ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 41–56

NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION

Abhijit Banerjee and Sonali Mukherjee

Abstract. Dealing with a question of Lahiri [6] we study the uniqueness problem of meromorphic functions concerning two nonlinear differential polynomials sharing a small function. Our results will not only improve and supplement the results of Lin-Yi [16], Lahiri Sarkar [12] but also improve and supplement a very recent result of the first author [1].

1. Introduction definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. A meromorphic function α is said to be a small function of f provided that $T(r,\alpha) = S(r,f)$, that is $T(r,\alpha) = o(T(r,f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. Clearly if f is rational then α is a constant and if f is transcendental then α is a nonconstant meromorphic function. We denote by $S(f)$ the set of all small functions of f.

If for some $\alpha \in S(f) \cap S(g)$, $f-\alpha$ and $g-\alpha$ have the same set of zeros with the same multiplicities, we say that f and g share α CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share α IM (ignoring multiplicities).

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

Let $N_E(r,\alpha;f,g)$ ($\overline{N}_E(r,\alpha;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-\alpha$ and $g-\alpha$ with the same multiplicities and $N_0(r,\alpha;f,g)$ ($\overline{N}_0(r,\alpha;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-\alpha$ and $g-\alpha$ ignoring multiplicities.

If

$$\overline{N}(r,\alpha;f) + \overline{N}(r,\alpha;g) - 2\overline{N}_E(r,\alpha;f,g) = S(r,f) + S(r,g)$$

then we say that f and g share α "CM".

On the other hand if

$$\overline{N}(r,\alpha;f) + \overline{N}(r,\alpha;g) - 2\overline{N}_0(r,\alpha;f,g) = S(r,f) + S(r,g)$$

2000 *Mathematics Subject Classification:* Primary: 30D35.

Key words and phrases: meromorphic function, uniqueness, nonlinear differential polynomials, small function, weakly weighted sharing.

Received June 2, 2007, revised September 2007. Editor O. Došlý.

then we say that f and g share α "IM".

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

In [6] Lahiri studied the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper [6] regarding the nonlinear differential polynomials Lahiri asked the following question. *What can be said if two nonlinear differential polynomials generated by two meromorphic functions share* 1 *CM?*

Naturally several authors investigate the possible answer to the above question and continuous efforts are being carried out to relax the hypothesis of the results. (cf. [1], [2], [3], [11], [12], [14], [15], [16]).

In 2002 Fang and Fang [2] and in 2004 Lin-Yi [15] independently proved the following result.

Theorem A. *Let f and g be two nonconstant meromorphic functions and $n\,(\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share* 1 *CM, then $f \equiv g$.*

In 2004 Lin-Yi [16] improved Theorem A by generalizing it in view of fixed point. Lin-Yi [16] proved the following result.

Theorem B. *Let f and g be two transcendental meromorphic functions and $n\,(\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share z CM, then $f \equiv g$.*

In the same paper Lin-Yi [16] mentioned that in Theorem B z can be replaced by $\alpha(z)$.

In 2001 an idea of gradation of sharing of values was introduced in ([8], [9]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1 ([8, 9])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of value recently the first author [1] improved Theorem A as follows.

Theorem C ([1])**.** *Let f and g be two nonconstant meromorphic functions and $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$, is an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $(1, 2)$ then $f \equiv g$.*

In the mean time Lahiri and Sarkar [12] also studied the uniqueness of meromorphic functions corresponding to nonlinear differential polynomials which are different from that of previously mentioned and proved the following.

Theorem D ([12]). *Let f and g be two nonconstant meromorphic functions such that $f^n(f^2-1)f'$ and $g^n(g^2-1)g'$ share $(1,2)$, where n (≥ 13) is an integer then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility of $f \equiv -g$ does not arise.*

From the above discussion it will be a natural query to investigate the uniqueness of meromorphic functions when two non linear differential polynomials of more general form namely $f^n(af^2+bf+c)f'$ and $g^n(ag^2+bg+c)g'$ where $a \neq 0$ and $|b|+|c| \neq 0$ share a small function.

In this paper we will study the above problem with the notion of weakly weighted sharing which has recently been introduced by Lin and Lin [13] generalizing the idea of weighted sharing of values. We are now giving the definition.

Definition 1.2 ([13]). Let f g share α "IM" for $\alpha \in S(f) \cap S(g)$ and k is a positive integer or ∞.

(i) $\overline{N}^E(r,\alpha;f,g \mid\leq k)$ denotes the reduced counting function of those α-points of f whose multiplicities are equal to the corresponding α-points of g, both of their multiplicities are not greater than k.

(ii) $\overline{N}^0(r,\alpha;f,g \mid> k)$ denotes the reduced counting function of those α-points of f which are α-points of g, both of their multiplicities are not less than k.

Definition 1.3 ([13]). For $\alpha \in S(f) \cap S(g)$, if k is a positive integer or ∞ and

$$\overline{N}(r,\alpha;f \mid\leq k) - \overline{N}^E(r,\alpha;f,g \mid\leq k) = S(r,f)\,,$$

$$\overline{N}(r,\alpha;g \mid\leq k) - \overline{N}^E(r,\alpha;f,g \mid\leq k) = S(r,g)\,,$$

$$\overline{N}(r,\alpha;f \mid\geq k+1) - \overline{N}^0(r,\alpha;f,g \mid\geq k+1) = S(r,f)\,,$$

$$\overline{N}(r,\alpha;g \mid\geq k+1) - \overline{N}^0(r,\alpha;f,g \mid\geq k+1) = S(r,g)$$

or if $k=0$ and

$$\overline{N}(r,\alpha;f) - \overline{N}_0(r,\alpha;f,g) = S(r,f)\,,$$

$$\overline{N}(r,\alpha;g) - \overline{N}_0(r,\alpha;f,g) = S(r,g)\,,$$

then we say f, g weakly share α with weight k. Here we write f, g share "(α,k)" to mean that f, g weakly share α with weight k.

Obviously if f, g share "(α,k)", then f, g share "(α,p)" for any integer p, $0 \leq p < k$. Also we note that f, g share α "IM" or "CM" if and only if f, g share "$(\alpha,0)$" or "(α,∞)" respectively.

We now state the following theorem which is the main result of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions such that $f^n(af^2+bf+c)f'$ and $g^n(ag^2+bg+c)g'$ where $a \neq 0$ and $|b|+|c| \neq 0$ share "$(\alpha,2)$". Then the following holds.*

(i) *If* $b \neq 0$, $c = 0$ *and* $n > \max\big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}, \frac{4}{\Theta(\infty;f)+\Theta(\infty;g)} - 2\big]$, *be an integer, where* $\Theta(\infty; f) + \Theta(\infty; g) > 0$, *then* $f \equiv g$.

(ii) *If* $b \neq 0$, $c \neq 0$, $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$, *the roots of the equation* $az^2 + bz + c = 0$ *are distinct and one of* f *and* g *is non entire meromorphic function having only multiple poles, then* $f \equiv g$.

(iii) *If* $b \neq 0$, $c \neq 0$, $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *and the roots of the equation* $az^2 + bz + c = 0$ *coincides, then* $f \equiv g$.

(iv) $b = 0$, $c \neq 0$, $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$, *then either* $f \equiv g$ *or* $f \equiv -g$. *If* n *is an even integer then the possibility* $f \equiv -g$ *does not arise.*

From Theorem 1.1 we can immediately deduce the following corollaries.

Corollary 1.1. *Let* f *and* g *be two transcendental meromorphic functions such that* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$, *and* $n\,(\geq 13)$ *be an integer. If* $f^n(af^2 + bf)f'$ *and* $g^n(ag^2 + bg)g'$ *share "*$(\alpha, 2)$*" then* $f \equiv g$.

Corollary 1.2. *Let* f *and* g *be two transcendental meromorphic functions and one of* f *and* g *is non entire meromorphic function having only multiple poles, such that* $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *be an integer. If* $af^n(f - \beta_1)(f - \beta_2)f'$ *and* $ag^n(g - \beta_1)(g - \beta_2)g'$ *share "*$(\alpha, 2)$*", where* β_1 *and* β_2 *are the distinct roots of the equation* $az^2 + bz + c = 0$ *with* $|\beta_1| \neq |\beta_2|$, *then* $f \equiv g$.

Corollary 1.3. *Let* f *and* g *be two transcendental meromorphic functions such that* $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *be an integer. If* $af^n(f + k)^2 f'$ *and* $ag^n(g + k)^2 g'$ *share "*$(\alpha, 2)$*" where* k *is a nonzero constant then* $f \equiv g$.

Corollary 1.4. *Let* f *and* g *be two transcendental meromorphic functions such that* $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *be an integer. If* $f^n(af^2 + c)f'$ *and* $g^n(ag^2 + c)g'$ *share "*$(\alpha, 2)$*" then* $f \equiv g$ *or* $f \equiv -g$. *If* n *is an even integer then the possibility* $f \equiv -g$ *does not arise.*

Though we use the standard notations and definitions of the value distribution theory available in [5], we explain some definitions and notations which are used in the paper.

Definition 1.4 ([7])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)$ $\big(N(r, a; f \mid\geq m)\big)$ the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $\big(\overline{N}(r, a; f \mid\geq m)\big)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.5 ([9], cf.[20]). We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;\ f\mid\geq 2)$.

Definition 1.6 ([9]). Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g for which $p>q$, each point in this counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$.

Definition 1.7 ([10]). Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.8 ([10]). Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f, g, F_1, G_1 be four nonconstant meromorphic functions. Henceforth we shall denote by h and H the following two functions.

$$h=\Big(\frac{f^{''}}{f'}-\frac{2f'}{f-1}\Big)-\Big(\frac{g^{''}}{g'}-\frac{2g'}{g-1}\Big)$$

and

$$H=\Big(\frac{F_1^{''}}{F_1'}-\frac{2F_1'}{F_1-1}\Big)-\Big(\frac{G_1^{''}}{G_1'}-\frac{2G_1'}{G_1-1}\Big).$$

Lemma 2.1. *If f, g be share "$(1,1)$" and $h\not\equiv 0$. Then*

$$N(r,1;f\mid\leq 1)\leq N(r,0;h)+S(r,f)\leq N(r,\infty;h)+S(r,f)+S(r,g)\,.$$

Proof. Since f, g share "$(1,1)$" it follows that if z_0 be a common simple 1-point of f and g, then in some neighborhoods of z_0 we have $h=(z-z_0)\phi(z)$, where $\phi(z)$ is analytic at z_0. Hence by the first fundamental theorem and Milloux theorem (p. 55 [5]) we get

$$\begin{aligned}N(r,1;f\mid\leq 1)&=N^E(r,1;f,g\mid\leq 1)+S(r,f)\\&\leq N(r,0;h)+S(r,f)\leq N(r,\infty;h)+S(r,f)+S(r,g)\end{aligned}$$ □

Lemma 2.2. *If f, g share "$(1,1)$" and $h\not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty;h)\leq{}&\overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,0;g\mid\geq 2)\\&+\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,\infty;g\mid\geq 2)\\&+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r)\,,\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g, (iii) the common zeros of $f-1$ and $g-1$ whose multiplicities are different, (iii) those 1-points of f (g) which are not the 1-points of g (f), (iv) zeros of f' which are not the zeros of $f(f-1)$, (v) zeros of g' which are not zeros of $g\,(g-1)$. Since all the poles of h are simple the lemma follows from above. This proves the lemma. □

Lemma 2.3. *If for a positive integer k, $N_k(r,0;f' \mid f\neq 0)$ denotes the counting function of those zeros of of f' which are not the zeros of f, where a zero of f' with multiplicity m is counted m times if $m\leq k$ and k times if $m>k$ then*

$$N_k(r,0;f' \mid f\neq 0)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\sum_{p=k+1}^{\infty}\overline{N}\Big(r,0;\frac{f'}{f}\mid\geq p\Big)+S(r,f)\,.$$

Proof. By the first fundamental theorem and Milloux theorem (p. 55 [5]) we get

$$\begin{aligned}N(r,0;f' \mid f\neq 0)=N\Big(r,0;\frac{f'}{f}\Big)&\leq N\Big(r,\infty;\frac{f'}{f}\Big)+S(r,f)\\&=\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\,.\end{aligned}$$

Now

$$\begin{aligned}N_k\Big(r,0;\frac{f'}{f}\Big)+\sum_{p=k+1}^{\infty}\overline{N}\Big(r,0;\frac{f'}{f}\mid\geq p\Big)&=N\Big(r,0;f' \mid f\neq 0\Big)\\&\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\,.\end{aligned}$$

The lemma follows from above as $N_k\big(r,0;\frac{f'}{f}\big)=N_k(r,0;f' \mid f\neq 0)$. □

Lemma 2.4. *Let f, g share "$(1,2)$" and $h\not\equiv 0$. Then*

$$\begin{aligned}T(r,f)\leq{}& N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\&-\sum_{p=3}^{\infty}\overline{N}\Big(r,0;\frac{g'}{g}\mid\geq p\Big)+S(r,f)+S(r,g)\,.\end{aligned}$$

Proof. Since f and g share "$(1,2)$" it follows that f and g share "$(1,1)$". Also we note that $\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)\leq \overline{N}(r,1;g\mid\geq 3)$. So by the second fundamental theorem Lemmas 2.1, 2.2 and 2.3 we get

$$\begin{aligned}T(r,f)&\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;f)-N_0(r,0;f')+S(r,f)\\&\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N(r,1;f\mid\leq 1)+\overline{N}(r,1;f\mid\geq 2)-N_0(r,0;f')\\&\leq N_2(r,0;f)+N_2(r,\infty;f)+\overline{N}(r,0;g\mid\geq 2)+\overline{N}(r,\infty;g\mid\geq 2)\\&\quad+\overline{N}(r,1;g\mid\geq 2)+\overline{N}(r,1;g\mid\geq 3)+S(r,f)+S(r,g)\end{aligned}$$

$$\begin{aligned}&\le N_2(r,0;f)+N_2(r,\infty;f)+\overline{N}(r,0;g\mid\ge 2)+\overline{N}(r,\infty;g\mid\ge 2)\\&\quad+N_2(r,0;g'\mid g\ne 0)+S(r,f)+S(r,g)\\&\le N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\&\quad-\sum_{p=3}^{\infty}\overline{N}\Big(r,0;\frac{g'}{g}\mid\ge p\Big)+S(r,f)+S(r,g)\,.\end{aligned}$$

□

Lemma 2.5 ([17])**.** *Let f be a nonconstant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\ne 0$ and $b_m\ne 0$. Then

$$T\big(r,R(f)\big)=dT(r,f)+S(r,f)\,,$$

where $d=\max\{n,m\}$.

Lemma 2.6. *Let $F_1=\frac{f^n(af^2+bf+c)f'}{\alpha}$ and $G_1=\frac{g^n(ag^2+bg+c)g'}{\alpha}$, where $a\ne 0$ and $|b|+|c|\ne 0$. Then $S(r,F_1)=S(r,f)$ and $S(r,G_1)=S(r,g)$.*

Proof. Using Lemma 2.5 we see that

$$T(r,F_1)\le(n+2)T(r,f)+T(r,f')+S(r,f)=(n+4)\,T(r,f)+S(r,f)$$

and

$$(n+2)T(r,f)=T(r,f^n(af^2+bf+c))+0(1)\le T(r,F_1)+T(r,f')+S(r,f)\,,$$

that is,

$$T(r,F_1)\ge n\,T(r,f)+S(r,f)\,.$$

Hence $S(r,F_1)=S(r,f)$. In the same way we can prove $S(r,G_1)=S(r,g)$. □

Lemma 2.7 ([21])**.** *If $h\equiv 0$ and*

$$\limsup_{r\to\infty}\frac{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)}{T(r)}<1\,,\quad r\in I$$

then $f\equiv g$ or $f\cdot g\equiv 1$.

Lemma 2.8. *Let f, g be two nonconstant meromorphic functions. Then*

$$f^n(af^2+bf+c)f'g^n(ag^2+bg+c)g'\not\equiv\alpha^2\,,$$

where $a\ne 0$ and $|b|+|c|\ne 0$ and n (>7) is an integer.

Proof. If possible, let

$$f^n(af^2+bf+c)f'g^n(ag^2+bg+c)g'\equiv\alpha^2\,. \tag{2.1}$$

We consider the following cases.

Case 1. The roots of the equation $az^2+bz+c=0$ are distinct and suppose they are β_1 and β_2.

Subcase 1.1. One of β_1 and β_2 say $\beta_2 = 0$. Then (2.1) reduces to

$$a^2 f^{n+1}(f-\beta_1)f' g^{n+1}(g-\beta_1)g'' \equiv \alpha^2 .$$

Let z_0 be a zero of f with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Clearly z_0 is a pole of g with multiplicity $q\,(\geq 1)$ such that

$$(n+1)p + p - 1 = (n+2)q + q + 1\,, \tag{2.2}$$

i.e.

$$q = (n+2)(p-q) - 2 \geq n\,.$$

Again from (2.2) we get

$$(n+2)p = (n+3)q + 2 = (n+2)q + q + 2 \geq (n+1)(n+2)\,, \quad \text{i.e.,} \quad p \geq n+1\,.$$

Noting that α is a small function we obtain

$$N(r,0;f) \geq (n+1)\overline{N}(r,0;f) + S(r,f)\,.$$

Next suppose z_1 be a zero of $f-\beta_1$ with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Then z_1 be a pole of g with multiplicity $q\,(\geq 1)$ such that

$$2p - 1 = (n+1)q + 2q + 1 \quad \text{i.e.,} \quad p \geq \frac{n+5}{2}\,.$$

Let $\overline{N}_{\otimes}(r,0;f')$ ($\overline{N}_{\otimes}(r,0;g')$) denotes the reduced counting function of those zeros of f' (g') which are not the zeros of $f(f-\beta_1)$ ($g(g-\beta_1)$). Since a pole of f is either a zero of $g(g-\beta_1)$ or a zero of g' or a zero or pole of α we note that

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,\beta_1;g) + \overline{N}_{\otimes}(r,0;g') + S(r)\\ &\leq \frac{1}{n+1}N(r,0;g) + \frac{2}{n+5}N(r,\beta_1;g) + \overline{N}_{\otimes}(r,0;g') + S(r)\\ &\leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,g) + \overline{N}_{\otimes}(r,0;g') + S(r)\,.\end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,\beta_1;f) + \overline{N}(r,\infty;f) - \overline{N}_{\otimes}(r,0;f') + S(r,f)\\ &\leq \frac{1}{n+1}N(r,0;f) + \frac{2}{n+5}N(r,\beta_1;f) + \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,g)\\ &\quad + \overline{N}_{\otimes}(r,0;g') - \overline{N}_{\otimes}(r,0;f') + S(r)\,,\end{aligned}$$

i.e.,

$$\begin{aligned}\left(1 - \frac{1}{n+1} - \frac{2}{n+5}\right) T(r,f) \leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,g)&\\ + \overline{N}_{\otimes}(r,0;g') - \overline{N}_{\otimes}(r,0;f') + S(r)\,.&\end{aligned} \tag{2.3}$$

In a similar manner we get

$$\begin{aligned}\left(1 - \frac{1}{n+1} - \frac{2}{n+5}\right) T(r,g) \leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,f)&\\ + \overline{N}_{\otimes}(r,0;f') - \overline{N}_{\otimes}(r,0;g') + S(r)\,.&\end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) we get

$$\left(1-\frac{2}{n+1}-\frac{4}{n+5}\right)\{T(r,f)+T(r,g)\}\leq S(r)\,,$$

which is a contradiction for $n>7$. Hence this subcase does not hold.

Subcase 1.2. Both the roots β_1 and β_2 are non zero.
Let z_0 be a zero of f with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Then from (2.1) we get z_0 is a pole of g with multiplicity $q\,(\geq 1)$ such that

$$np+p-1=(n+3)q+1 \tag{2.5}$$

i.e., $q\geq\frac{n-1}{2}$. So from (2.5) we get

$$(n+1)p\geq\frac{(n+3)(n-1)+4}{2}\,,\quad \text{i.e.,}\quad p\geq\frac{n+1}{2}\,.$$

So from above we have

$$N(r,0;f)\geq\frac{n+1}{2}\overline{N}(r,0;f)+S(r,f)\,,\quad \text{and so}\quad \Theta(0;f)\geq 1-\frac{2}{n+1}\,.$$

Next suppose z_1 be a zero of $f-\beta_1$ with multiplicity $p\,(\geq 1)$ and it is not a zero or pole of α. Then z_1 be a pole of g with multiplicity $q\,(\geq 1)$ such that

$$2p-1=(n+3)q+1\,,\quad \text{i.e.,}\quad p=\frac{(n+3)q+2}{2}\geq\frac{n+5}{2}\,.$$

$$N(r,\beta_1;f)\geq\frac{n+5}{2}\overline{N}(r,0;f)+S(r,f)\,,\quad \text{and so}\quad \Theta(\beta_1;f)\geq 1-\frac{2}{n+5}\,.$$

Similarly we can deduce that

$$\Theta(\beta_2;f)\geq 1-\frac{2}{n+5}\,.$$

Since $\Theta(0;f)+\Theta(\beta_1;f)+\Theta(\beta_2;f)\leq 2$, it follows that

$$3-\frac{4}{n+5}-\frac{2}{n+1}\leq 2\,,\quad \text{or}\quad \frac{4}{n+5}+\frac{2}{n+1}\geq 1$$

which is a contradiction for $n>7$. Hence this subcase also does not hold.

Case 2. The roots of the equation $az^2+bz+c=0$ are equal say $\beta_1=\beta_2=\beta$. Let z_0 be a zero of f with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Then z_0 is a pole of g with multiplicity $q\,(\geq 1)$ such that $np+p-1=(n+3)q+1$, i.e.

$$q\geq\frac{n-1}{2}\quad \text{and so}\quad p\geq\frac{n+1}{2}\,.$$

Hence

$$N(r,0;f)\geq\frac{n+1}{2}\overline{N}(r,0;f)+S(r,f)\,.$$

Next suppose z_1 be a zero of $f-\beta$ with multiplicity $p(\geq 1)$ which is not a zero or pole of α. Then z_1 be a pole of g with multiplicity $q(\geq 1)$ such that

$$3p-1=(n+3)q+1\geq n+4\,,\quad \text{i.e.,}\quad p\geq\frac{n+5}{3}\,.$$

Let $\overline{N}_{\oplus}(r,0;f')$ ($\overline{N}_{\oplus}(r,0;g')$) denotes the reduced counting function of those zeros of f' (g') which are not the zeros of $f(f-\beta)$ ($g(g-\beta)$). Now proceeding in the same way as done in Subcase 1.1 we note that

$$\overline{N}(r,\infty;f) \le \left(\frac{2}{n+1}+\frac{3}{n+5}\right) T(r,g) + \overline{N}_{\oplus}(r,0;g') + S(r)\,.$$

By the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,\beta;f) + \overline{N}(r,\infty;f) - \overline{N}_{\oplus}(r,0;f') + S(r,f) \\ &\le \frac{2}{n+1} N(r,0;f) + \frac{3}{n+5} N(r,\beta;f) + \left(\frac{2}{n+1}+\frac{3}{n+5}\right) T(r,g) \\ &\quad + \overline{N}_{\oplus}(r,0;g') - \overline{N}_{\oplus}(r,0;f') + S(r)\,, \end{aligned}$$

i.e.,

$$\begin{aligned} \left(1-\frac{2}{n+1}-\frac{3}{n+5}\right) T(r,f) &\le \left(\frac{2}{n+1}+\frac{3}{n+5}\right) T(r,g) \\ &\quad + \overline{N}_{\oplus}(r,0;g') - \overline{N}_{\oplus}(r,0;f') + S(r)\,. \end{aligned} \tag{2.6}$$

In a similar manner we get

$$\begin{aligned} \left(1-\frac{2}{n+1}-\frac{3}{n+5}\right) T(r,g) &\le \left(\frac{2}{n+1}+\frac{3}{n+5}\right) T(r,f) \\ &\quad + \overline{N}_{\oplus}(r,0;f') - \overline{N}_{\oplus}(r,0;g') + S(r)\,. \end{aligned} \tag{2.7}$$

Adding (2.6) and (2.7) we get

$$\left(1-\frac{4}{n+1}-\frac{6}{n+5}\right) \{T(r,f)+T(r,g)\} \le S(r)\,,$$

which is a contradiction for $n>7$. This proves the lemma. □

Lemma 2.9. *Let $F = f^{n+1}\left[\frac{af^2}{n+3}+\frac{bf}{n+2}+\frac{c}{n+1}\right]$ and $G = g^{n+1}\left[\frac{ag^2}{n+3}+\frac{bg}{n+2}+\frac{c}{n+1}\right]$, where $n(\ge 5)$ is an integer $a\ne 0$, $|b|+|c|\ne 0$. Then $F'\equiv G'$ implies $F\equiv G$.*

Proof. Let $F'\equiv G'$, then $F=G+d$ where d is a constant. If possible let $d\ne 0$. Then by the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned} (n+3)T(r,f) &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,d;F) + S(r,F) \\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,\beta_1;f) + \overline{N}(r,\beta_2;f) \\ &\quad + \overline{N}(r,0;g) + \overline{N}(r,\beta_1;g) + \overline{N}(r,\beta_2;g) + S(r,f) \\ &\le 4T(r,f) + 3T(r,g) + S(r,f)\,, \end{aligned} \tag{2.8}$$

where β_1 and β_2 are the roots of the equation $az^2+bz+c=0$. In a similar manner we get

$$(n+3)T(r,g) \le 3T(r,f) + 4T(r,g) + S(r,g). \tag{2.9}$$

Adding (2.8) and (2.9) we get

$$(n-4)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which is a contradiction for $n\ge 5$. So $d=0$ and the lemma follows. □

Lemma 2.10 ([4]). *Let*

$$Q(\omega) = (n-1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n-2)\,(\omega^{n-1} - 1)^2\,,$$

then

$$Q(\omega) = (\omega - 1)^4(\omega - \beta_1)\,(\omega - \beta_2)\dots(\omega - \beta_{2n-6})\,,$$

where $\beta_j \in C \setminus \{0,1\}$ $(j = 1, 2, \dots, 2n-6)$*, which are distinct respectively.*

Lemma 2.11. *Let* F *and* G *be given as in Lemma 2.9 and* $n\,(\geq 3)$ *be an integer. Suppose* $F \equiv G$*. Then the following holds.*

(i) *If* $b \neq 0$*,* $c = 0$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$ *then* $f \equiv g$*.*

(ii) *If* $b \neq 0$*,* $c \neq 0$*, and the roots of the equation* $az^2+bz+c = 0$ *are distinct and one of* f *and* g *is non entire meromorphic functions having only multiple poles then* $f \equiv g$*.*

(iii) *If* $b \neq 0$*,* $c \neq 0$*, and the roots of the equation* $az^2 + bz + c = 0$ *coincides then* $f \equiv g$*.*

(iv) *If* $b = 0$*,* $c \neq 0$ *then either* $f \equiv g$ *or* $f \equiv -g$*.*
If n *is an even integer then the possibility* $f \equiv -g$ *does not arise.*

Proof. We consider the following cases.

Case 1. Suppose $c = 0$ and $b \neq 0$. Then $F \equiv G$ implies

$$f^{n+2}(\frac{a}{n+3}f + \frac{b}{n+2}) \equiv g^{n+2}(\frac{a}{n+3}g + \frac{b}{n+2})\,. \tag{2.10}$$

Let us assume $f \not\equiv g$. We consider two cases:

Subcase 1.1. Let $y = \frac{g}{f}$ be a constant. Since $y \neq 1$, from (2.10) it follows that $y^{n+2} \neq 1$, $y^{n+3} \neq 1$ and $f \equiv -\frac{b(n+3)(1-y^{n+2})}{a(n+2)(1-y^{n+3})}$, a constant, which is impossible.

Subcase 1.2. Let $y = \frac{g}{f}$ be nonconstant. Noting that $f \not\equiv g$ clearly the poles of f comes from the zeros of $y - u_k$ where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \dots, n+2$. So we have

$$\sum_{k=1}^{n+2} \overline{N}(r, u_k; y) \leq \overline{N}(r, \infty; f)\,.$$

By the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned} n\,T(r, y) &\leq \sum_{k=1}^{n+2} \overline{N}(r, u_k; y) + S(r, y) \leq \overline{N}(r, \infty; f) + S(r, y) \\ &\leq (1 - \Theta(\infty; f) + \varepsilon)\,T(r, f) + S(r, y) \\ &= (n+2)\,(1 - \Theta(\infty; f) + \varepsilon)\,T(r, y) + S(r, y)\,, \end{aligned}$$

i.e.,

$$\Big[\frac{n}{n+2} - 1 + \Theta(\infty; f) - \varepsilon\Big]T(r, y) \leq S(r, y)\,, \tag{2.11}$$

where $\varepsilon > 0$ be arbitrary. In a similar manner we can obtain

$$\left[\frac{n}{n+2} - 1 + \Theta(\infty; g) - \varepsilon\right] T(r, y) \leq S(r, y)\,. \tag{2.12}$$

Adding (2.11) and (2.12) we get

$$\left(\Theta(\infty; f) + \Theta(\infty; g) - \frac{4}{n+2} - 2\varepsilon\right) T(r, y) \leq S(r, y)\,. \tag{2.13}$$

Since $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$ we can choose a $\delta > 0$ such that

$$\Theta(\infty; f) + \Theta(\infty; g) = \frac{4}{n+2} + \delta\,.$$

So for $0 < \varepsilon < \frac{\delta}{2}$ from (2.13) we can deduce a contradiction. Hence $f \equiv g$.

Case 2. Suppose $b \neq 0$ and $c \neq 0$. Then $F \equiv G$ implies

$$Af^{n+3} + Bf^{n+2} + Cf^{n+1} \equiv Ag^{n+3} + Bg^{n+2} + Cg^{n+1}\,, \tag{2.14}$$

where $A = \frac{a}{n+3}$, $B = \frac{b}{n+2}$ and $C = \frac{c}{n+1}$.

Let us assume $f \not\equiv g$.

Subcase 2.1. Suppose the roots of the equation $az^2 + bz + c = 0$ are distinct. Since (2.14) implies f, g share (∞, ∞) without loss of generality we may assume that g has some multiple poles. Putting $\eta = \frac{f}{g}$ in (2.14) we get

$$Ag^2(\eta^{n+3} - 1) + Bg(\eta^{n+2} - 1) + C(\eta^{n+1} - 1) \equiv 0\,,$$

i.e.,

$$Ag^2 \equiv -Bg\ \frac{\eta^{n+2} - 1}{\eta^{n+3} - 1} - C\ \frac{\eta^{n+1} - 1}{\eta^{n+3} - 1}\,. \tag{2.15}$$

Let z_0 be a pole of g which is not a root of $\eta - u_k = 0$, where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \ldots, n+2$ with multiplicity p. Then from (2.15) we have

$$2p = p \quad \text{i.e.,} \quad p = 0\,,$$

which is impossible. The other poles of the right hand side of (2.15) are the roots of $\eta - u_k = 0$ where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \ldots, n+2$. Suppose z_1 is a zero of $\eta - u_k$ of multiplicity r. From (2.15) we see that z_1 is a pole of g with multiplicity s (say) such that

$$2s = r + s \quad \text{i.e.,} \quad r = s\,.$$

Since g has no simple pole it follows that $\eta - u_k$ has no simple zero for $k = 1, 2, \ldots, n+2$. Hence

$$\Theta(u_k; \eta) \geq \frac{1}{2}$$

for $k = 1, 2, \ldots, n+2$. Since $\sum_{k=1}^{n+2} \Theta(u_k; \eta) \leq 2$ and $n \geq 3$ we arrive at a contradiction.

Subcase 2.2. Suppose the roots of the equation $az^2 + bz + c = 0$ coincides and so we obtain $b^2 = 4ac$. Putting $\eta = \frac{f}{g}$ in (2.14) we get

$$\begin{aligned} &a(n+2)(n+1)g^2(\eta^{n+3}-1) + b(n+3)(n+1)g(\eta^{n+2}-1) \\ &\qquad + c(n+3)(n+2)(\eta^{n+1}-1) \equiv 0\,. \end{aligned} \tag{2.16}$$

Since η is not constant using Lemma 2.10 we get from (2.16) that

$$\begin{aligned} &\Big[(n+2)(n+1)g(\eta^{n+3}-1) + \frac{b}{2a}(n+3)(n+1)(\eta^{n+2}-1)\Big]^2 \\ &= -\,(n+3)(n+1)\Big[\frac{c}{a}(n+2)^2(\eta^{n+3}-1)(\eta^{n+1}-1) \\ &\quad - \frac{b^2}{4a^2}(n+3)(n+1)(\eta^{n+2}-1)^2\Big] = -\frac{c}{a}(n+3)(n+1)Q(\eta)\,, \end{aligned}$$

where $Q(\eta) = (\eta-1)^4(\eta-\beta_1)\ (\eta-\beta_2)\dots(\eta-\beta_{2n})$ and $\beta_j \in C \setminus \{0,1\}$ $(j = 1,2,\dots,2n)$ which are distinct. This implies that every zero of $\eta - \beta_j$ $(j = 1,2,\dots,2n)$ has a multiplicity of at least 2, i.e., $\Theta(\beta_j;\eta) \geq \frac{1}{2}$ for $(j = 1,2,\dots,2n)$. But $\sum_{j=1}^{2n} \Theta(\beta_j;\eta) \leq 2$ which implies $n \leq 2$. This is a contradiction. So η is constant and from (2.15) we have $(\eta^{n+1}-1) = 0$ and $(\eta^{n+2}-1) = 0$ which implies $\eta = 1$ and so $f \equiv g$.

Case 3. Suppose $b = 0$ and $c \neq 0$. Then (2.14) reduces to

$$\Big[\frac{a}{n+3}f^2 + \frac{c}{n+1}\Big]\ f^{n+1} \equiv \Big[\frac{a}{n+3}g^2 + \frac{c}{n+1}\Big]\ g^{n+1}\,.$$

Now proceeding in the line of Lemma 2.4 in [12] we can prove $f \equiv g$ and $f \equiv -g$ and if n is an even integer then the possibility of $f \equiv -g$ does not arise. □

Lemma 2.12 ([19]). *Let f be a nonconstant meromorphic function. Then*

$$N(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N(r,0;f) + S(r,f)\,.$$

Lemma 2.13. *Let F and G be given as in Lemma 2.9 and F_1, G_1 be given by Lemma 2.6. If γ_1, γ_2 are the roots of $\frac{a}{n+3}z^2 + \frac{b}{n+2}z + \frac{c}{n+1} = 0$ and β_1, β_2 are the roots of $az^2 + bz + c = 0$. Then*

$$\begin{aligned} T(r,F) \leq\ & T(r,F_1) + N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) \\ & - N(r,\beta_1;f) - N(r,\beta_2;f) - N(r,0;f') + S(r)\,. \end{aligned}$$

Proof. Clearly $F' = \alpha\, F_1$ and $G' = \alpha\, G_1$. By the first fundamental theorem and Lemmas 2.5, 2.6 we obtain

$$\begin{aligned}
T(r,F) &= T(r,\frac{1}{F}) + O(1) = N(r,0;F) + m(r,\frac{1}{F}) + O(1)\\
&\le N(r,0;F) + m(r,\frac{F'}{F}) + m(r,0;F') + O(1)\\
&= T(r,F') + N(r,0;F) - N(r,0;F') + S(r,F)\\
&\le T(r,F_1) + (n+1)N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) - nN(r,0;f)\\
&\quad - N(r,\beta_1;f) - N(r,\beta_2;f) - N(r,0;f') + S(r)\\
&= T(r,F_1) + N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) - N(r,\beta_1;f)\\
&\quad - N(r,\beta_2;f) - N(r,0;f') + S(r)\,. \qquad \square
\end{aligned}$$

3. Proof of the theorem

Proof of Theorem 1.1. Let F, G be defined as in Lemma 2.9 and F_1 and G_1 be defined as in Lemma 2.6. Then it follows that F' and G' share "$(\alpha;2)$" and hence F_1 and G_1 share "$(1,2)$". Suppose $H \not\equiv 0$. Then by Lemmas 2.4, 2.6 and (2.6) we get

$$\begin{aligned}
T(r,F_1) &\le N_2(r,0;F_1) + N_2(r,\infty;F_1) + N_2(r,0;G_1)\\
&\quad + N_2(r,\infty;G_1) + S(r,f) + S(r,g)\\
&\le 2\overline{N}(r,0;f) + N(r,\beta_1;f) + N(r,\beta_2;f) + 2\overline{N}(r,0;g)\\
&\quad + N(r,\beta_1;g) + N(r,\beta_2;g) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g)\\
&\quad + N(r,0;f') + \overline{N}(r,0;g') + S(r)\,. \qquad (3.1)
\end{aligned}$$

Now from Lemmas 2.5, 2.12 and 2.13 we can obtain from (3.1) for $\varepsilon(>0)$

$$\begin{aligned}
(n+3)T(r,f) &\le 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + 3T(r,f) + 2\overline{N}(r,0;g)\\
&\quad + 2\overline{N}(r,\infty;g) + 2T(r,g) + N(r,0;g') + S(r)\\
&\le 5T(r,f) + 5T(r,g) + 2\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + S(r)\\
&\le (15 - 2\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon)\ T(r) + S(r)\,. \qquad (3.2)
\end{aligned}$$

In a similar manner we can obtain

$$(n+3)T(r,g) \le (15 - 3\Theta(\infty;f) - 2\Theta(\infty;g) + 2\varepsilon)\,T(r) + S(r)\,. \qquad (3.3)$$

From (3.2) and (3.3) we get

$$\big[n - 12 + 2\Theta(\infty;f) + 2\Theta(\infty;g) + \min\{\Theta(\infty;f);\Theta(\infty;g)\} - 2\varepsilon\big]T(r) \le S(r)\,. \qquad (3.4)$$

Since ε (> 0) is arbitrary, (3.4) implies a contradiction. Hence $H \equiv 0$.
Since for $\varepsilon > 0$ we have

$$\begin{aligned}\overline{N}(r,0;f') &\le T(r,f') - m\Big(r,\frac{1}{f'}\Big)\\ &\le m(r,f) + N(r,\infty;f) + \overline{N}(r,\infty;f) - m\Big(r,\frac{1}{f'}\Big) + S(r,f)\\ &\le (2-\Theta(\infty;f)+\varepsilon)T(r,f) - m\Big(r,\frac{1}{f'}\Big) + S(r,f)\,.\end{aligned}$$

We note that

$$\begin{aligned}&\overline{N}(r,0;F_1) + \overline{N}(r,\infty;F_1) + \overline{N}(r,0;G_1) + \overline{N}(r,\infty;G_1)\\ &\quad\le \overline{N}(r,0;f) + \overline{N}(r,\beta_1;f) + \overline{N}(r,\beta_2;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;f')\\ &\qquad + \overline{N}(r,0;g) + \overline{N}(r,\beta_1;g) + \overline{N}(r,\beta_2;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;g')\\ &\quad\le \big(12 - 2\Theta(\infty;f) - 2\Theta(\infty;g) + 2\varepsilon\big)\,T(r)\\ &\qquad - m(r,0;f') - m(r,0;g') + S(r)\,. \qquad (3.5)\end{aligned}$$

Also using Lemma 2.5 we get

$$\begin{aligned}T(r,F') + m\Big(r,\frac{1}{f'}\Big) &= m\big(r,f^n(af^2+bf+c)f'\big) + m\Big(r,\frac{1}{f'}\Big)\\ &\quad + N(r,\infty;f^n(af^2+bf+c)f') \ge m\left(r,f^n(af^2+bf+c)\right)\\ &\quad + N(r,\infty;f^n\left(af^2+bf+c\right)) = T(r,f^n\left(af^2+bf+c\right))\\ &= (n+2)T(r,f) + O(1)\,. \qquad (3.6)\end{aligned}$$

Similarly

$$T(r,G') + m\Big(r,\frac{1}{g'}\Big) \ge (n+2)T(r,g) + O(1)\,. \qquad (3.7)$$

From (3.6) and (3.7) we get

$$\max\left\{T(r,F_1),T(r,G_1)\right\} \ge (n+2)T(r) - m\Big(r,\frac{1}{f'}\Big) - m\Big(r,\frac{1}{g'}\Big) + O(1)\,. \qquad (3.8)$$

By (3.5) and (3.8) applying Lemma 2.7 we get either $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$.

Now from Lemma 2.8 it follows that $F_1G_1 \not\equiv 1$. Again $F_1 \equiv G_1$ implies $F' \equiv G'$. So from Lemmas 2.9 and 2.11 the theorem follows. ☐

Acknowledgement. The authors are grateful to the referee for his/her valuable comments and suggestions towards the improvement of the paper. The first author is also thankful to Prof. M. L. Fang for supplying him the electronic file of the paper [2].

References

[1] Banerjee, A., *On uniqueness for nonlinear differential polynomials sharing the same* 1*-point*, Ann. Polon. Math. **89** (3) (2006), 259–272.

[2] Fang, C. Y., Fang, M. L., *Uniqueness of meromorphic functions and differential polynomials*, Comput. Math. Appl. **44** (5–6) (2002), 607–617.

[3] Fang, M. L., Hong, W., *A unicity theorem for entire functions concerning differential polynomials*, Indian J. Pure Appl. Math. **32** (9) (2001), 1343–1348.

[4] Frank, G., Reinders, M., *A unique range set for meromorphic functions with 11 elements*, Complex Variables Theory Appl. **37** (9) (1998), 185–193.

[5] Hayman, W. K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[6] Lahiri, I., *Uniqueness of meromorphic functions when two linear differential polynomials share the same* 1*-points*, Ann. Polon. Math. **71** (2) (1999), 113–128.

[7] Lahiri, I., *Value distribution of certain differential polynomials*, Internat. J. Math. Math. Sci. **28** (2) (2001), 83–91.

[8] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161** (2001), 193–206.

[9] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Variables Theory Appl. **46** (2001), 241–253.

[10] Lahiri, I., Banerjee, A., *Weighted sharing of two sets*, Kyungpook Math. J. **46** (1) (2006), 79–87.

[11] Lahiri, I., Pal, R., *Nonlinear differential polynomials sharing* 1*-points*, Bull. Korean Math. Soc. **43** (1) (2006), 161–168.

[12] Lahiri, I., Sarkar, A., *Nonlinear differential polynomials sharing* 1*-points with weight two*, Chinese J. Contemp. Math. **25** (3) (2004), 325–334.

[13] Lin, S., Lin, W. C., *Uniqueness of meromorphic functions concerning weakly weighted sharing*, Kodai Math. J. **29** (2006), 269–280.

[14] Lin, W. C., *Uniqueness of differential polynomials and a problem of Lahiri*, Pure Appl. Math. **17** (2) (2001), 104–110, (in Chinese).

[15] Lin, W. C., Yi, H. X., *Uniqueness theorems for meromorphic function*, Indian J. Pure Appl. Math. **35** (2) (2004), 121–132.

[16] Lin, W. C., Yi, H. X., *Uniqueness theorems for meromorphic functions concerning fixed points*, Complex Variables Theory Appl. **49** (11) (2004), 793–806.

[17] Mohon'ko, A. Z., *On the Nevanlinna characteristics of some meromorphic functions. Theory of Functions*, Funct. Anal. Appl. **14** (1971), 83–87.

[18] Yang, C. C., *On deficiencies of differential polynomials II*, Math. Z. **125** (1972), 107–112.

[19] Yi, H. X., *Uniqueness of meromorphic functions and a question of C. C. Yang*, Complex Variables Theory Appl. **14** (14) (1990), 169–176.

[20] Yi, H. X., *On characteristic function of a meromorphic function and its derivative*, Indian J. Math. **33** (2) (1991), 119–133.

[21] Yi, H. X., *Meromorphic functions that share one or two values*, Complex Variables Theory Appl. **28** (1995), 1–11.

Department of Mathematics, Kalyani Government Engineering College
West Bengal 741235, India
E-mail: abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com