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SLANT HANKEL OPERATORS

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Abstract. In this paper the notion of slant Hankel operator K_φ, with symbol φ in L^∞, on the space $L^2(\mathbb{T})$, $\mathbb{T}$ being the unit circle, is introduced. The matrix of the slant Hankel operator with respect to the usual basis $\{z^i : i \in \mathbb{Z}\}$ of the space L^2 is given by $\langle \alpha_{ij} \rangle = \langle a_{-2i-j} \rangle$, where $\sum_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of φ. Some algebraic properties such as the norm, compactness of the operator K_φ are discussed. Along with the algebraic properties some spectral properties of such operators are discussed. Precisely, it is proved that for an invertible symbol φ, the spectrum of K_φ contains a closed disc.

1. Introduction

Let $\varphi = \sum_{i=-\infty}^{\infty} a_i z^i$ be a bounded measurable function on the unit circle $\mathbb{T}$. Mark C. Ho in his paper [4] has introduced the notion of slant Toeplitz operator A_φ with symbol φ on the space L^2 and it is defined as follows

$$A_\varphi(z^i) = \sum_{i=-\infty}^{\infty} a_{2i-j} z^i$$

for all j in $\mathbb{Z}$, $\mathbb{Z}$ being the set of integers.

Also, it is shown that if (α_{ij}) is the matrix of A_φ with respect to the usual basis $\{z^i : i \in \mathbb{Z}\}$ of L^2, then $\alpha_{ij} = a_{2i-j}$. Moreover if $W : L^2 \to L^2$ be defined as

$$W(z^{2n}) = z^n$$

and

$$W(z^{2n-1}) = 0\,,$$

for each $n \in \mathbb{Z}$, then he has proved that $A_\varphi = WM_\varphi$, where M_φ is the multiplication operator induced by φ.

The Hankel operators H_φ are usually defined on the space H^2 but they can be extended to the space L^2 as follows.

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The Hankel operator S_φ on L^2 is defined as

$$S_\varphi(z^j) = \sum_{i=-\infty}^{\infty} a_{-i-j} z^i$$

for all j in $\mathbb{Z}$. Moreover, if $J : L^2 \to L^2$ is the reflection operator defined by $J(f(z)) = f(\overline{z})$, then we can see here that $S_\varphi = JM_\varphi$ and $M_\varphi = JS_\varphi$.

Motivated by Mark C. Ho, we here in this paper introduce the notion of slant Hankel operator on the space L^2 as follows.

The slant Hankel operator K_φ on L^2 is defined as

$$K_\varphi(z^j) = \sum_{i=-\infty}^{\infty} a_{-2i-j} z^i$$

for all j in $\mathbb{Z}$. That is, if $\langle \beta_{ij} \rangle$ is the matrix of K_φ with respect to the usual basis $\{z^i : i \in \mathbb{Z}\}$ of L^2 then $\beta_{ij} = a_{-2i-j}$. Therefore if A_φ is the slant Toeplitz operator then we can easily see that $A_\varphi = JK_\varphi$ and $K_\varphi = JA_\varphi$. Moreover, we also observe that J reduces W as

$$JW(z^{2n}) = Jz^n = \overline{z}^n \qquad JW(z^{2n-1}) = J0 = 0$$

and

$$WJz^{2n} = W\overline{z}^{2n} = \overline{z}^n \qquad WJz^{2n-1} = Wz^{-2n+1} = 0\,.$$

Also

$$JW^*(z^n) = Jz^{2n} = \overline{z}^{2n} = J(z^{2n}) = JW^* z^n\,.$$

Hence

$$JW = WJ \qquad \text{and} \qquad JW^* = W^* J\,.$$

We begin with the following

Theorem 1. $K_\varphi = WS_\varphi$.

Proof. If S_φ is the Hankel operator on L^2 then

$$S_\varphi(z^j) = \sum_{i=-\infty}^{\infty} a_{-i-j} z^i\,.$$

Therefore,

$$WS_\varphi(z^j) = W\Big(\sum_{i=-\infty}^{\infty} a_{-i-j} z^i\Big) = \sum_{i=-\infty}^{\infty} a_{-2i-j} z^i = K_\varphi(z^j)\,.$$

This is true for all j in $\mathbb{Z}$. Therefore we can conclude that $K_\varphi = WS_\varphi$. From here we can see that $K_\varphi = WS_\varphi = WJM_\varphi = JWM_\varphi = JA_\varphi$. □

As a consequence of the above we can prove the following

Corollary 2. *A slant Hankel operator K_φ with φ in L^∞ is a bounded linear operator on L^2 with $\|K_\varphi\| \le \|\varphi\|_\infty$.*

Proof. Since $\|K_\varphi\| = \|WS_\varphi\| = \|WJM_\varphi\| \le \|W\|\,\|J\|\,\|M_\varphi\| \le \|M_\varphi\| = \|\varphi\|_\infty$. This completes the proof. □

If we denote L_φ, the compression of K_φ on the space H^2, then L_φ is defined as

$$L_\varphi f = PK_\varphi f$$

for all f in H^2, where P is the orthogonal projection of L^2 onto H^2. Equivalently

$$\begin{aligned} L_\varphi &= PK_\varphi \mid H^2 = PJA_\varphi \mid H^2 = PJWM_\varphi \mid H^2 \\ &= PWJM_\varphi \mid H^2 = PWS_\varphi \mid H^2 = WPS_\varphi \mid H^2 = WH_\varphi\,. \end{aligned}$$

That is $L_\varphi = WH_\varphi$, where H_φ is the Hankel operator on H^2. If (β_{ij}) is the matrix of K_φ with respect to the usual basis $\{z^i : i \in \mathbb{Z}\}$ then this matrix is given by

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & \cdots \\ \cdots & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & \cdots \\ \cdots & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ \cdots & a_{-1} & a_{-2} & a_{-3} & a_{-4} & a_{-5} & a_{-6} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The lower right quarter of the matrix is the matrix of L_φ. That is

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-4} & a_{-5} & a_{-6} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

We know obtain a characterization of slant Hankel operator as follows

Theorem 3. *A bounded linear operator K on L^2 is a slant Hankel operator if and only if $M_{\overline{z}}K = KM_{z^2}$.*

Proof. Let K be a slant Hankel operator. Then by definition $K = WS_\varphi$, for some φ in L^∞. Then,

$$\begin{aligned} M_{\overline{z}}K &= M_{\overline{z}}WS_\varphi = WM_{\overline{z}^2}S_\varphi = WM_{\overline{z}^2}JM_\varphi \\ &= WJM_{z^2}M_\varphi = WJM_\varphi M_{z^2} = WS_\varphi M_{z^2} = KM_{z^2}\,. \end{aligned}$$

Conversely, suppose that K satisfies $M_{\overline{z}}K = KM_{z^2}$. Let f be in L^2 and let $\sum_{i=-\infty}^{\infty} b_i z^i$ be its Fourier expansion. Then from the equation $M_{\overline{z}}K = KM_{z^2}$, we get

$$K\big(f(\overline{z}^2)\big) = K\Big(\sum_{i=-\infty}^{\infty} b_i \overline{z}^{2i}\Big) = \sum_{i=-\infty}^{\infty} b_i K M_{\overline{z}^{2i}}(1)$$
$$= \sum_{i=-\infty}^{\infty} b_i M_{z^i} K(1) = \sum_{i=-\infty}^{\infty} b_i z^i K(1) = f(z)K(1)\,.$$

This implies that

$$\|f(z)K(1)\| = \|K(f(\overline{z}^2))\| \le \|K\|\,\|f(\overline{z}^2)\| = \|K\|\,\|f(z)\|\,.$$

Let $\varphi_0 = K1$. Let $\epsilon > 0$ be any real number and $A_\epsilon = \{z : |\varphi_0(z)| > \|K\| + \epsilon\}$. Let χ_{A_ϵ} denote the characteristic function of A_ϵ. Then

$$\|K(\chi_{A_\epsilon})\|^2 = \int_{\mathbb{T}} |K\big(\chi_{A_\epsilon}(z)\big)|^2\, d\mu = \int_{A_\epsilon} |K(1)|^2\, d\mu = \int_{A_\epsilon} |\varphi_0|^2\, d\mu$$
$$\ge (\|K\| + \epsilon)^2 \mu(A_\epsilon) = (\|K\| + \epsilon)^2 \|\chi_{A_\epsilon}\|^2\,.$$

Therefore if $\|\chi_{A_\epsilon}\| \neq 0$ then we get $\|K\| + \epsilon \le \|K\|$, a contradiction. Thus $\|\chi_{A_\epsilon}\| = 0$ and $\mu(A_\epsilon) = 0$, where μ is the normalized Lebesgue measure on $\mathbb{T}$. This is true for all $\epsilon > 0$. Hence if $A = \{z : |\varphi_0| \ge \|K\|\}$ then $\mu(A) = 0$. Thus $|\varphi_0(z)| \le \|K\|$ a.e. This implies that φ_0 is in L^∞. Again if we consider

$$K\big(\overline{z} f(\overline{z}^2)\big) = K\Big(\overline{z} \sum_{i=-\infty}^{\infty} b_i z^{-2i}\Big) = K\Big(\sum_{i=-\infty}^{\infty} b_i z^{-2i-1}\Big)$$
$$= \sum_{i=-\infty}^{\infty} b_i K M_{z^{-2i}} M_{\overline{z}} = \sum_{i=-\infty}^{\infty} b_i M_{z^i} K M_{\overline{z}}$$
$$= \sum_{i=-\infty}^{\infty} b_i z^i K(\overline{z}) = f(z) K(\overline{z})\,.$$

So by the same arguments as above, we can see that $K\overline{z}$ is also bounded. Let $\varphi_1 = K\overline{z}$ and let $\varphi(z) = \varphi_0(\overline{z}^2) + z\varphi_1(\overline{z}^2)$. Since φ_0 and φ_1 are bounded, therefore φ is also bounded and hence is in L^∞. Now we will show that $K = WS_\varphi$. Let f be in L^2, then f can be written as

$$f(z) = f_0(\overline{z}^2) + \overline{z} f_1(\overline{z}^2)\,.$$

This implies that

$$\begin{aligned} WS_\varphi f &= WJM_\varphi f = WJ(\varphi f) = W(\varphi(\overline{z})f(\overline{z})) \\ &= W[(\varphi_0(z^2) + \overline{z}\varphi_1(z^2))(f_0(z^2) + zf_1(z^2))] \\ &= W\left[\varphi_0(z^2)f_0(z^2) + \varphi_1(z^2)f_1(z^2)\right] \\ &\qquad \{\text{as W eliminates the odd powers of } z\} \\ &= W\left[\varphi_0(z^2)f_0(z^2)\right] + W\left[\varphi_1(z^2)f_1(z^2)\right] = \varphi_0(z)f_0(z) + \varphi_1(z)f_1(z) \\ &= f_0(z)K1 + f_1(z)K\overline{z} = K(f_0(\overline{z}^2)) + K(\overline{z}f_1(\overline{z}^2)) \\ &= K(f_0(\overline{z}^2) + \overline{z}f_1(\overline{z}^2)) = Kf\,. \end{aligned}$$

Hence K is a slant Hankel operator. This completes the proof. □

Corollary 4. *The set of all slant Hankel operators is weakly closed and hence strongly closed.*

Proof. Suppose that for each α, K_α is a slant Hankel operator and $K_\alpha \to K$ weakly, where $\{\alpha\}$ is a net. Then for all f, g in $L^2 \langle K_\alpha f, g\rangle \to \langle Kf, g\rangle$. This implies that

$$\langle M_z K_\alpha M_{z^2} f, g\rangle = \langle K_\alpha z^2 f, \overline{z}g\rangle \to \langle Kz^2 f, \overline{z}g\rangle = \langle M_z K M_{z^2} f, g\rangle$$

Since K_φ is a slant Hankel operator, therefore from its characterization, we have $M_z K_\alpha M_{z^2} = K_\alpha$ for each α. Thus $K = M_z K M_{z^2}$ and so K is slant Hankel operator. This completes the proof. □

Definition : The slant Hankel matrix is defined as a two way infinite matrix (a_{ij}) such that

$$a_{i-1,j+2} = a_{ij}\,.$$

This definition gives the characterization of the slant Hankel operator K_φ in terms of its matrix as follows

A necessary and sufficient condition for a bounded linear operator on L^2 to be a slant Hankel operator is that its matrix (with respect to the usual basis $\{z^i : i \in \mathbb{Z}\}$) is a slant Hankel matrix.

The adjoint K^*_φ, of the operator K_φ, is defined by

$$K^*_\varphi(z^j) = \sum_{i=-\infty}^{\infty} \overline{a}_{-2j-i} z^i.$$

That is, $K^*_\varphi = JA^*_{\varphi(\overline{z})}$. Moreover if J is the reflection operator then $JK^*_\varphi(z^j) = \sum\limits_{i=-\infty}^{\infty} \overline{a}_{-2j+i} z^i$ and therefore $WJK^*_\varphi(z^j) = \sum\limits_{i=-\infty}^{\infty} \overline{a}_{-2j+2i} z^i$. That is the matrix of

WJK_φ^* is given by

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & \overline{a}_2 & \overline{a}_0 & \overline{a}_{-2} & \overline{a}_{-4} & \overline{a}_{-6} & \dots \\ \dots & \overline{a}_4 & \overline{a}_2 & \overline{a}_0 & \overline{a}_{-2} & \overline{a}_{-4} & \dots \\ \dots & \overline{a}_6 & \overline{a}_4 & \overline{a}_2 & \overline{a}_0 & \overline{a}_{-2} & \dots \\ \dots & \overline{a}_8 & \overline{a}_6 & \overline{a}_4 & \overline{a}_2 & \overline{a}_0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

which is constant on diagonals and therefore is the matrix of the multiplication operator M_ψ where $\psi = W(\overline{\varphi}(\overline{z}))$. This helps us in proving the following

Theorem 5. *K_φ is compact if and only if $\varphi = 0$.*

Proof. Let K_φ be compact, then K_φ^* is also compact. Since W and J are bounded linear operators, therefore WJK_φ^* is also compact. But $WJK_\varphi^* = W(\overline{\varphi}(\overline{z})) = M_\psi$ where $\psi = W(\overline{\varphi}(\overline{z}))$. This implies that M_ψ is compact and therefore $\langle \psi, z^n \rangle = 0$ for all n. That is

$$\langle \psi, z^n \rangle = \langle \overline{\varphi}(\overline{z}), W^* z^n \rangle = \langle \Sigma \overline{a}_i z^i, z^{2n} \rangle = \overline{a}_{2n} = 0\,.$$

On the other hand, since $K_\varphi M_{\overline{z}}$ is also compact and therefore

$$\begin{aligned} WJ(K_\varphi M_{\overline{z}})^* &= WJ(JA_\varphi M_{\overline{z}})^* = WJ(JWM_{\varphi\overline{z}})^* \\ &= WJ(K_{\varphi\overline{z}})^* = M_{\psi_0}\,. \end{aligned}$$

where $\psi_0 = W\big(z\overline{\varphi}(\overline{z})\big)$, is also compact. This further yields that for each n in $\mathbb{Z}$

$$\begin{aligned} 0 = \langle \psi_0, z^n \rangle &= \langle W(\overline{\varphi}(\overline{z})z), z^n \rangle = \langle \overline{\varphi}(\overline{z})z, z^{2n} \rangle \\ &= \Big\langle \sum_{i=-\infty}^{\infty} \overline{a}_i z^{i+1}, z^{2n} \Big\rangle = \Big\langle \sum_{i=-\infty}^{\infty} \overline{a}_{i-1} z^i, z^{2n} \Big\rangle = \overline{a}_{2n-1}\,. \end{aligned}$$

Thus $a_i = 0$ for all i which concludes that $\varphi = 0$. This completes the proof. □

The next result deals with the norm of K_φ as follows

Theorem 5. $\|K_\varphi\| = \|A_\varphi\| = \sqrt{\|W|\varphi|^2\|_\infty}$.

Proof. Consider,

$$\begin{aligned} K_\varphi K_\varphi^* &= JA_\varphi(JA_\varphi)^* = JWM_\varphi(JWM_\varphi)^* = JWM_\varphi M_{\overline{\varphi}} W^* J^* \\ &= JWM_{|\varphi|^2} W^* J^* = WJ(JWM_{|\varphi|^2})^* = WJK_{|\varphi|^2}^* = M_\psi \end{aligned}$$

where $\psi = W(|\varphi|^2)$. It follows that

$$\|K_\varphi\|^2 = \|K_\varphi K_\varphi^*\| = \|M_\psi\| = \|\psi\|_\infty = \|W|\varphi|^2\|_\infty = \|A_\varphi\|^2.$$

This completes the proof. □

2. Spectrum of K_φ

In [4] Mark C. Ho has proved that the spectrum of slant Toeplitz operator contains a closed disc, for any invertible φ in $L^\infty(\mathbb{T})$. The same is true for slant Hankel operator. We begin with the following

Lemma 6. *If φ is invertible in L^∞, then $\sigma_p(K_\varphi) = \sigma_p(K_{\varphi(\overline{z}^2)})$, where $\sigma_p(K_\varphi)$ denotes the point spectrum of K_φ.*

Proof. Let $\lambda \in \sigma_p(K_\varphi)$. Therefore there exists a non zero f in L^2 such that $K_\varphi f = \lambda f$. Consider $F = \varphi f$. Then

$$\begin{aligned} K_{\varphi(\overline{z}^2)}F &= K_{\varphi(\overline{z}^2)}\varphi f = JA_{\varphi(\overline{z}^2)}(\varphi f) = JWM_{\varphi(\overline{z}^2)}\varphi f = JM_{\varphi(\overline{z})}WM_\varphi f \\ &= M_{\varphi(z)}JA_\varphi f = \varphi(z)K_\varphi(f) = \varphi\lambda f = \lambda\varphi f = \lambda F\,. \end{aligned}$$

Since φ is invertible and $f \neq 0$, therefore $F \neq 0$ and hence $\lambda \in \sigma_p(K_{\varphi(\overline{z}^2)})$. This implies that $\sigma_p(K_\varphi) \subset \sigma_p(K_{\varphi(\overline{z}^2)})$.

Conversely, let $\mu \in \sigma_p(K_{\varphi(\overline{z}^2)})$. Thus there exists some $0 \neq g$ in L^2 such that $K_{\varphi(\overline{z}^2)}g = \mu g$. Let $G = \varphi^{-1}g$. This gives that

$$\begin{aligned} K_\varphi G = K_\varphi(\varphi^{-1}g) &= JA_\varphi(\varphi^{-1}g) = JWM_\varphi(\varphi^{-1}g) = WJ(\varphi\varphi^{-1}g) = WJg \\ &= \varphi^{-1}\varphi WJg = \varphi^{-1}WJ\varphi(\overline{z}^2)g = \varphi^{-1}K_{\varphi(\overline{z}^2)}g \\ &= \varphi^{-1}\mu g = \mu\varphi^{-1}g = \mu G\,. \end{aligned}$$

By the same reasons φ is invertible, $g \neq 0$, we must have $G \neq 0$ and therefore the result follows. □

Lemma 7. $\sigma(K_\varphi) = \sigma(K_{\varphi(\overline{z}^2)})$ *for any φ in L^∞, where $\sigma(K_\varphi)$ denotes the spectrum of K_φ.*

Proof. We know the if A and B are two bounded linear operators then

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}\,.$$

Consider

$$K_\varphi^* = (JA_\varphi)^* = A_\varphi^* J^* = M_{\overline{\varphi}}W^*J^* = M_{\overline{\varphi}}(JW)^*\,.$$

Therefore,

$$\sigma(K_\varphi^*) \cup \{0\} = \sigma\big[(M_{\overline{\varphi}})(JW)^*\big] \cup \{0\} = \sigma\big[(JW)^*(M_{\overline{\varphi}})\big] \cup \{0\}$$

Again since,

$$\begin{aligned} (JW)^*M_{\overline{\varphi}} = W^*J^*M_{\overline{\varphi}(z)} &= W^*M_{\overline{\varphi}(\overline{z})}J^* = M_{\overline{\varphi}(\overline{z}^2)}W^*J^* \\ &= (WM_{\varphi(\overline{z}^2)})^*J^* = A^*_{\varphi(\overline{z}^2)}J^* = K^*_{\varphi(\overline{z}^2)}\,. \end{aligned}$$

So,

$$\sigma(K_\varphi^*) \cup \{0\} = \sigma(K_{\varphi(\overline{z}^2)}) \cup \{0\}\,.$$

This gives that

$$\sigma(K_\varphi) \cup \{0\} = \overline{\sigma(K_\varphi^*)} \cup \{0\} = \overline{\sigma(K^*_{\varphi(\overline{z}^2)})} \cup \{0\} = \sigma(K_{\varphi(\overline{z}^2)}) \cup \{0\}\,.$$

We assert the $0 \in \sigma_p(K_{\varphi(\overline{z}^2)})$. We can see that $R(W^*)$ = the range of $W^* = P_e(L^2)$= the closed linear span of $\{z^{2n} : n \in \mathbb{Z}\}$ in $L^2 \neq L^2$. Hence W^* is not onto. This gives that $\overline{R(W^*J^*M_{\overline{\varphi}})} \neq L^2$. As $W^*L^*M_{\overline{\varphi}} = K^*_{\varphi(\overline{z}^2)}$, therefore $\ker K_{\varphi(\overline{z}^2)} \neq 0$. This implies that $0 \in \sigma_p(K_{\varphi(\overline{z}^2)})$. If φ is invertible in L^∞, then by the above Lemma $0 \in \sigma_p(K_\varphi)$ and we are done.

Let φ be not invertible in L^∞. As the set $\{\varphi \in L^\infty : \varphi^{-1} \in L^\infty\}$ is dense in L^∞ [4], therefore we can have a sequence $\{\varphi_n\}$ of invertible functions such that $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. Since φ_n is invertible for each n, therefore $0 \in \sigma_p(K_{\varphi_n})$ for each n. Hence for each n we can find $f_n \neq 0$ such that $K_{\varphi_n} f_n = 0$. Without loss of generality, we can assume that $\|f_n\| = 1$. Now

$$\begin{aligned}\|K_\varphi f_n\| &= \|K_\varphi f_n - K_{\varphi_n} f_n + K_{\varphi_n} f_n\| \\ &\leq \|K_\varphi f_n - K_{\varphi_n} f_n\| + \|K_{\varphi_n} f_n\| \leq \|\varphi - \varphi_n\| \to 0\end{aligned}$$

as $n \to \infty$. Hence $0 \in \Pi(K_\varphi)$, the approximate point spectrum of K_φ and hence is in the spectrum of K_φ. Also 0 is in the approximate point spectrum of $K_{\varphi(\overline{z}^2)}$. This completes the proof. □

Theorem 8. *The spectrum of K_φ contains a closed disc, for any invertible φ in $L^\infty(\mathbb{T})$.*

Proof. Let $\lambda \neq 0$ and suppose that $K^*_{\varphi(\overline{z}^2)} - \lambda$ is onto. For f in $L^2(\mathbb{T})$, we have

$$\begin{aligned}(K^*_{\varphi(\overline{z}^2)} - \lambda)f &= K^*_{\varphi(\overline{z}^2)} f - \lambda f = M_{\overline{\varphi}(\overline{z}^2)} W^* J^* f - \lambda f \\ &= \overline{\varphi}(\overline{z}^2) f(\overline{z}^2) - \lambda(P_e f \oplus P_0 f) = (W^* J^*(\overline{\varphi} f) - \lambda P_e f) \oplus (-\lambda P_0 f) \\ &= (J^* W^*(\overline{\varphi} f) - \lambda P_e f) \oplus (-\lambda P_0 f) = (J^* W^* \overline{\varphi} - \lambda P_e) f \oplus (-\lambda P_0 f) \\ &= \lambda J^* W^* M_{\overline{\varphi}}(\lambda^{-1} - M_{\overline{\varphi}^{-1}} JW) f \oplus (-\lambda P_0 f)\end{aligned}$$

where $P_0 = I - P_e$, that is $P_0 = \{z^{2k-1} : k \in \mathbb{Z}\}$. Let $0 \neq g_0$ be in $P_0(L^2)$. Since $K^*_{\varphi(\overline{z}^2)} - \lambda$ is onto, there exists a non zero vector f in $L^2(\mathbb{T})$ such that $(K^*_{\varphi(\overline{z}^2)} - \lambda)f = g_0$. That is,

$$\lambda J^* W^* M_{\overline{\varphi}}(\lambda^{-1} - M_{\overline{\varphi}^{-1}} JW) \oplus (-\lambda P_0 f) = g_0\,.$$

Since $g_0 \in P_0(L^2)$ and $g_0 \neq 0$, therefore, we must have

$$\lambda J^* W^* M_{\overline{\varphi}}(\lambda^{-1} - M_{\overline{\varphi}-1} JW) f = 0\,.$$

Since $\lambda \neq 0$, W^* and J^* are isometries and $M_{\overline{\varphi}}$ being invertible, this implies that

$$(\lambda^{-1} - M_{\overline{\varphi}^{-1}} JW) f = 0\,.$$

Since $M_{\overline{\varphi}^{-1}} JW = K_{\overline{\varphi}^{-1}(z^2)}$, therefore we have

$$(\lambda^{-1} - K_{\overline{\varphi}^{-1}(z^2)}) f = 0\,.$$

Thus $\lambda^{-1} \in \sigma_p(K_{\overline{\varphi}^{-1}(\overline{z}^2)})$. Now let $\lambda \in \rho(K^*_{\varphi(\overline{z}^2)})$, the resolvent of $K^*_{\varphi(\overline{z}^2)}$, the operator $K^*_{\varphi(\overline{z}^2)} - \lambda$ is invertible and hence onto, therefore, $\lambda^{-1} \in \sigma_p(K_{\overline{\varphi}^{-1}(\overline{z}^2)})$. That is

$$D = \{\lambda^{-1} : \lambda \in \rho(K^*_{\varphi(\overline{z}^2)})\} \subseteq \sigma_p(K_{\overline{\varphi}^{-1}(\overline{z}^2)})\,.$$

By Lemma 7, we get $D \subseteq \sigma_p(K_{\overline{\varphi}^{-1}})$. So replacing $\overline{\varphi}^{-1}$ by φ, we get that $D \subseteq \sigma_p(K_\varphi) \subset \sigma(K_\varphi)$ and therefore we have proved that for any invertible φ in L^∞, the spectrum of K_φ contains a disc consisting of eigenvalues of K_φ. Since spectrum of any operator is compact, it follows that $\sigma(K_\varphi)$ contains a closed disc. □

Remark 1. The radius of the closed disc contained in $\sigma(K_\varphi)$ is $\big(r(K_{\overline{\varphi}-1})\big)^{-1}$, where $r(A)$ denote the spectral radius of the operator A. For,

$$\begin{aligned}\max\{|\lambda^{-1}| : \lambda \in \rho\big(K^*_{\varphi(\overline{z}^2)}\big)\} &= \big[\{|\lambda| : \lambda \in \rho\big(K^*_{\varphi(\overline{z}^2)}\big)\}\big]^{-1} \\ &= \big[r\big(K^*_{\varphi(\overline{z}^2)}\big)\big]^{-1} = \big[r\big(K_{\varphi(\overline{z}^2)}\big)\big]^{-1}.\end{aligned}$$

Replacing φ by φ^{-1} we get that the radius of the disc is $\big(r(K_{\varphi(\overline{z}^2)})\big)^{-1}$ and therefore

$$r(K_\varphi) \geq \big(r(K_{\overline{\varphi}^{-1}})\big)^{-1}.$$

References

[1] Arora, S. C. and Ruchika Batra, *On Slant Hankel Operators*, to appear in Bull. Calcutta Math. Soc.

[2] Brown, A. and Halmos, P. R., *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1964), 89–102.

[3] Halmos, P. R., *Hilbert Space Problem Book*, Springer Verlag, New York, Heidelberg-Berlin, 1979.

[4] Ho, M. C., *Properties of Slant Toeplitz operators*, Indiana Univ. Math. J. **45** (1996), 843–862.

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