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ITERATIVE SOLUTION OF NONLINEAR EQUATIONS OF THE PSEUDO-MONOTONE TYPE IN BANACH SPACES

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Abstract. The weak convergence of the iterative generated by $J(u_{n+1} - u_n) = \tau(Fu_n - Ju_n)$, $n \geq 0$, $\left(0 < \tau = \min\left\{1, \frac{1}{\lambda}\right\}\right)$ to a coincidence point of the mappings $F, J\colon V \to V^\star$ is investigated, where V is a real reflexive Banach space and $V^\star$ its dual (assuming that $V^\star$ is strictly convex). The basic assumptions are that J is the duality mapping, $J - F$ is demiclosed at 0, coercive, potential and bounded and that there exists a non-negative real valued function $r(u, \eta)$ such that

$$\sup_{u,\eta \in V} \{r(u, \eta)\} = \lambda < \infty$$

$$r(u, \eta)\|J(u - \eta)\|_{V^\star} \geq \|(J - F)(u) - (J - F)(\eta)\|_{V^\star}\,, \quad \forall\, u, \eta \in V\,.$$

Furthermore, the case when V is a Hilbert space is given. An application of our results to filtration problems with limit gradient in a domain with semipermeable boundary is also provided.

1. Introduction

A map $\Phi\colon [0, \infty) \to [0, \infty)$ is said to be a *gauge function* if Φ is continuous and strictly increasing, $\Phi(0) = 0$, and $\lim_{t \to +\infty} \Phi(t) = +\infty$. Suppose V is a real Banach space with a strictly convex dual $V^\star$. A map $J\colon V \to V^\star$ is said to be a duality map with gauge function Φ if for each $u \in V$, $\langle Ju, u\rangle = \Phi(\|u\|_V)\|u\|_V$ and $\|Ju\|_{V^\star} = \Phi(\|u\|_V)$, where $\langle \cdot, \cdot\rangle$ denotes the duality relation between V and $V^\star$. It is well known that (see, e.g. [7]) if $V^\star$ is strictly convex, then J is single-valued and if $V^\star$ is uniformly convex and V is a reflexive Banach space, then J is uniformly continuous on bounded sets (see e.g. [5, Chapter 8]).

When $\Phi(t) = t$, J is called a *normalized duality map*. If V is a Hilbert space, then the normalized duality map J is the identity map I.

It is known (see, e.g. [7]) that $Ju = \Phi(\|u\|_V)u_0^\star$ where $u_0^\star \in V^\star$, $\|u_0^\star\|_{V^\star} = 1$ and $\langle u_0^\star, u_0\rangle = \|u_0\|_V = 1$ $(u_0 = \frac{u}{\|u\|_V}$, $u \neq 0)$.

We always use the symbols "$\to$" and "$\rightharpoonup$" to indicate strong and weak convergence, respectively.

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A map $F\colon V \to V^*$ is called *demiclosed at* 0 (see, e.g. [4]) if for any sequence $\{u_n\}_{n=0}^{\infty}$ in V the following implication holds: $u_n \rightharpoonup u$ and $Fu_n \to 0$ as $n \to \infty$ implies $u \in V$ and $Fu = 0$.

According to [2] and [11], the mapping $F\colon V \to V^\star$ is said to be *pseudo-monotone* if it is bounded and

$$u_n \rightharpoonup u \in V \quad \text{and} \quad \lim_{n\to\infty} \sup \langle Fu_n, u_n - u\rangle \le 0$$

imply

$$\langle Fu, u-\eta\rangle \;\le\; \lim_{n\to\infty} \inf \langle Fu_n, u_n - \eta\rangle \quad \text{for all} \quad \eta \in V\,.$$

Recall that a map $A\colon V \to V^\star$ is said to be *bounded Lipschitz continuous* (see, e.g. [3]) if

$$\|Au - A\eta\|_{V^\star} \le \mu(R)\Phi(\|u-\eta\|_V) \quad \forall\; u, \eta \in V\,,$$

where $R = \max\{\|u\|_V, \|\eta\|_V\}$, μ is nondecreasing function on $[0,\infty)$ and Φ is the gauge function.

An operator $A\colon V \to V^\star$ is said to be *coercive* (see, e.g. [7]) if

$$\langle Au, u\rangle \ge \rho(\|u\|_V)\|u\|_{V^\star}\,; \qquad \lim_{\xi\to+\infty} \rho(\xi) = +\infty\,.$$

According to [3] the mapping A is said to be *potential* if

$$\int_0^1 \big(\langle A(t(u+\eta)), u+\eta\rangle - \langle A(tu), u\rangle\big)\, dt = \int_0^1 \langle A(u+t\eta), \eta\rangle\, dt \quad \forall\; u, \eta \in V\,.$$

The main objective of this work is the construction and investigation of approximation methods for solving the nonlinear equation

$$(\star) \qquad Au = f$$

in Banach and Hilbert spaces, where A is a bounded Lipschitz continuous, potential coercive, pseudo monotone operator from V into $V^\star$ and $f \in V^\star$. The problem $(\star)$ arises in the description of steady-state filtration processes (see, e.g. [8]).

2. Main results

We now establish the main results of this section:

Theorem 1. *Let V be a real reflexive Banach space with a strictly convex dual space $V^\star$, and let $F, J\colon V \to V^\star$ (where J is the duality map) be two mappings. Suppose $J - F$ is demiclosed at* 0*, coercive, potential and bounded, and there exists a non-negative real valued function $r(u,\eta)$ such that*

$$(1) \qquad \sup_{u,\eta\in V} \{r(u,\eta)\} = \lambda < \infty$$

$$(2) \qquad r(u,\eta)\|J(u-\eta)\|_{V^\star} \ge \|(J-F)(u) - (J-F)(\eta)\|_{V^\star}\,, \quad \forall\; u, \eta \in V\,.$$

Then the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$(3) \qquad J(u_{n+1} - u_n) = \tau(Fu_n - Ju_n)\,, \quad n \ge 0\,,$$

where u_0 is a point in V and $0 < \tau = \min\left\{1, \frac{1}{\lambda}\right\}$, is bounded in V and all its weak limit points are elements of $\mathfrak{F} = \{u \in V : Fu = Ju\}$.

Proof. Let us first prove the boundedness of the iterative sequence; more precisely, let us show that

$$\{u_n\}_{n=0}^{\infty} \subset S_0\,, \quad \|u_n\|_V \le R_0\,, \quad n = 0, 1, 2, \dots, \tag{4}$$

where $R_0 = \sup_{u \in S_0} \|u\|_V$, $S_0 = \{u \in M \colon F_1(u) \le F_1(u_0)\}$, and $F_1 \colon V \to R \cup \{+\infty\}$ is a functional defined by the formula

$$F_1(u) = \int_0^1 \langle (J - F)(tu), u\rangle\, dt \quad \forall\, u \in V\,. \tag{5}$$

By definition, $u_0 \in S_0$. Let $u_n \in S_0$; we claim that $u_{n+1} \in S_0$.
Indeed, substituting $u = u_{n+1} + t(u_n - u_{n+1})$, $\eta = u_n$ in (2) and writing r for $r(u_{n+1}, u_n)$, we obtain

$$r\|J((t-1)(u_n - u_{n+1}))\|_{V^\star} \ge \|(J-F)(u_{n+1} + t(u_n - u_{n+1})) - (J-F)(u_n)\|_{V^\star}\,.$$

Using the definition of J, we get

$$\begin{aligned} r\Phi(\|u_n - u_{n+1}\|_V) &\ge r\Phi(\|(t-1)(u_n - u_{n+1})\|_V) \\ &\ge \|(J-F)(u_{n+1} + t(u_n - u_{n+1})) - (J-F)(u_n)\|_{V^\star}\,, \end{aligned} \tag{6}$$

for $t \in [0, 1]$. Consequently, it follows that

$$\begin{aligned} &|\langle (J-F)(u_{n+1} + t(u_n - u_{n+1})) - (J-F)(u_n), u_n - u_{n+1}\rangle| \\ &\qquad \le r\Phi(\|u_n - u_{n+1}\|_V)\|u_n - u_{n+1}\|_V\,. \end{aligned} \tag{7}$$

Or

$$\begin{aligned} &-|\langle (J-F)(u_{n+1} + t(u_n - u_{n+1})) - (J-F)(u_n), u_n - u_{n+1}\rangle| \\ &\qquad \ge -\,r\Phi(\|u_n - u_{n+1}\|_V)\|u_n - u_{n+1}\|_V\,. \end{aligned} \tag{8}$$

Further, following [3], from (5), we obtain

$$\begin{aligned} F_1(u_n) - F_1(u_{n+1}) &= \int_0^1 (\langle\langle (J-F)(t(u_n), u_n\rangle - \langle (J-F)(tu_{n+1}), u_{n+1}\rangle)\, dt \\ &= \int_0^1 \langle (J-F)(u_{n+1} + t(u_n - u_{n+1})), u_n - u_{n+1}\rangle)\, dt \\ &= \int_0^1 \langle (J-F)(u_{n+1} + t(u_n - u_{n+1})) \\ &\quad - (J-F)(u_n), u_n - u_{n+1}\rangle)\, dt + \langle (J-F)(u_n), u_n - u_{n+1}\rangle \\ &\ge -\int_0^1 \big|\langle (J-F)(u_{n+1} + t(u_n - u_{n+1})) \\ &\quad - (J-F)(u_n), u_n - u_{n+1}\rangle\big|\, dt + \langle (J-F)(u_n), u_n - u_{n+1}\rangle\,. \end{aligned}$$

This, together with [8] and (3), implies that

$$\begin{aligned} F_1(u_n) - F_1(u_{n+1}) &\geq -r\Phi(\|u_n - u_{n+1}\|_V)\|u_n - u_{n+1}\|_V \\ &\quad + \tau^{-1}\langle J(u_{n+1} - u_n), u_{n+1} - u_n\rangle \\ &\geq -\lambda\Phi(\|u_{n+1} - u_n\|_V)\|u_{n+1} - u_n\|_V \\ &\quad + \tau^{-1}\langle J(u_{n+1} - u_n), u_{n+1} - u_n\rangle \\ &= \mu\Phi(\|u_{n+1} - u_n\|_V)\|u_{n+1} - u_n\|_V\,, \quad \mu = (\tau^{-1} - \lambda) > 0\,. \end{aligned} \tag{9}$$

Therefore, $F_1(u_{n+1}) \leq F_1(u_n) \leq F_1(u_0)$, i.e., $u_{n+1} \in S_0$, which completes the proof of (4).

Since the iterative sequence is bounded and the operator $J - F$ is bounded, it follows from the definition of F_1 that $\{F_1(u_n)\}_{n=0}^{\infty}$ is a bounded sequence; by (9), it is monotone. Therefore, the numerical sequence $\{F_1(u_n)\}_{n=0}^{\infty}$ has a finite limit. Consequently, from (9), we obtain

$$\lim_{n\to+\infty} \mu\Phi(\|u_n - u_{n+1}\|_V)\|u_n - u_{n+1}\|_V = 0\,.$$

This, together with the continuity and the strictly monotone growth of Φ, implies that

$$\lim_{n\to+\infty} \|u_n - u_{n+1}\|_V = 0\,. \tag{10}$$

Using the definition of J again, it follows from (3) and (10) that

$$\lim_{n\to\infty} \|Ju_n - Fu_n\|_{V^*} = 0\,.$$

Since V is reflexive and $\{u_n\}_{n=0}^{\infty}$ is bounded, we find some subsequence $\{u_{n_j}\}_{j=0}^{\infty}$ of $\{u_n\}_{n=0}^{\infty}$ which converges weakly to some $u^\star \in V$. Moreover, $u^\star$ is a coincidence point of F and J, since $Ju_{n_j} - Fu_{n_j} \to 0$ and $J - F$ is demiclosed at 0. Hence $Ju^\star = Fu^\star$. This completes the proof. □

We close this section with the case when the space V is a Hilbert space

Theorem 2. *Let $V = H$ be a real Hilbert space, and let F be a self-mapping of H such that $I - F$ is demiclosed at* 0*, coercive, potential and bounded and there exists a nonnegative real-valued function $r(u, \eta)$ such that* (1) *holds,*

$$r(u,\eta)\|u - \eta\|_H \geq \|(I - F)(u) - (I - F)(\eta)\|_H\,, \quad \forall\ u,\ \eta \in H\,. \tag{11}$$

Then the sequence $\{u_n\}_{n=0}^{\infty}$ of Mann iterates (see, e.g. [9]*) defined by*

$$u_{n+1} = (1 - \tau)u_n + \tau F u_n\,, \quad n \geq 0\,,$$

where $0 < \tau = \min\{1, \frac{1}{\lambda}\}$, converges weakly to a fixed point of F.

Proof. By Theorem 1, it follows that there exists a subsequence $\{u_{n_j}\}_{j=0}^{\infty}$ of $\{u_n\}_{n=0}^{\infty}$ which converges weakly to a fixed point of F. The rest of the argument now follows exactly as in ([10, p.70]) to yield that $\{u_n\}_{n=0}^{\infty}$ converges weakly to a fixed point of F. □

3. An application to filtration problems with limit gradient in a domain with semipermeable boundary

In this section we apply our results to the stationary problem on the filtration of an incompressible fluid governed by a discontinuous filtration law with the limit gradient (see, e.g. [8]).

We consider the nonlinear stationary problem of filtration theory for the case of a discontinuous law with the limit gradient (see., e.g. [6])

$$\overrightarrow{v}(u) = -g(|\nabla u|^2)\nabla u\,,$$

where $\overrightarrow{v}(u)$ is the filtration velocity, u the pressure, $\nabla u = \mathrm{grad} u$, $g(\xi^2)\xi$ is the function describing the filtration law. Let Ω be a bounded domain in R^n, $n \geq 1$, with the Lipschitz continuous boundary Γ.

We assume that $g(\xi^2)\xi = g_0(\xi^2)\xi + g_1(\xi^2)\xi$, where $\xi \to g_i(\xi^2)\xi$, $i = 0, 1$, are nonnegative functions, equal to zero when $\xi \leq \beta$, ($\beta \geq 0$ is the limit gradient), $\xi \to g_0(\xi^2)\xi$ is continuous and strictly increasing when $\xi > \beta$,

$$c_1(\xi - \beta)^{p-1} \leq g_0(\xi^2)\xi \leq c_2(\xi - \beta)^{p-1} \tag{12}$$

when $\xi \geq \beta$, $p > 1$, c_1, $c_2 > 0$, and $g_1(\xi^2)\xi = \vartheta > 0$ for $\xi > \beta$. We also assume that

$$\frac{(g_0(\xi^2)\xi - g_0(\eta^2)\eta)}{(\xi - \eta)} \leq c_0(1 + \xi + \eta)^{p-2} \quad \text{for all} \quad \xi, \eta \in R \cup \{+\infty\}\,. \tag{13}$$

Following [6], we define the solution of stationary filtration problem with a discontinuous law as the function $u \in W_0^{1,p}(\Omega)$, which satisfies the nonlinear equation

$$Au = f\,,$$

where the operator $A\colon W_0^{1,p}(\Omega) \to W_q^{-1}(\Omega)$, $q = \frac{p}{p-1}$ is induced by the form

$$Au = -\mathrm{div}(g(\mid \nabla u \mid^2)\nabla u)\,,$$

and $f \in W_q^{-1}(\Omega)$ is the density of external sources.

It is known that the operator A is pseudo-monotone potential coercive (see, e.g. [8]).

The following lemma is proved in [1].

Lemma 1 (see [1])**.** *Let $V = W_0^{1,p}(\Omega)$, $p \geq 2$. Then A is bounded Lipschitz continuous with*

$$\mu(\xi) = c_3(1 + 2\xi)^{p(2-q)}\,, \quad c_3 > 0\,, \quad \text{and} \quad \Phi(\xi) = \xi\,.$$

Remark 1. If we set $p = q = 2$ in Lemma 1, the bounded Lipschitz continuous condition reduces to

$$\|Au - A\eta\|_{V^\star} \leq c_3\|u - \eta\|_V \quad \forall\ u,\ \eta \in V$$

which is exactly the condition of Lipschitz continuity of the operator A with constant $c_3 > 0$.

Theorem 3. *Let $V = W_0^{1,p}(\Omega)$, $p \geq 2$, $V^\star = W_q^{-1}(\Omega)$, $q = \frac{p}{p-1}$. Suppose $A\colon V \to V^\star$ is a bounded Lipschitz continuous pseudo-monotone potential coercive mapping. Then the sequence $\{u_n\}_{n=0}^\infty$ generated from a suitable $u_0 \in V$ by*

$$J(u_{n+1} - u_n) = \tau(f - Au_n)\,, \quad n \geq 0\,, \tag{14}$$

where $0 < \tau = \min\left\{1, \frac{1}{\lambda}\right\}$, $f \in V^\star$, with

$$\sup\{(1 + \|u\|_V + \|\ \eta\|_V)^{p(2-q)}\} = \lambda < \infty\,, \quad p \geq 2$$

is bounded in V and all its weak limit points are solutions of the equation

$$Au = f\,. \tag{15}$$

Proof. Note that (15) has at least one solution because of conditions on A (see, e.g. [7]). We apply Theorem 1 with $F\colon V \to V^\star$ defined by $Fu = Ju - Au + f$. If we set

$$\Omega_\eta^+ = \{x \in \Omega \mid \nabla\eta(x) \mid > \beta\}\,, \quad \Omega_\eta^- = \Omega/\Omega_\eta^+$$

and

$$\langle Ju, \eta\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\eta)\,dx\,.$$

Then taking into account (12) for all $u,\ \eta \in V$, we get

$$\begin{aligned}\langle A_0 u, \eta\rangle &= \int_\Omega [g_0(|\nabla u|^2)(\nabla u, \nabla\eta)]\,dx \\ &\leq c_2 \int_{\Omega_u^+} \frac{(|\nabla u| - \beta)^{p-1}}{|\nabla u|}(\nabla u, \nabla\eta)\,dx\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle A_0 u, \eta\rangle &\leq c_2 \int_{\Omega_u^+} ||\nabla u| - \beta|^{p-1}|\nabla\eta|\,dx \\ &\leq c_2[\int_{\Omega_u^+} ||\nabla u| - \beta|^p\,dx]^q \|\nabla\eta\|_V \\ &\leq c_2\|\nabla u\|_V^{p-1}\|\nabla\eta\|_V \\ \langle A_1 u, \eta\rangle &= \int_\Omega g_1(|\nabla u|^2)(\nabla u, \nabla\eta)\,dx \\ &\leq \vartheta\int_{\Omega_u^+} |\nabla\eta|\,dx \leq \vartheta\int_\Omega |\nabla\eta|\,dx = \vartheta\|\nabla\eta\|_V\,.\end{aligned}$$

Thus

$$\langle Au, \eta\rangle \leq [c_2\|\nabla u\|_V^{p-1} + \vartheta]\|\nabla\eta\|_V\,.$$

This implies

$$\begin{aligned}\langle Au - Ju, \eta\rangle &\le [c_2\|\nabla u\|_V^{p-1} + \vartheta]\|\nabla\eta\|_V + \beta^{p-1}\int_{\Omega_u^+} |\nabla\eta|\,dx \\ &\le [c_2\|\nabla u\|_V^{p-1} + \vartheta]\|\nabla\eta\|_V + \beta^{p-1}\int_{\Omega} |\nabla\eta|\,dx \\ &\le [c_2\|\nabla u\|_V^{p-1} + \vartheta + \beta^{p-1}]\|\nabla\eta\|_V\end{aligned}$$

Consequently,

$$\begin{aligned}\|Au - Ju\|_{V^\star} &= \sup_{\eta\neq 0}\frac{\langle Au - Ju, \eta\rangle}{\|\eta\|_V} \\ &\le [c_2\|\nabla u\|_V^{p-1} + \vartheta + \beta^{p-1}] \quad \forall\, u \in V\end{aligned}$$

which implies that

$$\begin{aligned}\|Fu\|_{V^\star} &\le \|Ju - Au\|_{V^*} + \|f\|_{V^\star} \\ &\le [c_2\|\nabla u\|_V^{p-1} + \vartheta + \beta^{p-1} + \|f\|_{V^\star}]\,.\end{aligned}$$

Now we are going to prove that condition (2) is satisfied. Since $p \ge 2$, it follows from Lemma 1 that

$$\begin{aligned}\|(J-F)u - (J-F)\eta\|_{V^\star} &= \|Au - A\eta\|_{V^\star} \\ &\le c_3(1 + \|u\|_V + \|\eta\|_V)^{p(2-q)}\|u-\eta\|_V\,, \quad \forall u,\ \eta \in V\,.\end{aligned}$$

Hence we see that condition (2) is satisfied for

$$r(u,v) = (1 + \|u\|_V + \|v\|_V) \quad \text{and} \quad \|J(u-\eta)\|_{V^\star} = c_3\|u-\eta\|_V\,.$$

Also from the pseudomonotonicity, coercivity, and the potentiality of A we obtain the boundedness, coercivity and the potentiality of $J - F$.

It remains to show that $J-F$ is demiclosed at 0. Let $\{u_{n_k}\}_{k=0}^{\infty}$ be a subsequence of $\{u_n\}_{n=0}^{\infty}$ such that $u_{n_k} \rightharpoonup u^\star$ and $\{Au_{n_k} - f\}_{k=0}^{\infty}$ converges strongly in V to zero. Suppose

(16) $$\lim_{k\to\infty}\sup\langle Au_{n_k}, u_{n_k} - u^\star\rangle \;\le\; 0\,.$$

Since A is pseudo-monotone, then

$$\lim_{k\to\infty}\inf\langle Au_{n_k}, u_{n_k} - \eta\rangle \ge \langle Au^\star, u^\star - \eta\rangle \quad \forall \eta \in V\,.$$

Or

(17) $$\lim_{k\to\infty}\sup\langle Au_{n_k}, \eta - u_{n_k}\rangle \le \langle Au^\star,\ \eta - u^\star)\rangle \quad \forall\, \eta \in V\,.$$

Now we prove that A satisfies condition (16).

Since $u_{n_k} \rightharpoonup u^\star$ in V, then it is bounded, consequently

$$\begin{aligned}\lim_{k\to+\infty}\sup\langle Au_{n_k} - f, u_{n_k} - u^\star\rangle &\le \lim_{k\to+\infty}\sup\|Au_{n_k} - f\|_{V^\star}\|u_{n_k} - u^\star\|_V \\ &\le \text{const}\lim_{k\to+\infty}\sup\|Au_{n_k} - f\|_{V^\star} = 0.\end{aligned}$$

Hence, from (16), we have

$$\langle Au^\star - f, \eta - u^\star\rangle \geq \lim_{k\to\infty} \sup\langle Au_{n_k} - f, \eta - u_{n_k}\rangle \quad \forall\, \eta \in V\,.$$

Analogically to the above argument, we get

$$\lim_{k\to\infty} \sup\langle Au_{n_k} - f, \eta - u_{n_k}\rangle = 0 \quad \forall\, \eta \in V\,,$$

that is $u^\star$ is the solution of the following variational inequality

$$\langle Au^\star - f, \eta - u^\star\rangle \ \geq 0 \quad \forall \eta \in V\,,$$

and consequently (see, e.g. [7]), $Au^\star - f = 0$. Therefore $J - F$ is demiclosed at 0 on V. □

An application of Theorem 1 now completes the proof of Theorem 3.

Remark 2. It follows from Remark 1 that relation (2) is satisfied with $r(u, v) = 1$ and

$$\|J(u-\eta)\|_{V^\star} = c_3\|u-\eta\|_V\,.$$

It is obvious that all conditions of Theorem 2 are satisfied. Therefore, the sequence $\{u_n\}_{n=0}^{\infty}$ generated by (14) converges weakly to a solution of (15).

Remark 3. It should be noted that at every step of the iterative process (14) it is necessary to solve the nonlinear problem

$$-\|w\|_V^{2-p}\mathrm{div}(|\nabla w|^{p-2}\nabla w) = \tau(f - Au_n)\,, \quad w = u_{n+1} - u_n \in V\,,\ p > 2$$

which, with the help of the substitution $w = \|w_1\|_V^{p-2} w_1$, reduces to the problem

$$-\mathrm{div}(|\nabla w_1|^{p-2}\nabla w_1) = \tau(f - Au_n)\,.$$

When $p = 2$, (14) reduces to solve

$$-\Delta w = \tau(f - Au_n)\,, \quad w = u_{n+1} - u_n \in V\,.$$

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