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THE JET PROLONGATIONS OF 2-FIBRED MANIFOLDS AND THE FLOW OPERATOR

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Abstract. Let r, s, m, n, q be natural numbers such that $s \geq r$. We prove that any 2-$\mathcal{FM}_{m,n,q}$-natural operator $A\colon T_{\text{2-proj}} \rightsquigarrow TJ^{(s,r)}$ transforming 2-projectable vector fields V on (m,n,q)-dimensional 2-fibred manifolds $Y \to X \to M$ into vector fields $A(V)$ on the (s,r)-jet prolongation bundle $J^{(s,r)}Y$ is a constant multiple of the flow operator $\mathcal{J}^{(s,r)}$.

All manifolds and maps are assumed to be of class C^∞. Manifolds are assumed to be finite dimensional and without boundaries.

The category of all manifolds and maps is denoted by $\mathcal{M}f$. The category of all fibred manifolds (surjective submersions $X \to M$ between manifolds) and fibred maps is denoted by $\mathcal{FM}$. The category of all fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred embeddings is denoted by $\mathcal{FM}_{m,n}$. The category of 2-fibred manifold (pairs of surjective submersions $Y \to X \to M$ between manifolds) and their 2-fibred maps is denoted by 2-$\mathcal{FM}$. The category of all fibred manifolds $Y \to X \to M$ such that $X \to M$ is an $\mathcal{FM}_{m,n}$-object and their 2-fibred maps covering $\mathcal{FM}_{m,n}$-maps is denoted by 2-$\mathcal{FM}_{m,n}$. The category of all fibred manifolds $Y \to X \to M$ such that $X \to M$ is an $\mathcal{FM}_{m,n}$-object and $Y \to X$ is an $\mathcal{FM}_{m+n,q}$-object and their 2-fibred embeddings is denoted by 2-$\mathcal{FM}_{m,n,q}$. The standard 2-$\mathcal{FM}_{m,n,q}$-object is denoted by $\mathbf{R}^{m,n,q} = (\mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^q \to \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m)$. The usual coordinates on $\mathbf{R}^{m,n,q}$ are denoted by $x^1, \dots, x^m$, $y^1, \dots, y^n$, $z^1, \dots, z^q$.

Taking into consideration some idea from [1] one can generalize the concept of jets as follows. Let r and s be integers such that $s \geq r$. Let $Y \to X \to M$ be a 2-$\mathcal{FM}_{m,n}$-object. Sections $\sigma_1, \sigma_2\colon X \to Y$ of $Y \to X$ have the same (s,r)-jet $j^{(s,r)}_x\sigma_1 = j^{(s,r)}_x\sigma_2$ at $x \in X$ iff

$$j^{s-r}_x\big(J^r\sigma_1 \mid X_{p_0(x)}\big) = j^{s-r}_x\big(J^r\sigma_2 \mid X_{p_0(x)}\big)\,,$$

where $J^r\sigma_i\colon X \to J^rY$ is the r-jet map $J^r\sigma_i(x) = j^r_x\sigma_i$, $x \in X$, and $X_{p_0(x)}$ is the fibre of $X \to M$ through x. Equivalently $j^{(s,r)}_x\sigma_1 = j^{(s,r)}_x\sigma_2$ iff (in some and then in every 2-$\mathcal{FM}_{m,n}$-coordinates) $D_{(\alpha,\beta)}\sigma_1(x) = D_{(\alpha,\beta)}\sigma_2(x)$ for all $\alpha \in \big(\mathbf{N} \cup \{0\}\big)^m$ and $\beta \in \big(\mathbf{N} \cup \{0\}\big)^n$ with $|\alpha| \leq r$ and $|\alpha| + |\beta| \leq s$, where $D_{(\alpha,\beta)}$ denotes the

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iterated partial derivative corresponding to (α,β). Thus we have the so called (s,r)-jets prolongation bundle

$$J^{(s,r)}Y=\left\{j^{(s,r)}_x\sigma \ \mid \sigma\colon X\to Y \text{ is a section of } Y\to X,\ x\in X\right\}.$$

Given a 2-$\mathcal{FM}_{m,n}$-map $f\colon Y_1\to Y_2$ of two 2-$\mathcal{FM}_{m,n}$-objects covering $\mathcal{FM}_{m,n}$-map $\underline{f}\colon X_1\to X_2$ we have the induced map $J^{(s,r)}f\colon J^{(s,r)}Y_1\to J^{(s,r)}Y_2$ given by $J^{(s,r)}f(j^{(s,r)}_x\sigma)=j^{(s,r)}_{\underline{f}(x)}(f\circ\sigma\circ\underline{f}^{-1})$, $j^{(s,r)}_x\sigma\in J^{(s,r)}Y_1$. The correspondence $J^{(s,r)}\colon 2\text{-}\mathcal{FM}_{m,n}\to\mathcal{FM}$ is a (fiber product preserving) bundle functor.

Let $Y\to X\to M$ be an 2-$\mathcal{FM}_{m,n,q}$-object. A vector field V on Y is called 2-projectable if there exist (unique) vector fields V_1 on X and V_0 on M such that V is related with V_1 and V_1 is related with V_0 (with respect to the 2-fibred manifold projections). Equivalently, the flow ExptV of V is formed by (local) 2-$\mathcal{FM}_{m,n,q}$-isomorphisms. Thus we can apply functor $J^{(s,r)}$ to ExptV and obtain new flow $J^{(s,r)}(\mathrm{ExptV})$ on $J^{(s,r)}Y$. Consequently we obtain vector field $\mathcal{J}^{(s,r)}V$ on $J^{(s,r)}Y$. The corresponding 2-$\mathcal{FM}_{m,n,q}$-natural operator $\mathcal{J}^{(s,r)}\colon T_{2\text{-proj}}\rightsquigarrow TJ^{(s,r)}$ is called the flow operator (of $J^{(s,r)}$).

The main result of the present note is the following classification theorem.

Theorem 1. *Let r, s, m, n, q be natural numbers such that $s\geq r$. Any 2-$\mathcal{FM}_{m,n,q}$-natural operator $A\colon T_{2\text{-proj}}\rightsquigarrow TJ^{(s,r)}$ is a constant multiple of the flow operator $\mathcal{J}^{(s,r)}$.*

Thus Theorem 1 extends the result from [2] on 2-fibred manifolds. More precisely, in [2] it is proved that any $\mathcal{FM}_{m,n}$-natural operator A lifting projectable vector fields V from fibred manifolds $Y\to M$ to vector fields $A(V)$ on J^rY is a constant multiple of the flow operator.

In the proof of Theorem 1 we will use the method from [4] (a Weil algebra technique). We start with the proof of the following lemma. Let $A\colon T_{2\text{-proj}}\rightsquigarrow TJ^{(s,r)}$ be a natural operator in question.

Lemma 1. *The natural operator A is determined by the restriction $A(\frac{\partial}{\partial x^1})\mid \big(J^{(s,r)}(\mathbf{R}^{m,n,q})\big)_{(0,0)}$, where $(0,0)\in\mathbf{R}^m\times\mathbf{R}^n$.*

Proof. The assertion is an immediate consequence of the naturality and regularity of A and the fact that any 2-projectable vector field which is not $(Y\to M)$-vertical is related with $\frac{\partial}{\partial x^1}$ by an 2-$\mathcal{FM}_{m,n,q}$-map. □

Now we prove

Lemma 2. *Let A be the operator. Let $\pi\colon J^{(s,r)}Y\to X$ be the projection. Then there exists the unique real number c and the unique π-vertical operator $\mathcal{V}\colon T_{2\text{-proj}}\rightsquigarrow TJ^{(s,r)}$ with $\mathcal{V}(0)=0$ such that $A=c\mathcal{J}^{(s,r)}+\mathcal{V}$.*

Proof. Define $C=T\pi\circ A(\frac{\partial}{\partial x^1})\colon \big(J^{(s,r)}(\mathbf{R}^{m,n,q})\big)_{(0,0)}\to T_{(0,0)}(\mathbf{R}^m\times\mathbf{R}^n)$. Using the invariance of A with respect to 2-$\mathcal{FM}_{m,n,q}$-maps

$$(x^1,\dots,x^m,y^1,\dots,y^n,\tau z^1,\dots,\tau z^q)$$

for $\tau > 0$ and putting $t \to 0$ we get that $C\big(j^{(s,r)}_{(0,0)}(\sigma)\big) = C\big(j^{(s,r)}_{(0,0)}(0)\big)$, where 0 is the zero section. Then using the invariance of A with respect to

$$(x^1, \tau x^2, \dots, \tau x^m, \tau y^1, \dots, \tau y^n, \tau z^1, \dots, \tau z^q)$$

for $\tau > 0$ and putting $t \to 0$ we get that $C\big(j^{(s,r)}_{(0,0)}(0)\big) = c\frac{\partial}{\partial x^1}_{|0}$ for some $c \in \mathbf{R}$. We put $\mathcal{V} = A - c\mathcal{J}^{(s,r)}$. Then $\mathcal{V}$ is of vertical type because of Lemma 1. Clearly, $A = c\mathcal{J}^{(s,r)} + \mathcal{V}$.

It remains to show that $\mathcal{V}(0) = 0$. Clearly, the flow of $\mathcal{V}(0)$ is a family of natural automorphisms $J^{(s,r)} \to J^{(s,r)}$. Since the 2-$\mathcal{FM}_{m,n,q}$-orbit of $j^{(s,r)}_{(0,0)}(0)$ is the whole $\big(J^{(s,r)}(\mathbf{R}^{m,n,q})\big)_{(0,0)}$ (any element $j^{(s,r)}_{(0,0)}\sigma \in \big(J^{(s,r)}(\mathbf{R}^{m,n,q})\big)_{(0,0)}$ is transformed by 2-$\mathcal{FM}_{m,n,q}$-map

$$\big(x, y, z - \sigma(x,y)\big)$$

into $j^{(s,r)}_{(0,0)}(0)$), then any natural automorphism $\mathcal{E}\colon J^{(s,r)} \to J^{(s,r)}$ is determined by $\mathcal{E}\big(j^{(s,r)}_{(0,0)}(0)\big)$. Then using the invariance of $\mathcal{E}$ with respect to

$$(\tau x^1, \dots, \tau x^m, \tau y^1, \dots, \tau y^n, \tau z^1, \dots, \tau z^q)$$

for $\tau > 0$ and putting $\tau \to 0$ we get $\mathcal{E}\big(j^{(s,r)}_{(0,0)}(0)\big) = j^{(s,r)}_{(0,0)}(0)$. Then $\mathcal{E} = \mathrm{id}$ and then $\mathcal{V}(0) = 0$. □

Define a bundle functor $F\colon \mathcal{M}f \to \mathcal{FM}$ by

$$FN = \big(J^{(s,r)}(\mathbf{R}^m \times \mathbf{R}^n \times N)\big)_{(0,0)}, \quad Ff = \big(J^{(s,r)}(\mathrm{id}_{\mathbf{R}^m} \times \mathrm{id}_{\mathbf{R}^n} \times f)\big)_{(0,0)}.$$

Lemma 3. *The bundle functor $F\colon \mathcal{M}f \to \mathcal{FM}$ is product preserving.*

Proof. It is clear. □

Let $B = F\mathbf{R}$ be the Weil algebra corresponding to F.

Lemma 4. *We have $B = \mathcal{D}^s_{m+n}/\underline{B}$, where $\mathcal{D}^s_{m+n} = J^s_{(0,0)}(\mathbf{R}^{m+n}, \mathbf{R})$ and $\underline{B} = \big\langle j^s_{(0,0)}(x^1), \dots, j^s_{(0,0)}(x^m)\big\rangle^{r+1}$ is the $(r+1)$-power of the ideal $\big\langle j^s_{(0,0)}(x^1), \dots, j^s_{(0,0)}(x^m)\big\rangle$, generated by the elements as indicate.*

Proof. It is a simple observation. □

We have the obvious action $H\colon G^s_{m,n} \times B \to B$,

$$H\big(j^s_{(0,0)}\psi, [j^s_{(0,0)}\gamma]\big) = \big[j^s_{(0,0)}(\gamma \circ \psi^{-1})\big]$$

for any $\mathcal{FM}_{,m,n}$-map $\psi\colon \big(\mathbf{R}^m \times \mathbf{R}^n, (0,0)\big) \to \big(\mathbf{R}^m \times \mathbf{R}^n, (0,0)\big)$ and $\gamma\colon \mathbf{R}^{m+n} \to \mathbf{R}$. This action is by algebra automorphisms.

Lemma 5. *For any derivation $D \in \mathrm{Der}(B)$ we have the implication: if*

$$H\big(j^s_{(0,0)}(\tau\,\mathrm{id})\big) \circ D \circ H\big(j^s_{(0,0)}(\tau^{-1}\,\mathrm{id})\big) \to 0 \quad \text{as} \quad \tau \to 0 \quad \text{then} \quad D = 0\,.$$

Proof. Let $D \in \mathrm{Der}(B)$ be such that

$$H\big(j^s_{(0,0)}(\tau\,\mathrm{id})\big) \circ D \circ H\big(j^s_{(0,0)}(\tau^{-1}\,\mathrm{id})\big) \to 0 \quad \text{as} \quad \tau \to 0\,.$$

For $i = 1, \dots, m$ and $j = 1, \dots, n$ write $D\big([j^s_{(0,0)}(x^i)]\big) = \sum a^i_{\alpha\beta}\big[j^s_{(0,0)}(x^\alpha y^\beta)\big]$ and $D\big([j^s_{(0,0)}(y^j)]\big) = \sum b^j_{\alpha\beta}\big[j^s_{(0,0)}(x^\alpha y^\beta)\big]$ for some (unique) real numbers $a^i_{\alpha\beta}$ and $b^j_{\alpha\beta}$, where the sums are over all $\alpha \in \big(\mathbf{N} \cup \{0\}\big)^m$ and $\beta \in \big(\mathbf{N} \cup \{0\}\big)^n$ with $|\alpha| \le r$ and $|\alpha| + |\beta| \le s$. We have

$$H\big(j^s_{(0,0)}(\tau\,\mathrm{id})\big)\circ D\circ H\big(j^s_{(0,0)}(\tau^{-1}\,\mathrm{id})\big)\,\big([j^s_{(0,0)}(x^i)]\big) = \sum a^i_{\alpha\beta}\frac{1}{\tau^{|\alpha|+|\beta|-1}}\big[j^s_{(0,0)}(x^\alpha y^\beta)\big]\,.$$

Then from the assumption on D it follows that $a^i_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq \big((0), (0)\big)$. Similarly, $b^j_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq \big((0), (0)\big)$. Then $D\big([j^s_{(0,0)}(x^i)]\big) = a^i_{(0)(0)}\big[j^s_{(0,0)}(1)\big]$ and $D\big([j^s_{(0,0)}(y^j)]\big) = b^j_{(0)(0)}\big[j^s_{(0,0)}(1)\big]$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. Then (since $\big[j^s_{(0,0)}((x^i)^{r+1})\big] = 0$ and D is a differentiation) we have

$$\begin{aligned} 0 = D\big([j^s_{(0,0)}((x^i)^{r+1})]\big) &= (r+1)\big[j^s_{(0,0)}((x^i)^r)\big]D\big([j^s_{(0,0)}(x^i)]\big) \\ &= (r+1)a^i_{(0)(0)}\big[j^s_{(0,0)}((x^i)^r)\big]\,. \end{aligned}$$

Then $a^i_{(0)(0)} = 0$ as $\big[j^s_{(0,0)}((x^i)^r)\big] \neq 0$. Similarly, $b^j_{(0)(0)} = 0$. Then $D = 0$ because the $\big[j^s_{(0,0)}(x^i)\big]$ and $[j^s_{(0,0)}(y^j)]$ generate the algebra B. □

Proof of Theorem 1. Operator $\mathcal{V}$ from Lemma 2 defines (by the restriction) $\mathcal{M}f_q$-natural vector fields $\tilde{\mathcal{V}}_t = \mathcal{V}\big(t\frac{\partial}{\partial x^1}\big)|FN$ on FN for any $t \in \mathbf{R}$. Clearly, $\mathcal{V}$ is determined by $\tilde{\mathcal{V}}_1$. By Lemma 2, $\tilde{\mathcal{V}}_0 = 0$. By [2], $\tilde{\mathcal{V}}_t = \mathrm{op}(D_t)$ for some $D_t \in \mathrm{Der}(B)$. Then using the invariance of $\mathcal{V}$ with respect to

$$(\tau x^1, \dots, \tau x^m, \tau y^1, \dots, \tau y^n, z^1, \dots, z^q)$$

for $\tau \neq 0$ and putting $\tau \to 0$ we obtain that

$$H\big(j^s_{(0,0)}(\tau\,\mathrm{id})\big) \circ D_t \circ H\big(j^s_{(0,0)}(\tau^{-1}\,\mathrm{id})\big) \to 0 \quad \text{as} \quad \tau \to 0\,.$$

Then $D_t = 0$ because of Lemma 5. Then $\mathcal{V} = 0$, and then $A = c\mathcal{J}^{(s,r)}$ as well. □

Remark 1. There is another (non-equivalent) generalization of jets. Let $s \ge r$. Let $Y \to X \to M$ be a 2-fibred manifold. By [2], sections $\sigma_1, \sigma_2 \colon X \to Y$ of $Y \to X$ have the same r,s-jets $j^{r,s}_x\sigma_1 = j^{r,s}_x\sigma_2$ at $x \in X$ iff

$$j^r_x\sigma_1 = j^r_x\sigma_2 \quad \text{and} \quad j^s_x\big(\sigma_1 \mid X_{p_o(x)}\big) = j^s_x\big(\sigma_2 \mid X_{p_o(x)}\big)\,,$$

where $X_{p_o(x)}$ is the fiber of $X \to M$ through x. Consequently we have the corresponding bundle $J^{r,s}Y$ and the corresponding (fiber product preserving) bundle functor $J^{r,s} \colon 2\text{-}\mathcal{FM}_{m,n} \to \mathcal{FM}$. In [3], we proved that any $2\text{-}\mathcal{FM}_{m,n,q}$-natural operator $A \colon T_{2\text{-proj}} \rightsquigarrow TJ^{r,s}$ is a constant multiple of the flow operator $\mathcal{J}^{r,s}$ corresponding to $J^{r,s}$ (we used quite different method than the one in [4] or in the present note).

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