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ON METRICS OF POSITIVE RICCI CURVATURE CONFORMAL TO $M \times \mathbf{R}^m$

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Abstract. Let (M^n, g) be a closed Riemannian manifold and g_E the Euclidean metric. We show that for $m > 1$, $(M^n \times \mathbf{R}^m, (g + g_E))$ is not conformal to a positive Einstein manifold. Moreover, $(M^n \times \mathbf{R}^m, (g + g_E))$ is not conformal to a Riemannian manifold of positive Ricci curvature, through a radial, integrable, smooth function, $\varphi\colon \mathbf{R^m} \to \mathbf{R}^+$, for $m > 1$. These results are motivated by some recent questions on Yamabe constants.

1. Introduction

We study the existence of positive Einstein metrics and of metrics of positive Ricci curvature on the conformal class of the product manifold $(M^n \times \mathbf{R}^m, g + g_E)$, where g is a metric on a closed manifold with positive scalar curvature, g_E the Euclidean metric of $\mathbf{R}^m$, and $m > 1$. The case $m = 1$ was studied recently by A. Moroianu and L. Ornea [7], who have shown that when (M^n, g) is compact and Einstein, then $\left(M^n \times \mathbf{R}, g + dt^2\right)$ is conformal to a positive Einstein manifold, in which case the function depends only on t, and is of the form $\alpha^2 Cosh^{-2}(\beta t + \gamma)$, for some real constants α, β, γ.

Our first result shows that a conformal positive Einstein metric does not exist when $m > 1$.

Theorem 1. *Let (M^n, g) be a closed Riemannian manifold, and g_E the Euclidean metric of $\mathbf{R}^m$, with $m > 1$. Then $(M^n \times \mathbf{R}^m, g + g_E)$ is not conformal to a positive Einstein manifold.*

Tensorial obstructions to the existence of Riemannian metrics that are conformally Einstein have been studied recently. See for instance the articles of Listing, [5], [6], and of Gover and Nurowski, [3]. These obstructions work only under some non-degeneracy hypothesis on the Weyl tensor, which do not apply in our case.

As our second result we show that in the conformal class of $(M^n \times \mathbf{R}^m, \tilde{g})$ there is no metric of positive Ricci curvature, at least for radial functions of the factor $\mathbf{R}^m$.

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Theorem 2. *Let (M^n, g) be a closed Riemannian manifold of dimension n. Consider $(\mathbf{R}^m, g_E)$, with g_E the Euclidean metric of $\mathbf{R}^m$. Then, for $m > 1$, there is no radial, smooth, positive, integrable function $\varphi : \mathbf{R^m} \to \mathbf{R}^+$, such that,*

$$\big(M^n \times \mathbf{R}^m, \tilde{h}\big) = \big(M^n \times \mathbf{R}^m, \varphi(g + g_E)\big)$$

has positive Ricci curvature.

It seems reasonable to believe that this result should extend from a radial function of $\mathbf{R}^m$ to any conformal factor. The inequality $m > 1$ is sharp, by the already mentioned results of A. Moroianu and L. Ornea [7], showing that when (M^n, g) is a compact, positive Einstein manifold, then $(M^n \times \mathbf{R}, g + dt^2)$ is conformal to a positive Einstein manifold.

As we mentioned above, sets of Einstein metrics, or of positive Ricci curvature metrics, on conformal classes have been recently studied on different contexts. But our interest was originally motivated by some recent results about Yamabe constants.

Let us recall that the Yamabe constant of the conformal class of a Riemannian metric g on a closed manifold M is defined as

$$Y(M, [g]) = \inf_{\hat{g} \in [g]} \frac{\int_M S_{\hat{g}} d\mu_{\hat{g}}}{\big(\int_M d\mu_{\hat{g}}\big)^{\frac{n-2}{n}}} \tag{1}$$

where $S_{\hat{g}}$ and $d\mu_{\hat{g}}$ are the scalar curvature and the volume element corresponding to $\hat{g}$, respectively. The existence of a conformal positive Einstein metric was used, for instance, by Petean [9] to compute in some cases the Yamabe constant of $M \times \mathbf{R}$. Moreover, Akutagawa, Florit and Petean showed in [1] that if $S_g > 0$ then

$$\lim_{t \to \infty} Y\big(M^n \times N^m, [g + th]\big) = Y\big(M^n \times \mathbf{R}^m, [g + g_E]\big) . \tag{2}$$

One gets lower bounds on positive Yamabe constants with conditions on the Ricci curvature. By a theorem of Obata [8] an Einstein metric is the unique unit volume metric of constant scalar curvature in the conformal class. Moreover, S. Ilias proved in [4] that if $R_g \geq \lambda g$, with $\lambda > 0$, then

$$Y\big(M, [g]\big) \geq n\lambda\big(\,\mathrm{Vol}(M, g)\big)^{\frac{2}{n}} . \tag{3}$$

Therefore, existence of metrics of positive Ricci curvature on $M \times \mathbf{R}^m$ would have give a way to obtain lower bounds for the Yamabe constants of $M \times \mathbf{R}^m$, but Theorems 1 and 2 say that this is not the case.

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2. Notation and general formulas for changes of metric

Let (N, g) be a Riemannian manifold of dimension k. For a function φ on N, we denote $\Delta\varphi = -\operatorname{div}(\nabla\varphi)$ the Laplacian of φ, $\nabla\varphi$ the gradient of φ and $D^2\varphi$ the Hessian of φ, given by $D^2\varphi(X, Y) = X(Y\varphi) - (\nabla_X Y)\varphi$ for any X, Y vector fields on the manifold. We denote the Ricci curvature tensor of the metric g by R_g, the scalar curvature by S_g and the trace free part of the Ricci tensor by Z_g. We recall that Z_g is given by $Z_g = R_g - \frac{S_g}{k} g$.

Consider a conformal change of metric $\tilde{g} = \varphi^{-2} g$. The conformal transformation of the trace free part of the Ricci tensor, Z_g, under this conformal transformation of the metric is given by (cf. in [8, page 255]):

$$Z_{\tilde{g}} = Z_g + \frac{k-2}{\varphi}\Big(D^2\varphi + \frac{\Delta\varphi}{k} g\Big) \tag{4}$$

Likewise, the conformal transformation of the scalar curvature S_g under this conformal transformation of the metric is given by (cf. in [8, page 255]):

$$S_{\tilde{g}} = \varphi^2 S_g - 2(k-1)\varphi\Delta\varphi - k(k-1)|\nabla\varphi|^2 \tag{5}$$

In the proof of Theorem 2, it will be useful to choose the scaling factor in a different form in order to simplify the expressions. Under the conformal transformation of the metric, $\tilde{g} = e^{2\psi} g$, the conformal transformation of the Ricci tensor is given by (cf. ([2, page 59]):

$$R_{\tilde{g}} = R_g - (k-2)(D^2\psi - d\psi \otimes d\psi) + \big(\Delta\psi - (k-2)|\nabla\psi|^2\big) g \tag{6}$$

3. Proof of Theorem 1

Proof. Let (M^n, g) be a closed Riemannian manifold of dimension n, and let g_E denote the Euclidean metric of $\mathbf{R}^m$, $m > 1$. Let $h = g + g_E$.

We proceed by contradiction. Suppose we have a smooth, positive function $u\colon M \times \mathbf{R}^m \to \mathbf{R}^+$, such that $(M \times \mathbf{R}^m, u^{-2}h)$ is positive Einstein.

Let $\tilde{h} = u^{-2}h$. Since $(M \times \mathbf{R}^m, \tilde{h})$ is Einstein, we have from (4) that

$$0 = Z_h + \frac{n+m-2}{u}(D^2 u + \frac{\Delta u}{n+m} h).$$

Since $Z_h = R_h - \frac{S_h}{n+m} h$, it follows that

$$D^2 u = \frac{-u}{n+m-2} R_h + \Big(\frac{u S_h}{(n+m-2)(n+m)} - \frac{\Delta u}{n+m}\Big) h\,. \tag{7}$$

Let $\{\partial_1, \ldots, \partial_m\}$ be the usual global orthonormal frame for $T\mathbf{R}^m$ and let $X \in TM$. We will denote by $\tilde{X}$ a vector field on M extending the tangent vector X. From (7) we have

$$D^2 u(\partial_i, \tilde{X}) = D^2 u(\tilde{X}, \partial_i) = 0\,, \tag{8}$$

and therefore,

$$0 = D^2 u(\tilde{X}, \partial_i) = \partial_i(\tilde{X} u) - (\nabla_{\partial_i} \tilde{X}) u\,,$$

$$0 = D^2 u(\partial_i, \tilde{X}) = \tilde{X}(\partial_i u) - (\nabla_{\tilde{X}} \partial_i) u\,.$$

Note that $\nabla_{\partial_i}\tilde{X} = \nabla_{\tilde{X}}\partial_i = 0$, because h is a product metric. It follows that for any vector field $\tilde{X}$ on M,

$$\partial_i(\tilde{X}u)) = 0\,, \tag{9}$$

$$\tilde{X}(\partial_i u) = 0\,. \tag{10}$$

From (10), if we write $u = u(x,t)$, where $x \in M$ and $t \in \mathbf{R}^n$, for any $i = 1, \dots, m$, we have

$$\partial_i u(x,t) = \partial_i u(x_0, t)\,,$$

for all x, $x_0 \in M$. Therefore

$$u(x,t) - u(x_0,t) = w(x)\,,$$

for some smooth function w on M. That is, u is the sum of a function that depends only on M and a function that depends only on $\mathbf{R}^m$. We write

$$u(x,t) = v(t) + w(x)\,. \tag{11}$$

Then, since h is a Riemannian product, $\Delta_h u = \Delta_g w + \Delta_{g_E} v$, $|\nabla u|^2 = |\nabla_g w|^2 + |\nabla_{g_E} v|^2$.

It is also a consequence of (7) that

$$D^2u(\partial_i,\partial_j) = \Big(\frac{uS_h}{(n+m-2)(n+m)} - \frac{\Delta_g w + \Delta_{g_E} v}{n+m}\Big)\delta_{ij} \tag{12}$$

for any $i, j \leq m$.

And since

$$D^2u(\partial_i,\partial_j) = \partial_i(\partial_j u) - (\nabla_{\partial_i}\partial_j)u\,,$$

where the last term vanishes because ∂_i and ∂_j belong to the orthonormal frame of $T\mathbf{R}^m$ with the Euclidean metric, (12) can be rewritten as

$$D^2u(\partial_i,\partial_j) = \partial_i(\partial_j v) = \Big(\frac{uS_h}{(n+m-2)(n+m)} - \frac{\Delta_g w + \Delta_{g_E} v}{n+m}\Big)\delta_{ij} \tag{13}$$

for any $i, j \leq m$.

Now, given $\tilde{X} \in TM$, $D^2u(\tilde{X},\tilde{X}) = D^2w(\tilde{X},\tilde{X})$ depends only on M, so

$$\partial_i\big(D^2u(\tilde{X},\tilde{X})\big) = 0\,. \tag{14}$$

Also for any $i = 1, \dots, m$, and any $k = 1, \dots, m, i \neq k$,

$$\partial_i\big(D^2u(\partial_k,\partial_k)\big) = 0\,. \tag{15}$$

Since

$$\partial_i\big(D^2u(\partial_k,\partial_k)\big) = \partial_i\big(\partial_k(\partial_k u)\big) = \partial_k\big(\partial_i(\partial_k u)\big) = 0\,,$$

where the last equality follows from (13).

Now, let

$$p = \Big(\frac{uS_h}{(n+m-2)(n+m)} - \frac{\Delta u}{n+m}\Big)\,,$$

and let $i \in \{1, \dots, m\}$. Since $m > 1$, choose $k \leq m$, such that $k \neq i$. (15) and (12) imply that

$$\partial_i\big(D^2u(\partial_k,\partial_k)\big) = \partial_i p = 0\,. \tag{16}$$

To finish the proof we have to consider two cases: when g is Ricci flat and when it is not.

Case 1: (M,g) is not Ricci flat.

Since (M,g) is not Ricci flat, we choose some $\tilde{X} \in TM$ such that $R_g(\tilde{X},\tilde{X}) \neq 0$. Evaluating (7) in $\tilde{X}$ we have

$$D^2 w(\tilde{X},\tilde{X}) = \frac{-u}{n+m-2} R_h(\tilde{X},\tilde{X}) + p\, g(\tilde{X},\tilde{X}) .$$

Differentiating this equation by ∂_i, for any $i \leq m$, we have

$$\begin{aligned} 0 = \partial_i\big(D^2 u(\tilde{X},\tilde{X})\big) &= \partial_i\Big(\frac{-u}{n+m-2} R_h(\tilde{X},\tilde{X})\Big) + \partial_i\big(p\, h(\tilde{X},\tilde{X})\big) \\ &= \frac{-\partial_i u}{n+m-2}\, R_h(\tilde{X},\tilde{X}) , \end{aligned} \tag{17}$$

where the first equality follows from (14), and the last equality from the fact that $R_h(\tilde{X},\tilde{X})$ and $h(\tilde{X},\tilde{X})$ do not depend on $\mathbf{R}^m$, and neither does p, by (16). This implies that v is constant and then we can write $u = w$ as in (11). Then $D^2 u(\partial_k,\partial_k) = 0$, $\forall k \leq m$, and (12) imply that

$$S_h = \frac{n+m-2}{w} \Delta_g w . \tag{18}$$

On the other hand, since $(M \times \mathbf{R}^m, \tilde{h})$ is Einstein, $S_{\tilde{h}} = \lambda(n+m)$, where λ is the Einstein constant. Thus from (5) we have

$$S_h = \frac{\lambda(n+m)}{w^2} + 2(n+m-1)\frac{\Delta_g w}{w} + (n+m)(n+m-1)\frac{|\nabla_g w|^2}{w^2} . \tag{19}$$

Combining (18) and (19) yields

$$\lambda + w\Delta_g w + (n+m-1)|\nabla_g w|^2 = 0 . \tag{20}$$

Finally, we integrate (20) over M,

$$\begin{aligned} 0 &= \int_M \big(w\Delta_g w + (n+m-1)|\nabla_g w|^2 + \lambda\big)\, dV_g \\ &= \int_M \big((n+m)|\nabla_g w|^2 + \lambda\big)\, dV_g . \end{aligned}$$

This shows that λ cannot be positive (and if $\lambda = 0$ the function u has to be a constant).

Case 2: (M,g) is Ricci flat.

Since (M,g) is Ricci flat, it follows from (7) that

$$D_g^2 w = \frac{-\Delta_g w - \Delta_{g_E} v}{n+m} g , \tag{21}$$

$$D_{g_E}^2 v = \frac{-\Delta_g w - \Delta_{g_E} v}{n+m} g_E . \tag{22}$$

Taking the trace of (21) with respect to g we have that

$$-\Delta_g w = \frac{-\Delta_g w - \Delta_{g_E} v}{n+m} n\,,$$

it follows that

$$\frac{m}{n}\Delta_g w = \Delta_{g_E} v = c\,,$$

for some constant c, since $\Delta_g w$ depends only on M and $\Delta_{g_E} v$, only on $\mathbf{R}^m$.

It follows that $c = 0$ since, by Green's first identity,

$$0 = \int_M \Delta_g w dV_g = c \int_M dV_g\,,$$

and therefore w is constant.

Finally, since $\Delta_g w = \Delta_{g_E} v = 0$, it follows from (22) that

$$\partial_i(\partial_j v) = 0\,,$$

for all $i, j \leq m$. This implies that v is an affine function of $\mathbf{R}^m$ and since u is positive, v has to be constant. Clearly if u is constant $\tilde{h}$ is Ricci flat.
This finishes the proof of Theorem 1. □

4. Proof of Theorem 2

Proof. Let (M^n, g) be a complete Riemannian manifold and g_E the Euclidean metric of $\mathbf{R}^m$, $m > 1$. Let $h = g + g_E$. We proceed by contradiction. Suppose Theorem 2 is not true; and let $\varphi = \varphi(r)$, $r = \sqrt{\sum_i x_i^2}$, be a radial, positive, integrable, C^2 function, $\varphi\colon \mathbf{R}^n \to \mathbf{R}^+$, such that $(M^n, \varphi h)$ is Ricci positive. Let $f(r) = -\frac{1}{2}\operatorname{Log}\left[\varphi(r)\right]$, so that $\varphi(r) = e^{2(-f(r))}$.

Let $\{\partial_1, \ldots, \partial_m\}$ denote the usual global orthonormal frame for $\mathbf{R}^m$. Let $X, Y \in TM$. We will denote by $\tilde{X}$ and $\tilde{Y}$ vector fields on M extending the tangent vectors X and Y respectively. From (6) we have that

$$R_{\tilde{h}}(\tilde{X}, \tilde{Y}) = R_h(\tilde{X}, \tilde{Y}) + \left(-\Delta f - (n+m-2)|\nabla f|^2\right) g(\tilde{X}, \tilde{Y})\,, \tag{23}$$

$$\begin{aligned} R_{\tilde{h}}(\partial_i, \partial_j) &= (n+m-2)\big(D^2 f(\partial_i, \partial_j) + df \otimes df(\partial_i, \partial_j)\big) \\ &\quad + \big(- \Delta f - (n+m-2)|\nabla f|^2\big)\delta_{ij}\,, \end{aligned} \tag{24}$$

and

$$R_{\tilde{h}}(\partial_i, \tilde{X}) = 0\,.$$

For $R_{\tilde{h}}$ to be positive, it is thus necessary that both (23) and (24), be positive definite.

Let $f_i = \partial_i f$ and $\partial_j(\partial_k f) = f_{jk}$. As $f = f(r)$, we have,

$$f_j = \frac{f'}{r} x_j\,,$$

$$f_{jk} = \frac{rf'' - f'}{r^3} x_j x_k + \frac{f'}{r}\delta_{jk}\,,$$

where the prime denotes the derivative with respect to r.
Thus,

$$f_j f_k = \frac{f'^2}{r^2} x_j x_k \,,$$
$$\Delta f = -f'' - (m-1)\frac{f'}{r} \,,$$
$$|\nabla f|^2 = f'^2 \,.$$

It follows that for the 2-tensor on $\mathbf{R}^m$, given by (24), to be positive definite it is necessary that the 2-tensor $\alpha T + \beta Id_m$ is positive definite, where α, β are the functions given by

$$\alpha = (n+m-2)\frac{-f' + rf'' + f'^2 r}{r^3} \,,$$
$$\beta = f'' + (m-1)\frac{f'}{r} - (n+m-2)f'^2 + (n+m-2)\frac{f'}{r} \,,$$

and T is the 2-tensor given in the orthonormal coordinates by

$$T_{jk} = x_j x_k \,.$$

Thus, in order to have a positive definite Ricci tensor $R_{\tilde{h}}$, we need the eigenvalues of the 2-tensor $\alpha T + \beta Id_m$ to be positive.

Note that the eigenvalues of T are $\{0, \ldots, 0, r^2\}$ and therefore the eigenvalues of $\alpha T + \beta Id_m$ are $\{\beta, \ldots, \beta, \alpha r^2 + \beta\}$. Therefore, if $\tilde{h}$ has positive Ricci curvature, then f must satisfy

$$\alpha r^2 + \beta = (n+m-1)f'' + (m-1)\frac{f'}{r} > 0 \,, \tag{25}$$

and

$$\beta = f'' + (2m+n-3)\frac{f'}{r} - (m+n-2)f'^2 > 0 \,. \tag{26}$$

We now collect some immediate observations:
a) The function in the hypothesis, $\varphi = e^{-2f}$, is integrable, so it approaches zero as $r \to \infty$. As a consequence, we must have $f \to \infty$ as $r \to \infty$.

b) As f cannot have local maximums, by (25), it can only have one local minimum. So $f' = 0$ can occur at most only once; since f is radial and smooth, this can only occur at $r = 0$.

c) Since $f(r) \to \infty$ as $r \to \infty$ (by **a**) and $f'(r) \neq 0$ for $r > 0$, then $f'(r) > 0$ for $r > 0$.

Next, we obtain an upper bound for $f(r)$.

Consider (26). Let $p=(2m+n-3)$, $q=(m+n-2)$. Since $f'>0$, we have

$$\frac{f''}{f'}+\frac{p}{r}>qf'>0\,.$$

Then for any $a>0$ and $r>0$, we integrate from a to r to get

$$\operatorname{Log}\left(\frac{f'(r)}{f'(a)}\right)+\operatorname{Log}\left(\frac{r^p}{a^p}\right)>qf(r)-qf(a)>0\,.$$

Since the exponential function is increasing we have

$$f'(r)r^p>e^{qf(r)}\big(e^{-qf(a)}a^pf'(a)\big)>1>0\,.$$

And then,

$$f'(r)e^{-qf(r)}>\frac{C_1}{r^p}>0\,,$$

with $C_1=(e^{-qf(a)}a^pf'(a))>0$.
For $s>a$, we now integrate from s to r to obtain

$$-\frac{1}{q}e^{-qf(r)}+\frac{1}{q}e^{-qf(s)}>C_1\frac{1}{(1-p)}\Big(\frac{1}{r^{p-1}}-\frac{1}{s^{p-1}}\Big)>0\,.$$

Since this works for all $r>s>a$, the inequality is preserved in the limit as $r\to\infty$,

$$\frac{1}{q}e^{-qf(s)}\geq\frac{C_1}{(p-1)}\Big(\frac{1}{s^{p-1}}\Big)\geq 0\,,$$

since $\frac{1}{r^{p-1}}\to 0$ and $e^{-f(r)}\to 0$, as we observed earlier.

We then have an upper bound for $f(s)$, $s>a>0$.

$$f(s)<\operatorname{Log}\big[C_2s^{\frac{p-1}{q}}\big]=K_1+K_2\operatorname{Log}[s] \tag{27}$$

for some constants K_1, K_2.

We now obtain a lower bound for $f(r)$. Let $m_0=(m-1)/(n+m-1)$, we note that $0<m_0<1$.
By (25),

$$f''(r)+m_0\frac{f'(r)}{r}>0\,,$$

and since $f'(r)>0$ we have

$$\frac{m_0}{r}>-\frac{f''(r)}{f'(r)}\,.$$

We fix $r_0>0$ and pick $r_0<a<r$. Integrating from a to r the previous inequality we get

$$m_0\ \operatorname{Log}\left[\frac{r}{a}\right]>-\operatorname{Log}\left[\frac{f'(r)}{f'(a)}\right].$$

Since the exponential is increasing we have

$$\frac{r^{m_0}}{a^{m_0}}>\frac{f'(a)}{f'(r)}\,,$$

or,

$$f'(r) > \frac{f'(a)a^{m_0}}{r^{m_0}}\,.$$

We integrate again, now from $b > a$ to $r > b$, to get

$$f(r) - f(b) > \frac{f'(a)a^{m_0}}{(1-m_0)}(r^{1-m_0} - b^{1-m_0})\,.$$

Thus, there are positive constants c_1 and c_2, such that

$$f(r) > c_1 r^{1-m_0} + c_2\,. \tag{28}$$

This lower bound contradicts the upper bound obtained in (27), because

$$c_1 r^{\frac{n}{n+m-1}} + c_2 < f(r) < K_1 + K_2 \operatorname{Log}[r]\,,$$

does not hold as $r \to \infty$.

We conclude that a function $\varphi = e^{-2f}$ as in Theorem 2 cannot exist. □

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