M)\ :\ \eta(X) = 0,\ u(X) = 0\}$$

and the endomorphism $J : H(M) \longrightarrow H(M)$ given by the restriction of $\widetilde{\phi}$ to $H(M)$, i.e.

$$(2.8) \qquad \iota_*JX = \widetilde{\phi}\ \iota_*X.$$

It can be proved that the definition is good (in the following sense: if $X \in H(M)$ then $JX \in H(M)$). Moreover, J has the property

$$(2.9) \qquad J^2 = -\mathrm{id}_{H(M)}.$$

The tangent space of M can be decomposed in the following direct sum

$$(2.10) \qquad T(M) = H(M) \oplus \ \mathrm{span}\ [U] \oplus \ \mathrm{span}\ [\xi].$$

[1] Many geometers use to consider ξ tangent to the submanifold because in the the theory of CR submanifolds the condition M normal to ξ leads to M anti-invariant submanifold (see [7], Proposition 1.1, p. 43) and the condition ξ oblique gives very complicated embedding equations.

Let us remark that $d\eta$ is non degenerate on $H(M)$.

Lemma 2.1. *We have*

$$[\xi, \Gamma(H(M))] \subset \Gamma(H(M)) \tag{2.11}$$

$$[U, \Gamma(H(M))] \in \ker \eta\,, \tag{2.12}$$

where $\Gamma(H(M))$ is the $C^\infty(M)$-module of smooth sections in $H(M)$.

Proof. It is obvious that $[\xi, \Gamma(H(M))] \subset \ker \eta$. Let's compute $u([\xi, X])$ where $X \in H(M)$. We have

$$u([\xi, X]) = g(U, [\xi, X]) = \widetilde{g}(U, \widetilde{\nabla}_{\widetilde{\xi}} X - \widetilde{\nabla}_X \widetilde{\xi})\ .$$

(Sometimes we will give up the notation ι_* for vector fields tangent to M even that these will be thought as vector fields tangent to $\widetilde{M}$.) We know that $\widetilde{\nabla}_X \widetilde{\xi} = \widetilde{\phi} X$. If X belongs to $H(M)$ then $\widetilde{\phi} X$ belongs to $H(M)$ too, thus it is orthogonal to U. We obtain, since $\widetilde{\nabla}_{\widetilde{\xi}}\, U = -N$ that

$$u([\xi, X]) = \widetilde{g}(U, \widetilde{\nabla}_{\widetilde{\xi}} X) = -\widetilde{g}(\widetilde{\nabla}_{\widetilde{\xi}}\, U, X) = 0$$

which means that $[\xi, X] \in \ker\ u$. Thus $[\xi, X] \in H(M)$ for all $X \in H(M)$.

In order to prove (2.12) we will show that $d\eta(U, X) = 0$ for all $X \in \Gamma(H(M))$. We have

$$d\eta(U, X) = (\iota^* d\widetilde{\eta})(U, X) = d\widetilde{\eta}(\iota_* U, \iota_* X) = \widetilde{g}(\tilde{\phi}^{\,2} N, \iota_* X) = -\widetilde{g}(N, \iota_* X) = 0$$

(since $\iota_* X$ is tangent to M while N is normal). □

Proposition 2.2. *The vector fields ξ and U are orthogonal and of norm 1. Moreover*

$$[U, \xi] = 0\,. \tag{2.13}$$

Proof. We will prove only the second part of this Proposition. Since the inner product $\xi \lrcorner d\eta$ is zero one gets $[U, \xi] \in \ker\ \eta$. Then we use the same technique as above:

$$u([U, \xi]) = g(U, [U, \xi]) = \widetilde{g}(U, \widetilde{\nabla}_U \xi - \widetilde{\nabla}_\xi U) = \widetilde{g}(U, \widetilde{\phi} U) - \widetilde{g}(U, \widetilde{\nabla}_\xi U)\,.$$

But U and $\widetilde{\phi} U$ are orthogonal (with respect to $\widetilde{g}$) and since $||U|| = 1$, by derivation with respect to ξ one gets $\widetilde{g}(U, \widetilde{\nabla}_\xi U) = 0$. It follows that $[U, \xi] \in\ \ker u$.

Consequently $[U, \xi] \in H(M)$. We have to compute now the $H(M)$-component. To do this we will use the non degeneracy of $d\eta$ on $H(M)$. For an arbitrary $X \in H(M)$ we have $2d\eta([U, \xi], X) = -\eta([[U, \xi], X])$. In the Jacobi identity

$$\big[[U, \xi], X\big] + \big[[\xi, X], U\big] + \big[[X, U], \xi\big] = 0$$

applying η and by taking into account the Lemma 2.1 one obtains $\eta([[U, \xi], X]) = 0$. We get the conclusion. □

Proposition 2.3. *The distribution $H(M)$ defines a CR-structure on M of CR-codimension 2.*

Proof. Let $X, Y \in H(M)$. We have to verify the two integrability conditions in order to obtain a CR-structure, namely

a) $[JX, JY] - [X, Y] \in H(M)$,

b) $N_J(X, Y) = 0$, where N_J is the Nijenhuis tensor of J.

An easy computation yields a). Then, from the normality condition of the Sasakian structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ on $\widetilde{M}$, i.e. $N_{\widetilde{\phi}} + 2d\widetilde{\eta} \otimes \widetilde{\xi} = 0$, one gets $\iota_* N_J(X, Y) = 0$ $\big($for $X, Y \in \Gamma(H(M))\big)$. □

In the next we define on M a tensor field f of type $(1, 1)$ as follows

$$\begin{aligned} &f : \chi(M) \longrightarrow \chi(M) \\ &fX = JX \quad \text{for} \quad X \in H(M)\,, \quad fU = 0\,,\ f\xi = 0\,. \end{aligned} \tag{2.14}$$

With respect to the decomposition (2.10), if X is an arbitrary vector field on M then it can be written in the form

$$X = X_{H(M)} + u(X)\, U + \eta(X)\xi$$

where $X_{H(M)}$ is the $H(M)$ component of X. Thus f verifies

$$f^3 + f = 0\,. \tag{2.15}$$

Proposition 2.4. *The structure* $(\phi,\ \xi,\ U,\ \eta,\ u)$ *defined on* M *is an* f *structure with complemented frame* (*see S.I.Goldberg and K.Yano,* [3]) *or, in other terminology, an* f *structure with parallelizable kernel* (f*-pk structure*), *i.e.*

$$\eta(\xi) = 1\,,\ \eta(U) = 0\,,\ u(\xi) = 0\,,\ u(U) = 1$$

$$f\xi = 0\,,\ fU = 0\,,\ \eta \circ f = 0\,,\ u \circ f = 0$$

$$f^2 = -I + \eta \otimes \xi + u \otimes U\,.$$

If X and Y are vector fields on M then we have

$$g(X, Y) = d\eta(X, fY) + u(X)u(Y) + \eta(X)\eta(Y)\,. \tag{2.16}$$

In [4] it is defined a linear connection on an almost $\mathcal{S}$-manifold which generalizes the Tanaka-Webster connection for strictly pseudoconvex CR-manifolds of hypersurface type. In the following we will construct a torsion free connection on M as being the analogue of Matzeu-Oproiu connection.

From now on we will suppose that the inner product

$$U \lrcorner\, du = 0 \tag{2.17}$$

holds on M; this condition yields $[U, \Gamma(H(M))] \subset \ker\ u$ and consequently, by virtue of (2.12) we obtain $[U, \Gamma(H(M))] \subset \Gamma(H(M))$.

The condition above is a weaker condition than the "$\mathcal{S}$" condition $du = \Phi$ (here $\Phi(X, Y) = g(fX, Y))$ and it is equivalent to: $H(M)$ *component of* $\dot{\nabla}_U U$ *vanishes, where* $\dot{\nabla}$ *is the Levi Civita connection of* g. Moreover, this means that $H(M)$ component of AU vanishes too, where A is the Weingarten operator.

There are some important cases in which this happens, namely

- (TCG) M is a totally contact geodesic submanifold in $\widetilde{M}$, i.e.

$$B(X, Y) = \widetilde{\eta}(X)B(Y, \xi) + \widetilde{\eta}(Y)B(X, \xi)$$

• (TCU) M is a totally contact umbilical submanifold in $\widetilde{M}$, i.e.

$$B(X,Y) = [\widetilde{g}(X,Y) - \widetilde{\eta}(X)\widetilde{\eta}(Y)] \ \alpha + \widetilde{\eta}(X)B(Y,\xi) + \widetilde{\eta}(Y)B(X,\xi)$$

where α is a vector field normal to M; it follows that $\alpha = \lambda N$ with λ non-vanishing smooth function on M (B is the second fundamental form of M in $\widetilde{M}$)

• (PUH) M is a pseudo umbilical hypersurface in $\widetilde{M}$, i.e.

$$AX = \lambda \ (X - \eta(X)\xi) + \mu \ u(X) \ U - \eta(X)U - u(X) \ \xi$$

with $\lambda, \mu \in C^\infty(M)$.

Let us remark that in the case (TCG) the 1-form u is closed, while in cases (TCU) and (PUH) we have $du(X,Y) = \lambda g(X, fY)$ which, in general, is different from 0.

Recall two formulas (Gauss and Weingarten):
(G) $\widetilde{\nabla}_X Y = \dot{\nabla}_X Y + b(X,Y)N, \ X, Y \in \chi(M)$
(W) $\widetilde{\nabla}_X N = -AX, \ X \in \chi(M)$
where $b(X,Y)$ is the scalar second fundamental form. Since the ambient is Sasakian one gets:
(1) $\dot{\nabla}_X \xi = fX \ , \ b(X,\xi) = -u(X)$
(2) $\dot{\nabla}_X U = -fAX \ , \ b(X,U) = u(AX)$
(3) $\dot{\nabla}_U U = 0$
(4) $\eta(AX) = -u(X) \ , \ u(AX) = -\eta(X) + b \ u(X)$, where $b = b(U,U)$
(5) $A\xi = -U \ , \ AU = -\xi + b \ U \ , \ AX \in H(M)$ if $X \in H(M)$
(6) $AX = -\eta(X)U - u(X)\xi + b \ u(X)U + H(M)$, with $X \in \chi(M)$.

Denote by $\psi = \frac{1}{2}(\mathcal{L}_\xi f)$ and $p = \frac{1}{2}(\mathcal{L}_U f)$ where $\mathcal{L}$ is the Lie derivative. If the ambient manifold $\widetilde{M}$ is only a contact manifold (not necessarily Sasakian) we give

Proposition 2.5. *The following statements hold:*
1) $\psi\xi = 0 \ , \ \psi U = 0 \ , \ p\xi = 0 \ , \ pU = 0$
2) $f\psi + \psi f = 0 \ , \ fp + pf = 0$
3) $\eta \circ \psi = 0 \ , \ \eta \circ p = 0 \ , \ u \circ \psi = 0 \ , \ u \circ p = 0$
4) $d\eta(\psi X, Y) = -d\eta(X, \psi Y) \ , \ d\eta(pX, Y) = -d\eta(X, pY)$
5) *the operators ψ and p are self-adjoint with respect to the metric g, i.e.*

$$g(\psi X, Y) = g(X, \psi Y) \ , \ g(pX, Y) = g(X, pY) \, .$$

Proof. The relations 1) - 3) are immediately. Let's prove the first relation in 4) (the second can be proved in the same way). Let $X, Y \in H(M)$. Then

$$2d\eta(\psi X, Y) = \xi(d\eta(fX, Y)) - d\eta([X,\xi], fY) - d\eta([Y,\xi], fX) \, .$$

Interchanging X and Y we get $d\eta(\psi Y, X) = d\eta(\psi X, Y)$ and hence the conclusion. The last statement can be obtained easily from 1) - 4). □

In our case $\widetilde{M}$ is Sasakian.

Proposition 2.6. *We have*
1) $\psi \equiv 0$
2) $2pX = (A + fAf)X + u(X) \ \xi + \eta(X) \ U - b \ u(X) \ U;$

p vanishes in cases (TCG), (TCU) and (PUH).

Proof. 1) is a direct consequence of $\mathcal{L}_{\widetilde{\xi}}\,\widetilde{\phi} = 0$ (since $\widetilde{M}$ is Sasakian).

For 2) the following relations hold:

$$\begin{aligned}[U, fX] &= \dot{\nabla}_U(fX) - \dot{\nabla}_{fX}U = \tan\left(\widetilde{\nabla}_U(\widetilde{\phi}X + u(X)\ N)\right) + fAfX \\ &= \eta(X)\ U + fAfX + \tan(\widetilde{\phi}\widetilde{\nabla}_U X) \\ f[U, X] &= f\dot{\nabla}_U X - AX + \eta(AX)\ \xi + u(AX)\ U\,.\end{aligned}$$

Now, by using (G) and (W) we get the statement.

If M is TCG in $\widetilde{M}$ then $b = 0$; if M is TCU then $b = \lambda$ and if M is PUH then $b = \lambda + \mu$. One gets $p = 0$. □

In the following we are looking for a torsion free connection on M related with the structures defined so far. We can state the following theorem.

Theorem 2.7. *There exists one and only one torsion free connection on M such that*

$$\begin{aligned}&(\nabla_X \eta)(Y) = d\eta(X, Y)\,, \qquad (\nabla_X u)(Y) = du(X, Y) \\ &\nabla_X d\eta = 0\,, \quad \nabla_X \xi = 0\,, \quad \nabla_X U = 0 \\ &(\nabla_X f)Y = u(X)\,\{(A + fAf)Y + u(Y)\xi + \eta(Y)U - b\ u(Y)U\} \\ &\qquad\qquad - d\eta(X, fY)\xi + d\eta(AX, fY)U\,.\end{aligned} \tag{2.18}$$

Remark 2.8. If $\widetilde{M}$ is only a contact manifold the last condition in (2.18) must be replaced by

$$(\nabla_X f)Y = 2\eta(X)\psi Y + 2u(X)pY - d\eta(X, fY)\xi - du(X, fY)U \tag{2.19}$$

which generalizes the fourth condition in (3.1) from [5] pg. 5.

Remark 2.9. If M is TCG, TCU or PUH then $du = -\lambda\ d\eta$ (λ can vanish) and the last condition in (2.18) becomes

$$(\nabla_X f)Y = d\eta(fX, Y)\ \zeta$$

where $\zeta = \xi - \lambda\ U$.

Proof (of Theorem 2.7)**.** Before starting with the proof of the theorem let us remark that the conditions in (2.18) have been suggested by (3.1) from [5]. We also assumed the compatibility conditions with the structure defined so far. To get the connection ∇ (in terms of the f-pk structure) we shall compute $\eta(\nabla_X Y)$, $u(\nabla_X Y)$ and $d\eta(\nabla_X Y, Z)$ for $X, Y, Z \in \chi(M)$.

We obtain immediately

$$\begin{aligned}\eta(\nabla_X Y) &= X(\eta(Y)) - d\eta(X, Y) \\ u(\nabla_X Y) &= X(u(Y)) - du(X, Y)\,.\end{aligned} \tag{2.20}$$

Let's compute now ∇g. First we have

$$(\nabla_X g)(Y,Z) = (\nabla_X d\eta)(Y,fZ) + d\eta(Y,(\nabla_X f)Z) + (\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) + (\nabla_X u)(Y)u(Z) + u(Y)(\nabla_X u)(Z)\,. \tag{2.21}$$

Taking into account (2.18) we get

$$(\nabla_X g)(Y,Z) = u(X)\ d\eta(Y,(A+fAf)Z) + d\eta(X,Y)\eta(Z) + d\eta(X,Z)\eta(Y) + du(X,Y)u(Z) + du(X,Z)u(Y)\,. \tag{2.22}$$

Let us remark that if we restrict to the distribution $H(M)$ we have

$$(\nabla g)_{|_{H(M)}} = 0\,. \tag{2.23}$$

In the following we will do similar computations as in the Levi Civita theorem (since the connection is torsion free). We can write

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) - (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + (\nabla_Z g)(X,Y)\,. \tag{2.24}$$

Since $d\eta(\nabla_X Y,Z) = -g(\nabla_X Y,fZ)$ for all $X,Y,Z \in \chi(M)$ we obtain

$$2d\eta(\nabla_X Y,Z) = Xd\eta(Y,Z) + Yd\eta(X,Z) + (fZ)d\eta(X,fY) + d\eta([X,Y],Z) + d\eta([X,fZ],fY) + d\eta([Y,fZ],fX) + u(X)d\eta(Z,(Af-fA)Y) + u(Y)d\eta(Z,(Af-fA)X)\,. \tag{2.25}$$

If $\widetilde{M}$ is only a contact manifold the relation before becomes

$$2d\eta(\nabla_X Y,Z) = Xd\eta(Y,Z) + Yd\eta(X,Z) + (fZ)d\eta(X,fY) + d\eta([X,Y],Z) + d\eta([X,fZ],fY) + d\eta([Y,fZ],fX) + 2\eta(X)d\eta(fZ,\psi Y) + 2\eta(Y)d\eta(fZ,\psi X) + 2u(X)d\eta(fZ,pY) + 2u(Y)d\eta(fZ,pX)\,. \tag{2.26}$$

The relations (2.20) and (2.25) completely define the connection ∇ (since $d\eta$ is non degenerate on $H(M)$).

Furthermore, if ∇' is another connection satisfying the hypotheses of the theorem, we have $\eta(\nabla_X Y) = \eta(\nabla'_X Y)$, $u(\nabla_X Y) = u(\nabla'_X Y)$ and $d\eta(\nabla_X Y,Z) = d\eta(\nabla'_X Y,Z)$ which imply $\nabla = \nabla'$. □

We are interested now to find a relation between the adapted connection and the Levi Civita connection $\dot{\nabla}$. Define and endomorphism S on $H(M)$ by

$$d\eta(SX,Y) = -du(X,fY)\,, \quad X,Y \in H(M) \tag{2.27}$$

(due the non degeneracy of $d\eta$ the endomorphism S is well defined).

An easy computation yields $du(X,Y) = \frac{1}{2}\ (g(AX, fY) - g(AY, fX))$, for all $X, Y \in \chi(M)$. Consequently we obtain $S = -\frac{1}{2}\ (fA + Af)$. We can extend S, if necessary, to span$[\xi] \oplus$ span$[U]$ by putting $S\xi = 0$ and $SU = 0$ and the previous formula remains true.

Taking into account that the difference between two torsion free connections is a symmetric $(1, 2)$ tensor field, after a straightforward computations one gets

$$\nabla - \dot{\nabla} = \alpha \otimes U + 2\,(u \odot fA - \eta \odot f) \tag{2.28}$$

where

$$(\alpha \otimes U)\,(X,Y) = \alpha(X,Y)U$$

with $\alpha(X,Y) = \frac{1}{2}\ (g(AX, fY) + g(AY, fX))$ and $\odot$ is the symmetric product, i.e.

$$(\eta \odot f)\,(X,Y) = \frac{1}{2}\ (\eta(X)fY + \eta(Y)fX)$$

and

$$(\eta \odot fA)\,(X,Y) = \frac{1}{2}\ (\eta(X)fAY + \eta(Y)fAX)$$

for all $X, Y \in \chi(M)$.

Remark 2.10. If $\widetilde{M}$ is only a contact manifold the relation between ∇ and $\dot{\nabla}$ is given by

$$\nabla - \dot{\nabla} = \alpha \otimes U + 2\,(\eta \odot h + u \odot k - u \odot S - \eta \odot f) \tag{2.29}$$

where $h = f\psi$ and $k = fp$. (The equivalence between (2.28) and (2.29) holds due the fact that on $H(M)$ we have $\mathcal{L}_U f = A + fAf$ if the ambient manifold is Sasakian.)

Remark 2.11. If M is TCG, TCU or PUH then $\alpha = 0$ and

$$\nabla - \dot{\nabla} = -2\ \theta \odot f$$

where $\theta = \eta - \lambda\ u$.

3. Curvature of the torsion free adapted connection

Consider the curvature tensor field R of ∇ defined by

$$R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,, \quad X,\ Y,\ Z \in \chi(M)\,. \tag{3.1}$$

We will find some general relations and properties of R and especially for the restriction of R on $H(M)$.

Equation (2.18) imply:

$$R_{XY}\xi = 0\ , \quad R_{XY}U = 0\,, \quad \forall\ X, Y \in \chi(M)\,. \tag{3.2}$$

Moreover, $R_{XY}Z$ belongs to $\ker\eta$ but it is not necessarily a section in $H(M)$ for all $X, Y, Z \in \chi(M)$. Yet, $R_{UX}Y \in \Gamma(H(M))$. We also have

$$\begin{aligned} R_{XY}fZ &= fR_{XY}Z + 4du(X,Y)pZ \\ &\quad - (\nabla_X du)(Y, fZ)\ U + (\nabla_Y du)(X, fZ)\ U\,, \quad X, Y, Z \in \Gamma(H(M))\,. \end{aligned} \tag{3.3}$$

Define now a 4 covariant tensor field $\mathcal{R}$ by

$$\mathcal{R}(W,Z,X,Y) = g(W, R_{XY}Z)\,, \quad X,Y,Z \in \chi(M) \tag{3.4}$$

($\mathcal{R}$ is a kind of Riemann Christoffel tensor).

We are interested now to find some symmetry properties for the tensor field $\mathcal{R}$ similar those for the usual Riemann Christoffel tensor (in Riemannian geometry).

Obviously $\mathcal{R}(W,Z,X,Y) = -\mathcal{R}(W,Z,Y,X)$ and $\underset{(X,Y,Z)}{\mathfrak{S}} \mathcal{R}(W,Z,X,Y) = 0$ (due to the first Bianchi identity fulfilled by R).

Using (3.3) we get

$$\mathcal{R}(Z,Z,X,Y) = d\eta(Z,AZ)\ (d\eta(X,AY) - d\eta(Y,AX)) \tag{3.5}$$

$\big($with $X,Y,Z \in \Gamma(H(M))\big)$ which implies

$$\begin{aligned} &\mathcal{R}(Z,W,X,Y) + \mathcal{R}(W,Z,X,Y) \\ &\qquad = (d\eta(Z,AW) + d\eta(W,AZ))(d\eta(X,AY) - d\eta(Y,AX))\,. \end{aligned} \tag{3.6}$$

As consequence we have

Proposition 3.1. *The Riemann Christoffel tensor field $\mathcal{R}$ of the linear connection ∇ satisfies the following equation*

$$\begin{aligned} &\mathcal{R}(W,Z,X,Y) - \mathcal{R}(X,Y,W,Z) \\ &\qquad = d\eta(X,AY)d\eta(Z,AW) - d\eta(Y,AX)d\eta(W,AZ) \\ &\qquad\quad + d\eta(W,AX)d\eta(Z,AY) - d\eta(X,AW)d\eta(Y,AZ) \\ &\qquad\quad + d\eta(X,AZ)d\eta(Y,AW) - d\eta(Z,AX)d\eta(W,AY)\,. \end{aligned} \tag{3.7}$$

From the relation above we easily obtain

$$\mathcal{R}(E,Z,E,Y) - \mathcal{R}(E,Y,E,Z) = d\eta(E,AE)\ (d\eta(Z,AY) - d\eta(Y,AZ)) \tag{3.8}$$

where E,Y,Z are sections in $H(M)$. Now, replacing E by fE and taking into account that $d\eta(fE,AfE) = -d\eta(E,AE)$ we get

$$\mathcal{R}(E,Y,E,Z) + \mathcal{R}(fE,Y,fE,Z) = \mathcal{R}(E,Z,E,Y) + \mathcal{R}(fE,Z,fE,Y)\,. \tag{3.9}$$

Remark 3.2. If M is TCG, TCU or PUH in $\widetilde{M}$ then $\mathcal{R}$ is skew-symmetric in first two arguments and pairs symmetric (i.e. $\mathcal{R}(W,Z,X,Y) = \mathcal{R}(X,Y,W,Z)$).

Consider the two times covariant tensor

$$\rho(R)(Y,Z) = \text{trace}\ (X \longmapsto R_{XY}Z)\,, \quad X,Y,Z \in \chi(M) \tag{3.10}$$

(where the trace is made by using the metric g) – the tensor defined above is a kind of Ricci tensor. If we take an orthonormal basis of the form $\{E_i, fE_i, \xi, U\}_{i=\overline{1,n}}$ on M, the Ricci tensor can be written as

$$\rho(R)(Y,Z) = \sum_{i=1}^{n} \{\mathcal{R}(E_i,Z,E_i,Y) + \mathcal{R}(fE_i,Z,fE_i,Y)\}\ .$$

As consequence of the relation (3.9) we have the symmetry of the Ricci tensor, namely

$$\rho(R)(Y,Z) = \rho(R)(Z,Y)\,, \quad Y,Z \in \Gamma(H(M))\,. \tag{3.11}$$

Moreover, we have

$$\rho(R)(fY,fZ)-\rho(R)(Y,Z)=4(du(pY,fZ)+du(pZ,fY))\,. \tag{3.12}$$

We will say that the CR-manifold is CR-*Einstein* if

$$\rho(R)(X,Y)=\lambda\, g(X,Y)$$

for all $X,Y\in H(M)$, where $\lambda\in C^\infty(M)$.

We will end this section by studying the situation

$$AX=\lambda(X-\eta(X)\xi)+\mu u(X)\,U-\eta(X)\,U-u(X)\xi$$

with λ and μ not necessarily non vanishing smooth functions on M. This case, let's call it $(\lambda-\mu)$ contains the three cases TCG, TCU and PUH.

We have $d\eta(X,AY)=\lambda d\eta(X,Y)$ and $(\dot\nabla_X f)Y=-g(X,Y)\ \zeta$ for $X,Y\in H(M)$. Consequently, the most of the relations involving the curvature tensor fields simplifies. For example we have for $X,Y,Z,W\in H(M)$

$$R_{XY}(fZ)=fR_{X,Y}Z-\{X(\lambda)d\eta(Y,fZ)-Y(\lambda)d\eta(X,fZ)\}\ U$$

$$\mathcal{R}(Z,W,X,Y)=-\mathcal{R}(W,Z,X,Y)\,,$$

$$\mathcal{R}(W,Z,X,Y)=\mathcal{R}(X,Y,W,Z)\,.$$

These yield to

$$\mathcal{R}(fW,fZ,X,Y)=\mathcal{R}(W,Z,X,Y)\,,\ \ \mathcal{R}(fW,fZ,fX,fY)=\mathcal{R}(W,Z,X,Y)$$

and hence

$$\rho(R)(fY,fZ)=\rho(R)(Y,Z)\,.$$

Let write now the relation between the curvature tensor R (of the adapted connection) and $\dot R$ (the curvature tensor of the Levi Civita connection) – in general this relation is very complicated:

$$\begin{aligned}\dot R_{XY}Z&=R_{XY}Z+2(1+\lambda^2)d\eta(X,Y)fZ+(u(X)Y(\lambda)-u(Y)X(\lambda))fZ\\&\quad-(\theta(X)d\eta(fY,Z)-\theta(Y)d\eta(fX,Z))\zeta\\&\quad+\theta(Z)(\theta(X)f^2Y-\theta(Y)f^2X)\\&\quad-((1+\lambda^2)d\eta(Y,Z)-Y(\lambda)u(Z))fX\\&\quad+((1+\lambda^2)d\eta(X,Z)-X(\lambda)u(Z))fY\,.\end{aligned} \tag{3.13}$$

If we consider $W=X=E$, with $E\in H(M)$ and of norm 1, one gets

$$\begin{aligned}\dot{\mathcal{R}}(E,Z,E,Y)&=\mathcal{R}(E,Z,E,Y)+\theta(Y)\theta(Z)-3(1+\lambda^2)g(E,fY)g(E,fZ)\\&\quad-(u(Z)g(E,fY)+u(Y)g(E,fZ))\ d\lambda(E)\,.\end{aligned}$$

It follows that

$$\begin{aligned}\rho(\dot R)(Z,Y)&=\rho(R)(Z,Y)-2(1+\lambda^2)g(fY,fZ)\\&\quad+2n\ \theta(Y)\theta(Z)-(u(Y)d\lambda(fZ)+u(Z)d\lambda(fY))\,.\end{aligned}$$

4. Examples

In this section we will give some examples. Let consider as ambient manifolds $\mathbf{R}^5$ with (global) coordinates $x,\ y,\ v,\ w,\ z$. On $\mathbf{R}^5$ we have the usual Sasakian structure as follows:

$$(4.1)\qquad \begin{cases} \widetilde{\eta} = \frac{1}{2}\ (dz - ydx - wdv)\,, \quad \widetilde{\xi} = 2\frac{\partial}{\partial z} \\ \widetilde{\phi}\frac{\partial}{\partial x} = \frac{\partial}{\partial y}\,, \ \ \widetilde{\phi}\frac{\partial}{\partial y} = -\frac{\partial}{\partial x} - y\ \frac{\partial}{\partial z}\,, \ \ \widetilde{\phi}\frac{\partial}{\partial v} = \frac{\partial}{\partial w}\,, \ \ \widetilde{\phi}\frac{\partial}{\partial w} = -\frac{\partial}{\partial v} - w\ \frac{\partial}{\partial z} \\ \widetilde{g} = \widetilde{\eta}\otimes\widetilde{\eta} + \frac{1}{4}\ (dx^2 + dy^2 + dv^2 + dw^2) \end{cases}$$

If the hypersurface M is given by $f(x, y, v, w, z) = 0$ (where f is a smooth function on $\mathbf{R}^5$) the tangency condition of the structure vector field $\widetilde{\xi}$ to M yields the fact that f does not depend on z. After the computations we obtain the expression of the gradient of f. We will denote $\frac{\partial}{\partial x}$ by ∂_x and similarly for the other coordinates. With these notations, the unitary vector field N normal to the hypersurface M is given by $N = \frac{2}{\mu}\ \big(\frac{\partial f}{\partial x}\ \partial_x + \frac{\partial f}{\partial y}\ \partial_y + \frac{\partial f}{\partial v}\ \partial_v + \frac{\partial f}{\partial w}\ \partial_w + \big(y\frac{\partial f}{\partial x} + w\frac{\partial f}{\partial v}\big)\partial_z\big)$ where $\mu = (f_x^2 + f_y^2 + f_v^2 + f_w^2)^{\frac{1}{2}}$. Hence we obtain the expression of U:

$$(4.2)\qquad U = \frac{2}{\mu}\ (-f_y\partial_x + f_x\partial_y - f_w\partial_v + f_v\partial_w - (yf_y + wf_w)\ \partial_z)\,.$$

Remark that $[U, \xi] = 0$.

A vector field $X \in \chi(\mathbf{R}^5)$ is tangent to M and belongs to ker η if it is of the following form

$$(4.3)\qquad X = A(\partial_x + y\partial_z) + B\partial_y + C(\partial_v + w\partial_z) + D\partial_w$$

where $A,\ B,\ C,\ D$ are smooth functions on M satisfying

$$(4.4)\qquad Af_x + Bf_y + Cf_v + Df_w = 0\,.$$

Then, if we ask $X \in$ ker u we get another condition, namely

$$(4.5)\qquad -Af_y + Bf_x - Cf_w + Df_v = 0\,.$$

Since $\mu \neq 0$ we may suppose that $f_v^2 + f_w^2 \neq 0$, otherwise we have $f_x^2 + f_y^2 \neq 0$ and the computations are similar. Let's make some notations

$$(4.6)\qquad a = f_xf_w - f_yf_v\,, \quad b = f_xf_v + f_yf_w\,, \quad \alpha = f_v^2 + f_w^2\,.$$

By using the relations (4.4) and (4.5) we obtain

$$C = \frac{1}{\alpha}\,(aB - bA) \quad , \quad D = -\frac{1}{\alpha}\,(aA + bB)\,.$$

Consequently, we obtain a basis in $H =$ ker $\eta \cap$ ker u

$$(4.7)\qquad \begin{cases} X_1 = \alpha\partial_x - b\partial_v - a\partial_w + (y\alpha - wb)\partial_z \\ X_2 = \alpha\partial_y + a\partial_v - b\partial_w + wa\partial_z = JX_1 \end{cases}$$

where J is the restriction of $\widetilde{\phi}$ to H. Let us remark that $|X_1| = |X_2| = \frac{\mu}{2}\ \sqrt{\alpha}$.

Example 1 (TCG)**.** Consider M the hyperplane (passing by the origin and being parallel with z axis) defined by $f(x,y,z,v,w) = ax+by+cv+dw \equiv 0$ where $a,\ b,\ c,\ d \in \mathbf{R}$ with $a^2+b^2+c^2+d^2 = \mu^2 \neq 0$. We have

$$U = \frac{2}{\mu}\ \{-b\partial_x + a\partial_y - d\partial_v + c\partial_w - (by+dw)\ \partial_z\}$$

and $X_1 = (c^2+d^2)\ \partial_x - (ac+bd)\ \partial_v - (ad-bc)\ \partial_w + [(c^2+d^2)y - (ac+bd)w]\ \partial_z$. The 2-form du vanishes identically and we have $[U,X_1]=0$, $[U,JX_1]=0$. Computing the connection ∇ we obtain $\nabla_{X_1}X_2 = -\frac{\mu^2(c^2+d^2)}{4}\ \xi$. As consequence, the connection is flat.

Example 2 (TCU)**.** Let M be defined by $f(x,y,v,w,z) = x^2+y^2+v^2+w^2-1 \equiv 0$ (a hyper cylinder $S^3\times\mathbf{R}$ in $\mathbf{R}^5$). The vector field U is given by

$$U = 2\{x\partial_y - y(\partial_x + y\partial_z) + v\partial_w - w(\partial_v + w\partial_z)\}\,.$$

Moreover, we have

$$\mu = 2\,,\quad a = 4(xw-yv)\,,\quad b = 4(xv+yw)\,,\quad \alpha = 4(v^2+w^2)$$

and consequently X_1 is determined by

$$X_1 = 4\{(v^2+w^2)\partial_x - (xv+yw)\partial_v - (xw-yv)\partial_w + v(yv-xw)\partial_z\}\,.$$

The following relations hold

$$[U,X_1] = -2X_2 \quad , \quad [U,X_2] = 2X_1$$

(where $X_2 = JX_1$) which means that $[U,\Gamma(H)]\subset\Gamma(H)$.

Moreover, since $du = \frac{1}{2}\ (dx\wedge dy + dv\wedge dw)$ we get $U\lrcorner du = 0$. We have also

$$[X_1,X_2] = -16(v^2+w^2)\ U - 8(v^2+w^2)\ \xi\,.$$

Now we are able to write the expression of the connection ∇. We have

$$\begin{cases} \nabla_{X_1}X_1 = -4xX_1 + 4yX_2\,, \quad \nabla_{X_2}X_2 = 4xX_1 - 4yX_2 \\ \nabla_{X_1}X_2 = -4yX_1 - 4xX_2 - 8(v^2+w^2)\,U - 4(v^2+w^2)\xi \\ \nabla_U X_1 = -2X_2\,, \quad \nabla_U X_2 = 2X_1\,. \end{cases}$$

Computing the curvature tensor of ∇ we obtain

$$R_{X_1X_2}X_1 = -64(v^2+w^2)X_2\,, \quad R_{X_1X_2}X_2 = 64(v^2+w^2)X_1$$

other components being zero. It follows

$$\rho(R)(X_1,X_1) = \rho(R)(X_2,X_2) = 256(v^2+w^2)$$

and $\rho(R)(X_1,X_2) = 0$.

Example 3. Consider now the following hypersurface M in $\mathbf{R}^5$

$$M = \{(x,y,z,v,w)\in\mathbf{R}^5\ :\ w = xy\}\,.$$

The tangent space of M is spanned by

$$\left[\begin{array}{l} U = \dfrac{2}{\mu}\,(-x\partial_x + y\partial_y + \partial_v)\,, \quad \xi = 2\partial_z \\ X_1 = \partial_x + x\partial_v + y\partial_w + y(1+x^2)\,\partial_z\,, \quad X_2 = \partial_y - y\partial_v + x\partial_w - xy^2\partial_z\,. \end{array}\right.$$

In order to verify if $U \lrcorner du = 0$ or not we compute $[U, X_1]$ (since $X_1 \in \Gamma(H)$). We get

$$[U, X_1] = -\frac{x^3 + 3x + xy^2}{\mu^2}\,U + \frac{2(1 - x^2 + y^2)}{\mu^3}\,X_1 + \frac{4xy}{\mu^3}\,X_2$$

so, $u([U, X_1]) \neq 0$, which means that $U \lrcorner du \neq 0$. This example proves that the condition (2.17) is not automatically satisfied.

The next example is inspired from the following theorem ([7], Th. 5.2, p. 185): *Let M be a compact orientable pseudo-umbilical hypersurface of S^{2n+1} ($n \geq 2$). Then M is*

$$S^{2n-1}(r_1) \times S^1(r_2)\,, \qquad r_1^2 + r_2^2 = 1\,.$$

Example 4 (PUH)**.** Let $M = S^3(r_1) \times S^1(r_2)$ with $r_1^2 + r_2^2 = 1$ be a pseudo-umbilical hypersurface in S^5 ($\subset \mathbf{R}^6$) as a Sasakian space form. On $\mathbf{R}^6$ consider global coordinates x, y, v, w, s, t so, on M we have $|p_1| = r_1$ and $|p_2| = r_2$ where $p_1 = (x, y, v, w)$, $p_2 = (s, t)$ and $|\cdot|$ denotes the usual Euclidean norm. Consider

$$\xi_1 = \frac{1}{r_1}(-y, x, -w, v)\,, \quad X_1 = \frac{1}{r_1}(-v, w, x, -y)\,, \quad X_2 = \frac{1}{r_1}(-w, -v, y, x)$$

which form an orthonormal frame on $S^3(r_1)$ and $\xi_2 = \frac{1}{r_2}(-s, t) \in \chi(S^1(r_2))$. Consider also the following contact forms on $S^3(r_1)$ and $S^1(r_2)$ respectively

$$\eta_1 = \frac{1}{r_1}(-y\,dx + x\,dy - w\,dv + v\,dw)\,, \quad \eta_2 = \frac{1}{r_2}(-s\,dt + t\,ds)\,.$$

With these notations, the (almost) contact structure on S^5 is given by

$$\xi = r_1\xi_1 + r_2\xi_2\,, \quad \eta = r_1\eta_1 + r_2\eta_2\,, \quad \phi X_1 = X_2\,, \quad \phi X_2 = -X_1\,.$$

The unit normal vector field on M is $N = -\frac{r_2}{r_1}\,p_1 + \frac{r_1}{r_2}p_2$ thus $U = -r_2\xi_1 + r_1\xi_2$. We have obtained a global frame on M satisfying

$$\begin{cases} [X_1, \xi] = 2X_2\,, \quad [X_1, U] = -\dfrac{2r_2}{r_1}\,X_2 \\ [X_2, \xi] = -2X_1\,, \quad [X_2, U] = \dfrac{2r_2}{r_1}\,X_1 \\ [X_1, X_2] = -2\xi + \dfrac{2r_2}{r_1}\,U\,. \end{cases}$$

Moreover, on M we have $du = -\frac{r_2}{r_1}\,d\eta$. Computing the torsion free adapted connection we obtain:

$$\nabla_{X_1}X_1 = 0\,, \quad \nabla_{X_1}X_2 = -\xi + \frac{r_2}{r_1}\,U\,, \quad \nabla_{X_2}X_2 = 0$$

the other expressions are easy deducible from the relations above. The non-vanishing components of the curvature tensor are

$$R_{X_1X_2}X_1 = -\frac{4}{r_1^2}\ X_2\ ,\ R_{X_1X_2}X_2 = \frac{4}{r_1^2}\ X_1$$

and hence

$$\rho(R)(X_1, X_1) = \rho(R)(X_2, X_2) = \frac{4}{r_1^2}\,, \quad \rho(R)(X_1, X_2) = 0\,.$$

Consequently we have a CR-Einstein manifold which is never flat.

Example 5. The condition (2.17) $U \lrcorner du = 0$ is a quite strong condition which in general yields a PDE system. Even that the dimension of the ambient manifold is small, this PDE system is rather complicate. Yet, if the submanifold M in $\mathbf{R}^5$ is defined by a function depending only on one or two variables, the condition (2.17) is automatically satisfied. Let us, for example, consider $f(x, y, v, w) = y^2 + w^2 - r^2$ (i.e. $M = S^1 \times \mathbf{R}^3$). On M we have:

$N = \frac{1}{r}\,(y\mathcal{B}_1 + w\mathcal{B}_2)$ - the unitary normal vector field;
$U = -\frac{1}{r}\,(y\mathcal{A}_1 + w\mathcal{A}_2)$;
$\left.\begin{array}{l} X_1 = \frac{1}{r}\,(w\mathcal{A}_1 - y\mathcal{A}_2) \\ X_2 = \frac{1}{r}\,(w\mathcal{B}_1 - y\mathcal{B}_2) \end{array}\right]$ -unitary, orthogonal and belonging to $H(M)$

where $\mathcal{A}_1 = 2\,(\partial_x + y\partial_z)$, $\mathcal{B}_1 = 2\partial_y$, $\mathcal{A}_2 = 2\,(\partial_v + w\partial_z)$, $\mathcal{B}_2 = 2\partial_w$ (see for more details [1], [2]). An easy computation gives the expression of Weingarten operator namely, $AX_1 = 0$, $AX_2 = -\frac{2}{r}X_2$, $A\xi = -U$ and $AU = -\xi$ so M does not belong to the case $(\lambda - \mu)$ described in the previous section.

Computing the torsion free adapted connection we obtain

$$\begin{array}{llll} \nabla_{X_1}X_1 = 0\,, & \nabla_{X_2}X_2 = 0\,, & \nabla_{X_1}X_2 = -\xi - \dfrac{1}{r}\,U & \\ \nabla_U X_1 = 0\,, & \nabla_U X_2 = \dfrac{2}{r}\,X_1\,, & \nabla_\xi X_1 = -\,, & \nabla_\xi X_2 = 0\,. \end{array}$$

Consequently

$$\rho(R)(X_1, X_1) = 0\,, \quad \rho(R)(X_1, X_2) = 0\,, \quad \rho(R)(X_2, X_2) = \frac{4}{r^2}$$

and hence M is not CR-Einstein.

Example 6. Looking at the examples 3 and 5 we are interested to study a submanifold in $\mathbf{R}^5$ defined by a function f depending on x, y and w (with $f_w \neq 0$) and having the property (2.17). In this case one gets the following PDE's system

$$\begin{cases} -f_w f_y f_{xy} + f_w f_x f_{yy} + f_y^2 f_{xw} - f_x f_y f_{yw} = 0 \\ -f_w f_x f_{xy} + f_w f_y f_{xx} + f_x^2 f_{yw} - f_x f_y f_{xw} = 0\,. \end{cases}$$

Since $f_w \neq 0$ let us consider that the submanifold M is given explicitely by $w = r(x, y)$, $r \in C^\infty(D)$, $D \subset \mathbf{R}^2$. Then we obtain the following PDE's system involving r:

$$\begin{cases} r_x r_{yy} - r_y r_{xy} = 0 \\ r_x r_{xy} - r_y r_{xx} = 0 \end{cases}$$

having the solution $r(x,y) = \mathbf{r}(y+\rho x)$ with an arbitrary function $\mathbf{r} \in C^\infty(I)$, $I \subset \mathbf{R}$, $r' \neq 0$ and $\rho \in \mathbf{R}$.
On M we have: $\mu^2 = 1 + (\rho^2+1)\mathbf{r}'^2$,

$N = \frac{1}{\mu}(-\rho\,\mathbf{r}'\mathcal{A}_1 - \mathbf{r}'\mathcal{B}_1 + \mathcal{B}_2)$ - the unitary normal vector field,

$U = \frac{1}{\mu}(\mathbf{r}'\mathcal{A}_1 - \rho\,\mathbf{r}'\mathcal{B}_1 - \mathcal{A}_2)$,

$\left.\begin{array}{l} X_1 = \frac{1}{\mu}(\mathcal{A}_1 + \mathbf{r}'\mathcal{A}_2 + \rho\,\mathbf{r}'\mathcal{B}_2) \\ X_2 = \frac{1}{\mu}(\mathcal{B}_1 - \rho\,\mathbf{r}'\mathcal{A}_2 + \mathbf{r}'\mathcal{B}_2) \end{array}\right]$ - unitary, orthogonal and belonging to $H(M)$.

After some easy computations, we get the expression of the Weingarten operator: $A\xi = -U$, $AU = -\xi$, $AX_1 = \frac{2\rho\,\mathbf{r}''}{\mu^3}\,(\rho X_1 + X_2)$ and $AX_2 = \frac{2\mathbf{r}''}{\mu^3}\,(\rho X_1 + X_2)$, so, M is TCG if and only if M is a hyperplane ($\mathbf{r}'' = 0$) and if and only if M is minimal (the mean curvature is $H = \frac{\mathbf{r}''(\rho^2+1)}{2\mu^3}\,N$).

Computing the adapted connection we obtain

$$\nabla_{X_1} X_2 = -\nabla_{X_2} X_1 = -\xi + \frac{(\rho^2+1)\mathbf{r}''}{\mu^3} U$$

$$\nabla_U X_1 = \rho\,\nabla_U X_2 = -\frac{2\rho\mathbf{r}''}{\mu^3}\,(X_1 - \rho X_2)$$

all other combinations being zero. Consequently, the adapted connection is flat if and only if M is a hyperplane.

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