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ON 4-DIMENSIONAL LOCALLY CONFORMALLY FLAT ALMOST KÄHLER MANIFOLDS

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Abstract. Using the fundamental notions of the quaternionic analysis we show that there are no 4-dimensional almost Kähler manifolds which are locally conformally flat with a metric of a special form.

I. Basic notions and the aim of the paper

Let M^{2n} be a real C^∞-manifold of dimension $2n$ endowed with an almost complex structure J and a Riemannian metric g. If the metric g is invariant by the almost complex structure J, i.e.

$$g(JX, JY) = g(X, Y)$$

for any vector fields X and Y on M^{2n}, then (M^{2n}, J, g) is called *almost Hermitian manifold*.

Define the fundamental 2-form Ω by

$$\Omega(X, Y) := g(X, JY)\,.$$

An almost Hermitian manifold (M^{2n}, J, g, Ω) is said to be *almost Kähler* if Ω is a closed form, i.e.

$$d\Omega = 0\,.$$

Suppose that

$$n = 2\,.$$

The aim of the paper is to prove the following:

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Theorem I. *If* (M^4, J, g, Ω) *is a* 4*-dimensional almost Kähler manifold which is locally conformally flat, i.e. in a neighbourhood of every point* $p_0 \in M^4$ *there exists a system of local coordinates* $(U_{p_0}; w, x, y, z)$ *such that the metric* g *is expressed by*

$$g = g_0(p)[dw^2 + dx^2 + dy^2 + dz^2], \quad p \in U_{p_0},$$

where $g_0(p)$ *is a real positive* C^∞*-function defined around* p_0*, then* g_0 *is a modulus of some quaternionic function left* (*right*) *regular in the sense of Fueter* [1] *uniquely determined by* J *and* Ω.

II. Proof of Theorem

Let us denote by the same letters the matrices of g, J and Ω with respect to the coordinate basis. These matrices satisfy the equality:

$$g \cdot J = \Omega .$$

The metric g, by the assumption, is proportional to the identity, so it has the form

$$g = g_0 \cdot I = g_0 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

An almost complex structure J satisfies the condition:

$$J^2 = -I .$$

Since Ω is skew-symmetric then J is a skew-symmetric and orthogonal 4×4-matrix.

It is easy to check that J is of the form

$$(1) \qquad a) \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} \qquad \text{or} \qquad b) \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{pmatrix}$$

with

$$a^2 + b^2 + c^2 = 1 .$$

Suppose that J is of the form (1a). Then the matrix Ω looks as follows:

$$\Omega = g_o \cdot \begin{pmatrix} 0 & a & -b & c \\ -a & 0 & c & b \\ b & -c & 0 & a \\ -c & -b & -a & 0 \end{pmatrix} := \begin{pmatrix} 0 & A & -B & C \\ -A & 0 & C & B \\ B & -C & 0 & A \\ -C & -B & -A & 0 \end{pmatrix} .$$

Since

$$\left(\frac{A}{g_0}\right)^2 + \left(\frac{B}{g_0}\right)^2 + \left(\frac{C}{g_0}\right)^2 = a^2 + b^2 + c^2 = 1$$

then we get

$$A^2 + B^2 + C^2 = g_0^2 \,. \tag{2}$$

By the assumption

$$d\Omega = 0 \,.$$

Using the following formula (see e.g. [4], p.36):

$$\begin{aligned} d\Omega(X,Y,Z) = \frac{1}{3}\{ & X\Omega(Y,Z) + Y\Omega(Z,X) + Z\Omega(X,Y) \\ & - \Omega([X,Y],Z) - \Omega([Z,X],Y) - \Omega([Y,Z],X)\} \,, \end{aligned}$$

the condition $d\Omega = 0$ can be written in the form:

$$\begin{aligned} 0 &= 3d\Omega(\partial_x, \partial_y, \partial_z) = A_x + B_y + C_z \,, \\ 0 &= 3d\Omega(\partial_x, \partial_y, \partial_w) = B_x - A_y + C_w \,, \\ 0 &= 3d\Omega(\partial_x, \partial_z, \partial_w) = C_x - A_z - B_w \,, \\ 0 &= 3d\Omega(\partial_y, \partial_z, \partial_w) = C_y - B_z + A_w \,. \end{aligned}$$

Then the components A, B and C of Ω satisfy the following system of first order partial differential equations:

$$\begin{aligned} A_x + B_y + C_z &= 0 \,, \\ B_x - A_y + C_w &= 0 \,, \\ C_x - A_z - B_w &= 0 \,, \\ C_y - B_z + A_w &= 0 \end{aligned} \tag{3}$$

and the condition (2).

The above system (3), although overdetermined, does have solutions. We will show that the system (3) has a nice interpretation in the quaternionic analysis.

III. Fueter's regular functions

Denote by $\mathbf{H}$ the field of quaternions. $\mathbf{H}$ is a 4-dimensional division algebra over $\mathbf{R}$ with basis $\{1, i, j, k\}$ and the quaternionic units i, j, k satisfy:

$$\begin{aligned} & i^2 = j^2 = k^2 = ijk = -1 \,, \\ & ij = -ji = k \,. \end{aligned}$$

A typical element q of $\mathbf{H}$ can be written as

$$q = w + ix + jy + kz \,, \qquad w, x, y, z \in \mathbf{R} \,.$$

The conjugate of q is defined by

$$\overline{q} := w - ix - jy - kz$$

and the modulus $\|q\|$ by

$$\|q\|^2 = q\cdot\overline{q} = \overline{q}\cdot q = w^2 + x^2 + y^2 + z^2 \,.$$

We will need the following relation (which is easy to check)

$$\overline{q_1\cdot q_2} = \overline{q_2}\cdot\overline{q_1} \,.$$

A function $F : \mathbf{H}\rightarrow\mathbf{H}$ of the quaternionic variable q can be written as

$$F = F_o + iF_1 + jF_2 + kF_3 \,.$$

F_o is called the *real part* of F and $iF_1 + jF_2 + kF_3$ - the *imaginary part* of F.

In [1] Fueter introduced the following operators:

$$\overline{\partial}_{\rm left} := \frac{1}{4}\Big(\frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\Big)\,,$$
$$\overline{\partial}_{\rm right} := \frac{1}{4}\Big(\frac{\partial}{\partial w} + \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\Big)\,,$$

analogous to $\frac{\partial}{\partial\overline{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\big)$ in the complex analysis, to generalize the Cauchy--Riemann equations.

A quaternionic function F is said to be *left regular* (respectively, *right regular*) (in the sense of Fueter) if it is differentiable in the real variable sense and

$$\overline{\partial}_{\rm left}\cdot F = 0 \qquad (\text{resp. } \overline{\partial}_{\rm right}\cdot F = 0)\,. \tag{4}$$

Note that the condition (4) is equivalent to the following system of equations:

$$\partial_w F_o - \partial_x F_1 - \partial_y F_2 - \partial_z F_3 = 0\,,$$
$$\partial_w F_1 + \partial_x F_o + \partial_y F_3 - \partial_z F_2 = 0\,,$$
$$\partial_w F_2 - \partial_x F_3 + \partial_y F_o + \partial_z F_1 = 0\,,$$
$$\partial_w F_3 + \partial_x F_2 - \partial_y F_1 + \partial_z F_o = 0\,.$$

There are many examples of left and right regular functions in the sense of Fueter. Many papers have been devoted studying the properties of those functions (e.g. [3]). One has found the quaternionic generalizations of the Cauchy theorem, the Cauchy integral formula, Taylor series in terms of special polynomials etc.

Now we need an important result of [5]. It can be described as follows.

Let ν be an unordered set of n integers $\{i_1, \ldots, i_n\}$ with $1\le i_r\le 3$; ν is determined by three integers n_1, n_2 and n_3 with $n_1 + n_2 + n_3 = n$, where n_1 is the number of 1's in ν, n_2 - the number of 2's and n_3 - the number of 3's.

There are $\frac{1}{2}(n+1)(n+2)$ such sets ν and we denote the set of all of them by σ_n.

Let e_{i_r} and x_{i_r} denote i, j, k and x, y, z according as i_r is $1, 2$ or 3, respectively. Then one defines the following polynomials

$$P_\nu(q) := \frac{1}{n!} \sum (we_{i_1} - x_{i_1})\cdot\ldots\cdot(we_{i_n} - x_{i_n}),$$

where the sum is taken over all $n!\cdot n_1!\cdot n_2!\cdot n_3!$ different orderings of n_1 1's, n_2 2's and n_3 3's; when $n = 0$, so $\nu = \emptyset$, we take $P_\emptyset(q) = 1$.

For example we present the explicit forms of the polynomials P_ν of the first and second degrees. Thus we have

$$\begin{aligned} P_1 &= wi - x\,, \\ P_2 &= wj - y\,, \\ P_3 &= wk - z\,, \end{aligned}$$

$$\begin{aligned} P_{11} &= \frac{1}{2}(x^2 - w^2) - xwi\,, \\ P_{12} &= xy - wyi - wxj\,, \\ P_{13} &= xz - wzi - wxk\,, \\ P_{22} &= \frac{1}{2}(y^2 - w^2) - ywj\,, \\ P_{23} &= yz - wzj - wyk\,, \\ P_{33} &= \frac{1}{2}(z^2 - w^2) - zwk\,. \end{aligned}$$

In [5] Sudbery proved the following

Proposition. *Suppose F is left regular in a neighbourhood of the origin $0 \in \mathbf{H}$. Then there is a ball $B = B(0, r)$ with center 0 in which $F(q)$ is represented by a uniformly convergent series*

$$F(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q) a_\nu\,, \qquad a_\nu \in \mathbf{H}\,.$$

IV. The end of the proof

Let us denote

$$F_{ABC}(q) := Ai + Bj + Ck\,,$$

where we have identified $q \in \mathbf{H}$ with $(w, x, y, z) \in \mathbf{R}^4$. Then (3) is nothing but the condition that F_{ABC} is left regular in the sense of Fueter. Then, by (2), we have

$$\|F_{ABC}\| = g_0. \qquad \square$$

V. Conclusions

Let F satisfy the assumptions of Proposition. Then

$$F(q) = a_0 + \sum_{i=1}^{3} P_i \cdot a_i + \sum_{i \le j} P_{ij} \cdot a_{ij} + \sum_{i \le j \le k} P_{ijk} \cdot a_{ijk} + \dots$$

and

$$\overline{F(q)} = \overline{a_o} + \sum_{i=1}^{3} \overline{a_i} \cdot \overline{P_i} + \sum_{i \le j} \overline{a_{ij}} \cdot \overline{P_{ij}} + \sum_{i \le j \le k} \overline{a_{ijk}} \cdot \overline{P_{ijk}} + \dots .$$

Multiplying the above expressions we get

$$\begin{aligned} \|F(q)\|^2 = \|a_o\|^2 &+ \sum_{i=1}^{3} (P_i a_i \overline{a_o} + a_o \overline{a_i} \overline{P_i}) \\ &+ \sum_{i \le j} (P_{ij} a_{ij} \overline{a_o} + a_o \overline{a_{ij}} \overline{P_{ij}}) + \sum_{i,j} P_i a_i \overline{a_j} \overline{P_j} \\ &+ \sum_{i \le j \le k} (P_{ijk} a_{ijk} \overline{a_o} + a_o \overline{a_{ijk}} \overline{P_{ijk}}) \\ &+ \sum_{m=1}^{3} \sum_{i \le j} (P_m a_m \overline{a_{ij}} \overline{P_{ij}} + P_{ij} a_{ij} \overline{a_m} \overline{P_m}) + \dots . \end{aligned} \tag{5}$$

Example 1. Let

$$g_0(w, x, y, z) = \frac{1}{1+r}, \qquad r^2 = w^2 + x^2 + y^2 + z^2,$$

then

$$g_0^2 = \frac{1}{(1+r)^2} = 1 - 2r + 3r^2 - 4r^3 + \dots + (-1)^n (n+1) r^n + \dots . \tag{6}$$

Comparing the right sides of (5) and (6) we see that

$$\begin{aligned} a_0 &\ne 0, \\ -2r &= \sum_{i=1}^{3} (P_i a_i \overline{a_0} + a_0 \overline{a_i} \overline{P_i}) \end{aligned}$$

but the second equality is impossible.

Example 2. Take

$$g_0(w,x,y,z) = \frac{1}{\sqrt{1+r^2}}\,, \qquad r^2 = w^2 + x^2 + y^2 + z^2\,,$$

then

$$g_0^2 = \frac{1}{1+r^2} = 1 - r^3 + r^6 - r^9 + \ldots + (-1)^k r^{3k} + \ldots . \tag{7}$$

Comparing the right sides of (5) and (7) we get

$$a_0 \neq 0\,, \quad a_i = 0\,, \quad a_{ij} = 0$$

and

$$-r^3 = \sum_{i\leq j\leq k} (P_{ijk} a_{ijk} \overline{a_0} + a_0 \overline{a_{ijk}}\,\overline{P_{ijk}})$$

but the last equality is impossible.

Example 3. Let

$$g_0(w,x,y,z) = \frac{1}{\sqrt{1-r^2}}\,, \qquad r^2 = w^2 + x^2 + y^2 + z^2\,,$$

then

$$g_0^2 = \frac{1}{1-r^2} = 1 + r^2 + \frac{4}{3} r^3 + \ldots \tag{8}$$

Comparing the right sides of (5) and (8) we have

$$a_0 \neq 0\,, \qquad a_i = 0$$

and

$$r^2 = \sum_{i\leq j} (P_{ij} a_{ij} \overline{a_0} + a_0 \overline{a_{ij}}\,\overline{P_{ij}})\,. \tag{9}$$

Set

$$d_{ij} := a_{ij}\overline{a_0} := d_{ij}^0 + d_{ij}^1 \mathbf{i} + d_{ij}^2 \mathbf{j} + d_{ij}^3 \mathbf{k}$$

(**i, j, k** denote the quaternionic units) and rewrite (9) in the form

$$w^2 + x^2 + y^2 + z^2 = 2 \sum_{i\leq j} Re\ (P_{ij} d_{ij})$$

then we get

$$\begin{aligned} w^2+x^2+y^2+z^2 &= 2Re\ \{[\frac{1}{2}(x^2-w^2)-xw\mathbf{i}]d_{11} \\ &\quad + 2Re\ \{[\frac{1}{2}(y^2-w^2)-yw\mathbf{j}]d_{22} \\ &\quad + 2Re\ \{[\frac{1}{2}(z^2-w^2)-zw\mathbf{k}]d_{33}+\dots \\ &= (x^2-w^2)d^0_{11}+(y^2-w^2)d^0_{22}+(z^2-w^2)d^0_{33}\,. \end{aligned}$$

Comparing the terms in x^2, y^2 and z^2 we get

$$d^0_{11}=d^0_{22}=d^0_{33}=1$$

but then

$$w^2=-3w^2$$

and this is impossible.

Example 4. Let

$$g_0(w,x,y,z)=\frac{1}{(1-r^2)^2}\,, \qquad r^2=w^2+x^2+y^2+z^2\,,$$

then

$$g_0^2=\frac{1}{(1-r^2)^4}=1+4r^2+\dots. \tag{10}$$

Comparing the right sides of (5) and (10) we obtain

$$a_0\neq 0\,, \qquad a_i=0$$

and

$$4r^2=\sum_{i\le j}(P_{ij}a_{ij}\overline{a_0}+a_0\overline{a_{ij}}\overline{P_{ij}})\,.$$

Analogously, like in the Example 3, we have

$$2w^2+2x^2+2y^2+2z^2=\sum_{i\le j}Re\ (P_{ij}d_{ij})\,.$$

This time, comparing the terms in x^2, y^2 and z^2, we get

$$\begin{gathered} a_0\neq 0\,, \qquad a_i=0\,, \\ d^0_{11}=d^0_{22}=d^0_{33}=4 \end{gathered}$$

but then

$$-6w^2=2w^2\,.$$

This is again impossible.

VI. General conclusion

There is no 4-dimensional almost Kähler manifold (M^4, J, g, Ω) which is locally conformally flat with the metric

$$g = g_0(p)[dw^2 + dx^2 + dy^2 + dz^2]\,,$$

where g_0 is expressed by the formulae (6), (7), (8) and (10). In particular the Poincaré model, i.e. the unit ball B^4 in $\mathbf{R}^4$ with the metric

$$g := \frac{4}{(1-r^2)^2}[dw^2 + dx^2 + dy^2 + dz^2]\,, \qquad r^2 := w^2 + x^2 + y^2 + z^2\,,$$

is not an almost Kähler manifold.

Remark. If J is of the form (1b) then the proof of Theorem is similar. One has to replace the left regular quaternionic function with the right one (see [3], p.10).

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