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APPROXIMATE MAPS, FILTER MONAD, AND A REPRESENTATION OF LOCALIC MAPS

Bernhard Banaschewski and Aleš Pultr

Abstract. A covariant representation of the category of locales by approximate maps (mimicking a natural representation of continuous maps between spaces in which one approximates points by small open sets) is constructed. It is shown that it can be given a Kleisli shape, as a part of a more general Kleisli representation of meet preserving maps. Also, we present the spectrum adjunction in this approximation setting.

Introduction

In the point-free topology one represents a classsical topological space X, as a rule, as the lattice (frame) $\mathfrak{O}(X)$ of its open sets, and a continuous map $f\colon X \to Y$ as the frame homomorphism $\mathfrak{O}(f) = (U \mapsto f^{-1}[U])\colon \mathfrak{O}(Y) \to \mathfrak{O}(X)$. This (contravariant) representation is satisfactory in the sense that for a broad class of spaces (the *sober* ones, including e.g. all the Hausdorff spaces, or most of the Scott spaces) $f \mapsto \mathfrak{O}(f)$ is a one-one correspondence between all the continuous maps $f\colon X \to Y$ and all the frame homomorphisms $h\colon \mathfrak{O}(Y) \to \mathfrak{O}(X)$. The drawback is the contravariance, which is often faced formally by simply taking the opposite category of the category of frames (the category of locales). If one wishes to have the localic morphisms represented as maps, one can do so by taking the right Galois adjoints of frame homomorphisms. This has turned out to be useful in particular in gaining insight into the structure of sublocales, but not only in that (see [12, 13]). But still we may wish to have a representation mimicking what is actually happening with (approximated) points in spaces. Such has been presented in [2], albeit heavily dependent on a uniform enrichment of the structure. Here we approach this point of view in the context of mere frames.

The lattice $\mathfrak{O}(X)$ can be viewed as the system of feasible places; points, entities with position but no extent, may be seen as approximated by their open neighbourhoods, preferably very small (one can pinpoint a point by the system of all of its open neighbourhoods; this idea is very old, going back at least as far as Caratheodory [3] - note that this paper even preceded Hausdorff [6] initiating modern

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topology). Now if such a representation $U \ni x$ is very small and if $f\colon X \to Y$ is continuous then $f[U]$ is a very small set containing $f(x)$. Typically it is not open, but it can be represented in $\mathfrak{O}(Y)$ by small $V \supseteq f[U]$; these possible representations constitute a filter $f^\circ(U)$ in the frame $\mathfrak{O}(Y)$. Thus we obtain a *covariant* representation of continuous maps $f\colon X \to Y$ by specific mappings $f^\circ\colon \mathfrak{O}(X) \to \mathsf{Flt}\mathfrak{O}(Y)$ which can be then viewed as approximate maps $\mathfrak{O}(X) \overset{\bullet}{\longrightarrow} \mathfrak{O}(Y)$ (see Section 3) or special Kleisli morphisms $\mathfrak{O}(X) \multimap \mathfrak{O}(Y)$.

Note that the relation of the frame homomorphism $\mathfrak{O}(f)\colon \mathfrak{O}(Y) \to \mathfrak{O}(X)$ with the original $f\colon X \to Y$, and with the approximate extension $f^\circ\colon \mathfrak{O}(X) \to \mathsf{Flt}\mathfrak{O}(Y)$ is basically the same, namely taking preimages: the natural preimage of V under f° is

$$
\begin{aligned}
(f^\circ)^{-1}\langle V\rangle &= \bigcup\{U \mid \exists W \in f^\circ(U),\ W \subseteq V\}\\
&= \bigcup\{U \mid V \in f^\circ(U)\} = \bigcup\{U \mid f[U] \subseteq V\} \qquad \text{(PREIM)}\\
&= \bigcup\{U \mid U \subseteq f^{-1}[V]\} = f^{-1}[V]\,.
\end{aligned}
$$

In this article we extend such representation to the general context of frames; thus we also obtain an intuitively satisfactory representation of localic morphisms as approximate maps resp. Kleisli morphisms.

1. Preliminaries

1.1. Posets. In a partially ordered set $(X, \leq)$ the standard notation such as $\uparrow M$ for the subset $\{x \mid x \geq m,\ m \in M\}$ and $\uparrow a = \uparrow\{a\}$ will be used. Similarly, the standard concepts like that of a filter (proper or not) will be used without further explaination.

Our posets will be mostly complete lattices, more often then not distributive.

1.1.1. Recall that a filter F in a lattice L is *prime* if $a \vee b \in F \quad \Rightarrow \quad (a \in F$ or $b \in F)$. It is *completely prime* resp. *α-prime* if

$$\bigvee_{i\in J} a_i \in F \quad (\text{resp. “\ldots and } |J| < \alpha\text{”}) \quad \Rightarrow \quad \exists j,\ a_j \in F\,.$$

1.2. A *frame* is a complete lattice L satisfying the distributivity law

$$a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* $h\colon L \to M$ preserves arbitrary joins (including the bottom 0) and all *finitary* meets (including the top 1). As usual, the resulting category will be denoted by

$$\mathbf{Frm}\,.$$

If X is a topological space we have the frame $\mathfrak{O}X$ of its open sets, and if $f\colon L \to M$ is a continuous map then $\mathfrak{O}f = (U \mapsto f^{-1}[U])\colon \mathfrak{O}Y \to \mathfrak{O}X$ is a frame homomorphism.

The dual category of **Frm** is called the *category of locales* and denoted by **Loc**. Thus, the correspondence $\mathfrak{O}$ can be viewed as a (covariant) functor $\mathfrak{O}\colon \mathbf{Top} \to \mathbf{Loc}$. The morphisms of **Loc** are referred to as *localic morphisms* or *localic maps*.

For more about frames see, e.g., [7] or [15].

1.2.1. Convention. By abuse of language we will sometimes speak of *frame homomorphisms* $h\colon L \to M$, preserving arbitrary joins and finite meets, if L and M are general complete lattices.

1.3. We will use standard concepts of general topology (such as in, e.g., [9])); since we will deal with phenomena relevant in point-free topology, we will consider T_0-spaces only.

1.3.1. For a point x of a topological space we will set

$$\mathcal{U}(x) = \{U \in \mathfrak{O}(X) \mid x \in U\}\,.$$

Note that $\mathcal{U}(x)$ is a completely prime filter in $\mathfrak{O}(X)$.

1.3.2. A space X is *sober* (see, e.g., [5],[7]) if (it is T_0 and) each meet irreducible $U \in \mathfrak{O}(X)$ (that is, such U in $\mathfrak{O}(X)$ that if $U = U_1 \cap U_2$ then $U = U_i$ for some i) is of the form $X \smallsetminus \overline{\{x\}}$.

Equivalently, X is sober if there are no completely prime filters in $\mathfrak{O}(X)$ but the $\mathcal{U}(x)$.

1.4. For standard images and preimages of subsets under mappings we will consistently use square brackets, as in $f[A]$ or $f^{-1}[B]$, to avoid confusion with values $f(x)$, but in particular with the formal preimage $f^{-1}\langle B\rangle$ (Introduction, 4.2).

1.5. From category theory we will use the standard facts as e.g. in the opening chapters of [10], and the basic facts on monads (see 2.2 below).

2. Approximate maps. Monads and Kleisli morphisms

2.1. A *set with aproximate equality* (briefly, *apeset*) is a pair $A = (X_A, \overset{A}{\doteq})$ consisting of a set X_A and a reflective symmetric relation $\overset{A}{\doteq}$ on X_A. If there is no danger of confusion we will write $\doteq$ for $\overset{A}{\doteq}$.

Note. Think of a metric space, a fixed $\varepsilon > 0$ and a precision given by $x \doteq y$ if $\rho(x, y) < \varepsilon$. Or (and this will be the case in which we are particularly interested) take a set of approximations of some entities and $x \doteq y$ if x, y have a common refinement (if they are able to approximate the same entity): e.g. (small) open intervals representing real numbers with $x \doteq y$ amounting to $x \cap y \neq \emptyset$.

2.2. An *approximate map* (briefly, *a-map*) $f\colon A \overset{\bullet}{\longrightarrow} B$ is a relation $f \subseteq X_A \times X_B$ such that

(A1) for each $x \in X_A$ there is a $y \in X_B$ such that $(x, y) \in f$, and

(A2) if $x_1 \overset{A}{\doteq} x_2$ and $(x_i, y_i) \in f$ then $y_1 \overset{B}{\doteq} y_2$.

2.2.1. Notes. 1. This definition is obtained from the standard definition of a mapping by replacing the equality by approximate equalities.

2. The reader may wonder about the following aspect of the definition. The condition (A2) suggests a sort of continuity: if x_1 is very close to x_2 then the

respective values y_1, y_2 (defined up to he given precision) are very close as well. The point is that in this perspective a standard discontinuous map appears as a multivalued one (take for instance the $f(x)$ defined as 0 for $x \leq 0$ and as 1 for $x > 0$ then in the argument "approximately 0" the values are both 0 and 1, not even approximately equal).

2.2.2. Obviously the identical map $X_A \to X_A$ is an approximate map $A \stackrel{\bullet}{\longrightarrow} A$, and a composition of a-maps (as relations) is an a-map again. Thus, apesets and a-maps constitute a category.

2.2.3. Although we do not wish to think of an a-map as a multivalued map we will write for $f\colon A \stackrel{\bullet}{\longrightarrow} B$

$$f(x) = \{y \mid (x,y) \in f\}\,.$$

Thus represented, the approximate map appears as a mapping $f\colon X_A \to \mathfrak{P}(X_B)$; in the sequel such maps will be naturally structured.

2.3. Kleisli maps. A monad $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathbf{C}$ consists of a functor $T\colon \mathbf{C} \to \mathbf{C}$ and natural transformations $\eta\colon \mathrm{Id} \to T$ and $\mu\colon TT \to T$ such that $\mu \cdot \eta T = \mu \cdot T\eta = \mathrm{id}$ and $\mu \cdot \mu T = \mu \cdot T\mu$ (see e.g. [10]). In the equivalent Manes representation ([11]) one has a mapping $T\colon \mathrm{obj}\mathbf{C} \to \mathrm{obj}\mathbf{C}$, a system of morphisms $\eta_A\colon A \to TA$ and a lifting

$$f\colon A \to TB \quad \mapsto \quad \widetilde{f}\colon TA \to TB$$

satisfying

(1) $\widetilde{\eta_A} = \mathrm{id}_{TA}$,

(2) $\widetilde{f}\eta_a = f$, and

(3) $\widetilde{\widetilde{g}f} = \widetilde{g}\widetilde{f}$.

(The monad in the previous sense is then obtained by setting $Tf = \widetilde{\eta_B f}$ for $f\colon A \to B$, and $\mu_A = \widetilde{\mathrm{id}_{TA}}$.)

With a monad one has associated two canonical categories: the category $\mathbf{C}^{\mathbb{T}}$ of Eilenberg-Moore algebras, and the Kleisli category $\mathbf{C}_{\mathbb{T}}$ (see, e.g., [10]). In the sequel we will use the latter. It is as follows.

- The objects are those of $\mathbf{C}$,
- the morphisms $f\colon A \multimap B$ in $\mathbf{C}_{\mathbb{T}}$ are the morphisms $f\colon A \to TB$ from $\mathbf{C}$,
- and one has the composition of $f\colon A \multimap B$ and $g\colon B \multimap C$ defined by

$$g \circ f = \mu_c \cdot Tg \cdot f \quad (= \widetilde{g} \cdot f)\,.$$

Note that the $\eta_A\colon A \to TA$, as $\eta_A\colon A \multimap A$, play the role of the units.

We will speak of the $f\colon A \multimap B$ as the Kleisli morphisms, or Kleisli maps.

3. Approximate maps in frames. The filter monads

3.1. For a frame L (more generally, for a complete lattice) set

$$L^{\bullet} = L \smallsetminus \{0\}$$

and on $L^\bullet$ define an approximate equality by

$$a \overset{L}{=} b \quad \text{iff} \quad a \wedge b \neq 0\,.$$

3.1.1. Notes. 1. Thus $1 \overset{L}{=} a$ for any $a \in L$ which is counterintuitive. But consider open sets in a topological space as approximations of points, the smaller they are (whatever sense one gives to the "smallness") the better. If two such U, V can approximate the same point they meet, and if they are (small enough to be) satisfactory this make them close indeed; if at least one of the approximations is bad then their approximate equality is unsatisfactory as well.

2. More generally, suppose one has approximations of some entities modelled as a poset $(X, \leq)$ with $x \leq y$ interpreted as "x is a finer approximation then y" (of whatever one approximates). Then one has $x \doteq y$ defined by the existence of a common refinement $z \leq x$, y ("x and y are able to approximate the same entity").

3.2. For a frame homomorphism $h\colon M \to L$ define

$$h^\bullet : L^\bullet \overset{\bullet}{\longrightarrow} M^\bullet$$

by setting

$$(a, b) \in h^\bullet \quad \text{iff} \quad a \leq h(b)\,.$$

($h^\bullet$ is indeed an approximate map: $(a, 1) \in h^\bullet$ for any a, and if $a_1 \overset{L}{=} a_2$ and $(a_i, b_i) \in h^\bullet$ then $a_i \leq h(b_i)$ and hence $0 \neq a_1 \wedge a_2 \leq h(b_1 \wedge b_2)$, and $b_1 \wedge b_2 \neq 0$, that is, $b_1 \overset{M}{=} b_2$. – Note that this holds, more generally for any h preserving $\wedge$ and 0.)

Obviously the correspondence $h \mapsto h^\bullet$ is (contravariantly) functorial, and if $h \neq g$ then $h^\bullet \neq g^\bullet$ (if $a = g(b) \nleq h(b)$ we have $a \neq 0$ and $(a, b) \in g^\bullet$ while $(a, b) \notin h^\bullet$). Thus, the approximate maps $h^\bullet : L^\bullet \overset{\bullet}{\longrightarrow} M^\bullet$ can be viewed as representatives of the localic morphisms $L \to M$.

3.3. In the convention of 2.2.3 we have

$$h^\bullet(a) = \{b \mid a \leq h(b)\}\,.$$

Obviously $h^\bullet(a)$ is a proper filter in M.

To avoid repeated clumsy exclusions of zero we will work with the entire frames, using the obvious extension

$$h^\bullet(0) = M \quad (= 0_{\mathsf{Flt}_M})\,.$$

We have

3.3.1. Observation. *For a $\wedge$-homomorphism we have $h^\bullet(\bigvee_{i \in J} a_i) = \bigcap_{i \in J} h^\bullet(a_i)$.*

(Indeed, $b \in h^\bullet(a_i)$ for all $i \in J$ iff $\forall i \in J$, $a_i \leq h(b)$ iff $\bigvee a_i \leq h(b)$.)

3.4. The categories we will use, and the filter monads. The basic category will be the category

$$\mathcal{A}$$

of complete distributive lattices with suprema preserving mappings. Then we will consider

$$\mathcal{A}^\circ$$

the subcategory of $\mathcal{A}$ given by the morpisms that preserve all suprema and, furthermore, reflect zero, that is,

$$f(a) = 0 \quad \text{implies} \quad a = 0\,.$$

Finally define categories

$$\mathcal{B} \quad \text{resp.} \quad \mathcal{B}(\alpha) \quad (\alpha \text{ a regular cardinal})$$

as follows:

- the objects are pairs (L, A) with L an object of $\mathcal{A}$ and A a subset of L, and
- the morphisms $f\colon (L, A) \to (M, B)$ are morphisms $f\colon L \to M$ from $\mathcal{A}$ reflecting joins resp. joins smaller than α, in the sense that

  whenever $f(x) \le \bigvee_{i\in J} b_i$ for $b_i \in B$, in the latter case with $|J| < \alpha$, we have $x \le \bigvee_{i\in J} a_i$ with $a_i \in A$ and $f(a_i) \le b_i$ for all i.

Note that because of the void J one has in particular that each morphism in $\mathcal{B}(\alpha)$ is in $\mathcal{A}^\circ$.

For $L \in \mathcal{A}$ set

$$\mathsf{Flt}(L) = \big(\{F \subseteq L \mid F \text{ a filter}\}, \supseteq\big)$$

(note that it is ordered by the *inverse* inclusion, and that it is a complete lattice since intersections of filter are filters) and consider

$$\eta_L\colon (a \to \uparrow a)\colon L \to \mathsf{Flt}L\,.$$

For a morphism $f\colon L \to \mathsf{Flt}M$ in $\mathcal{A}$ define

$$\widetilde{f}\colon \mathsf{Flt}L \to \mathsf{Flt}M \quad \text{by} \quad \widetilde{f}(F) = \bigcup\{f(a) \mid a \in F\}\,.$$

The same formulas can be used in $\mathcal{A}^\circ$ ($\eta(a) = L = \uparrow 0$ yields $a = 0$ and if $\widetilde{f}(F) \ni 0$ there is an $a \in F$ such that $0 \in f(a)$, and hence $a = 0$).

Furthermore, in the context of $\mathcal{B}(\alpha)$ we will set

$$\mathsf{Flt}(L, A) = \big(\mathsf{Flt}L, \eta_L[A]\big) = \big(\mathsf{Flt}L, \{\uparrow a \mid a \in A\}\big)$$

and take the same formulas for η and $\widetilde{f}$ as before. This is correct:

if $\eta(a) = \uparrow a \le \bigvee_i \uparrow a_i$, $a_i \in A$, we have $\uparrow a \supseteq \bigcap \uparrow a_i = \uparrow(\bigvee a_i)$ and hence $a \le \bigvee a_i$;

if $\widetilde{f}(F) \le \bigvee_i \uparrow b_i$, $b_i \in B$, that is, $\widetilde{f}(F) \supseteq \bigcap \uparrow b_i = \uparrow(\bigvee a_i)$ we have $\bigvee b_i \in f(a)$ for some $a \in F$; then $f(a) \le \bigvee \uparrow b_i$ and since f is a morphism in $\mathcal{B}(\alpha)$ we have $a \le \bigvee a_i$ with $a_i \in A$ and $f(a_i) \le \uparrow b_i$; since $a \in F$ we can conclude that $F \supseteq \uparrow a$, that is, $F \le \uparrow a \le \bigvee \uparrow a_i$.

Finally set

$$\mathbb{F} = \big(\mathsf{Flt}, \eta, \widetilde{(-)}\big)$$

(it will be always obvious in which of the categories we are).

3.4.1. Note. Our category $\mathcal{A}$ is a full subcategory of the well-known category of *sup-lattices* ([8]). One might wish to use just the full subcategory generated by the frames, but that would not work. We need a category inhabited also by the filter lattices, and $\mathsf{Flt}L$ (with the inverse inclusion order, but this is necessary because of the η) is a co-frame but not a frame. In fact $\mathsf{Flt}L$ is typically not even pseudocomplemented. Take $L = \mathfrak{O}(X)$ with X a regular T_1-space that is not discrete, an $x \in X$ that is not isolated, and the filter $\mathcal{U}(x) = \{U \mid x \in U\}$. For any neighbourhood U of x the meet of $\mathcal{U}(x)$ and $\uparrow\{X \smallsetminus \overline{U}\}$ in $(\mathsf{Flt}L, \supseteq)$, that is, $\mathcal{U}(x) \vee \uparrow\{X \smallsetminus \overline{U}\}$, is the zero of $(\mathsf{Flt}L, \supseteq)$ ($\equiv L$, since it contains $\emptyset$). Then $\mathcal{U}(x)$ has no pseudocomplement: if F were such we had $F \subseteq\uparrow\{X \smallsetminus \overline{U}\}$ for all $U \ni x$. For $V \in F$, $V \supseteq X \smallsetminus \overline{U}$ for any $U \ni x$ and hence $V \supseteq X \smallsetminus \{x\}$ while $V \cap U \neq \emptyset$ for any $U \in \mathcal{U}(x)$ since x is not isolated.

3.4.2. Proposition. $\mathbb{F}$ *is a monad in any of the categories* $\mathcal{A}$, $\mathcal{A}^\circ$, $\mathcal{B}$ *or* $\mathcal{B}(\alpha)$.

Proof. Obviously any $\widetilde{f}(F)$ is a filter. We have to prove that for any system of filters F_i, $i \in J$,

$$(*) \qquad \widetilde{f}\big(\sup\{F_i \mid i \in J\}\big) = \widetilde{f}\Big(\bigcap_{i\in J} F_i\Big) = \bigcap_{i \in J} \widetilde{f}(F_i) = \sup\{\widetilde{f}(F_i) \mid i \in J\}\,.$$

Since obviously $F \subseteq G$ implies $\widetilde{f}(F) \subseteq \widetilde{f}(G)$ we have $\widetilde{f}(\bigcap_{i\in J} F_i) \subseteq \bigcap_{i\in J} \widetilde{f}(F_i)$. On the other hand, if $x \in \bigcap_{i\in J} \widetilde{f}(F_i)$ we have $x \in \widetilde{f}(F_i)$ for all i and there exist $a_i \in F_i$ with $x \in f(a_i)$. Thus, $x \in \bigcap_{i \in J} f(a_i) = f(\bigvee_{i\in J} a_i)$. Now $\bigvee a_i \in F_i$, and consequently $x \in \bigcap F_i$, and $(*)$ is proved.

Further, $\eta(\bigvee a_i) =\uparrow(\bigvee a_i) = \bigcap(\uparrow a_i) = \bigvee \eta(a_i)$, $\widetilde{\eta}_L(F) = \bigcup\{\uparrow a \mid a \in F\} = F$ and $\widetilde{f}\eta_L(a) = \bigcup\{f(b) \mid b \geq a\} = f(a)$ (as $b \geq a \;\Rightarrow\; f(b) \leq f(a)$).

Finally $x \in (\widetilde{g} \cdot \widetilde{f})(F)$ iff $\exists b \in \widetilde{f}(F)$ with $x \in g(x)$, that is, iff

$$(**) \qquad \exists a \in F \exists b \in f(a),\ x \in g(b)\,,$$

and also $x \in (\widetilde{\widetilde{g}f})(F)$ iff $\exists a \in F,\ x \in \widetilde{g}(f(a))$ iff $(**)$. $\square$

3.4.3. By 3.3.1 we have

Observation. *The approximate maps* $h^\bullet : L^\bullet \overset{\bullet}{\longrightarrow} M^\bullet$ *are morphisms* $h^\bullet : L \multimap M$ *in* $\mathcal{A}_{\mathbb{F}}$.

4. Dual representations

4.1. Besides the category of frames we will be interested in the categories of complete α-frames (where the distributivity is assumed for joins of less than α summands), in particular also in complete distributive lattices (that is, ω_0-frames), and in the categories

$$\mathbf{CLat}(\wedge) \quad \text{resp.} \quad \mathbf{CLat}(\wedge, 0)$$

of complete lattices with $\wedge$-homomorphisms resp. with $\wedge$ homomorphisms preserving 0.

4.2. Preimage of an a-map. Recall the observation (PREM) in the Introduction. More generally we will set for any $f\colon L \multimap M$ in $\mathcal{A}_{\mathbb{F}}$ (that is, $f\colon L \to \mathsf{Flt}M$ resp. $f\colon L^\bullet \overset{\bullet}{\longrightarrow} M^\bullet$)

$$f^{-1}\langle b\rangle = \bigvee\{a \in L \mid b \in f(a)\}\,.$$

4.2.1. Lemma. $a \leq f^{-1}\langle b\rangle$ *iff* $b \in f(a)$.

Proof. $\Leftarrow$ is trivial.

$\Rightarrow$: Let $a \le f^{-1}\langle b\rangle = \bigvee\{c \mid b \in f(c)\}$. Then by 3.3.1,

$$f(a) \supseteq f\big(\bigvee\{c \mid b \in f(c)\}\big) = \bigcap\{f(c) \mid b \in f(c)\} \ni b\,.$$

□

4.2.2. Proposition. *The mapping*

$$f^{-1}\langle - \rangle \colon M \to L$$

preserves finite meets.

Proof. Set $a = f^{-1}\langle b_1\rangle \wedge f^{-1}\langle b_2\rangle$. Then $a \le f^{-1}\langle b_i\rangle$, $i = 1, 2$, and by 4.2.1, $b_1, b_2 \in f(a)$ Since $f(a)$ is a filter we have $b_1 \wedge b_2 \in f(a)$ and, again by 4.2.1, $f^{-1}\langle b_1\rangle \wedge f^{-1}\langle b_2\rangle = a \le f^{-1}\langle b_1 \wedge b_2\rangle$. The other inequality is trivial. □

4.3. Recall the correspondence from the definition in 3.2 (and 3.4.3)

$$h \colon M \to L \text{ in } \mathbf{CLat}(\wedge) \quad \mapsto \quad h^{\bullet} : L \multimap M \ \ (\text{in } \mathcal{A}_{\mathbb{F}})\,.$$

Theorem. *The formulas $h \mapsto h^{\bullet}$ and $f \mapsto f^{-1}\langle - \rangle$ are mutually inverse and constitute two dual equivalences*

$$\mathbf{CLat}(\wedge) \ \cong^{op} \ \mathcal{A}_{\mathbb{F}} \quad \textit{and} \quad \mathbf{CLat}(\wedge, 0) \ \cong^{op} \ \mathcal{A}^{\circ}_{\mathbb{F}}\,.$$

Proof. Set $f = h^{\bullet}$. Then $f^{-1}\langle b\rangle = \bigvee\{a \mid b \in f(a)\} = \bigvee\{a \mid a \le h(b)\} = h(b)$. Thus, $f^{-1}\langle - \rangle = h$.

For $f \colon L \multimap M$ set $h = f^{-1}\langle - \rangle$. By 4.2.1, $a \le h(b)$ iff $b \in f(a)$. But by the definition of $h^{\bullet}$ we also have $a \le h(b)$ iff $b \in h^{\bullet}(a)$.

Now for the latter. If f reflects 0 then

$$f^{-1}\langle 0\rangle = \bigvee\{a \in L \mid 0 \in f(a)\} = 0$$

since $0 \in f(a)$ only if $a = 0$.

If $h(0) = 0$ and $h^{\bullet}(a) = L$ then $0 \in h^{\bullet}(a)$ and $a \le h(0) = 0$. □

4.4. Recall from the introduction the approximate extension

$$\varphi^{\circ} \colon \mathfrak{O}(X) \to \mathsf{Flt}\,\mathfrak{O}(Y) \quad (\text{that is, } \mathfrak{O}(X) \xrightarrow{\bullet} \mathfrak{O}(Y))$$

defined by

$$V \in \varphi^{\circ}(U) \quad \text{iff} \quad \varphi[U] \subseteq V \quad (\text{iff} \quad U \subseteq \varphi^{-1}[V])\,.$$

The filters $\varphi^{\circ}(U)$ are (of course) not completely prime, but as a collection they have a sort of "completely prime behaviour". Namely,

If $\bigcup V_i \in \varphi^{\circ}(U)$ we have $U \subseteq \bigcup \varphi^{-1}[V_i]$ and hence, if we set $U_i = U \cap \varphi^{-1}[V_i]$, we have $U = \bigcup U_i$ and $V_i \in \varphi^{\circ}(U_i)$.

This leads to the following definition. An a-map $f \colon L \multimap M$ (Kleisli map $f \colon L \to \mathsf{Flt}M$ from $\mathcal{A}_{\mathbb{F}}$ resp. $\mathcal{A}^{\circ}_{\mathbb{F}}$) is *collectionwise completely prime* (briefly, *cc-prime*) if

(ccp) whenever $\bigvee_{i \in J} b_i \in f(a)$ there is a decomposition $a = \bigvee a_i$ such that $b_i \in f(a_i)$.

More generally, $f \colon L \multimap M$ is *collectionwise α-prime* (briefly, *cα-prime*) if (cαp) whenever $\bigvee_{i\in J} b_i \in f(a)$ and $|J| < \alpha$ there is a decomposition $a = \bigvee a_i$ such that $b_i \in f(a_i)$.

If $\alpha = \omega_0$ (finite index sets J) one speaks of collectionwise prime (c-prime) f, and if $\alpha = \omega_1$ (countable index sets J) one speaks of σ-prime f.

4.4.1. Observation. *Let $h\colon M \to L$ preserve all joins (resp. all joins of less than α elements). Then $h^\bullet : L \multimap M$ is cc-prime (resp. cα-prime).*

(If $\bigvee b_i \in h^\bullet(a)$ then $a \le h(\bigvee b_i) = \bigvee h(b_i)$ and $a_i = a \wedge h(b_i) \le h(b_i)$.)

4.4.2. Theorem. *In the dualities from 4.3, the frame homomorphisms (resp. $\wedge$-homomorphisms preserving joins of less then α elements, in particular bounded lattice homomorphisms) correspond precisely to the cc-prime (resp. cα-prime, in particular c-prime) a-maps $f\colon L \multimap M$.*

Proof. It remains to be proved that for an $f\colon L \multimap M$ in $\mathcal{A}_{\mathbb{F}}$ the preimage $f^{-1}\langle - \rangle$ preserves joins. We have

$$f^{-1}\langle \bigvee_J b_i \rangle = \bigvee \{a \in L \mid \bigvee_J b_i \in f(a)\}\,.$$

Now, if $\bigvee b_i \in f(a)$ then for some $K \subseteq J$ and $i \in K$ there are a_i, $a = \bigvee_K a_i$ and $b_i \in f(a_i)$, hence $a_i \le f^{-1}\langle b_i \rangle$, and $a \le \bigvee_i f^{-1}\langle b_i \rangle$. Thus,

$$f^{-1}\langle \bigvee_J b_i \rangle \le \bigvee f^{-1}\langle b_i \rangle\,.$$

The other inequality is trivial. □

4.5. The behaviour of the Manes extension $\widetilde{f}\colon \mathsf{Flt}L \to \mathsf{Flt}M$ associated with an a-map $f\colon L \multimap M$ from 3.4 corroborates our terminology. We have

Proposition. *Let $f\colon L \multimap M$ be a cc-prime resp. cα-prime a-map and let F be a completely prime resp. an α-prime filter in L. Then $\widetilde{f}(F)$ is completely prime resp. α-prime.*

Proof. Let $\bigvee_J b_i \in \widetilde{f}(F)$ (in the latter case, $|J| < \alpha$). Then for some $a \in F$, $\bigvee_J b_i \in f(a)$. Take $a = \bigvee a_i$ as in (ccp) resp. (cαp). Since a is in F we have for some i, $a_i \in F$ and hence $b_i \in f(a_i) \subseteq \widetilde{f}(F)$. □

4.5.1. The question naturally arises whether the statement above can be reversed. That is, suppose $f\colon L \multimap M$ is such that $\widetilde{f}$ sends completely prime filters to completely prime ones; is then f cc-prime? Of course this cannot hold quite generally: a frame may lack completely prime filters so that the condition may be void, or simply weak in other cases. One does have, however, a positive result if they abound.

4.5.2. First observe that for any continuous $\varphi\colon X \to Y$ and $f = \varphi^\circ$ as in 4.4 one has

$$\widetilde{f}(\mathcal{U}(x)) = \mathcal{U}(\varphi(x))$$

(indeed, $V \in \widetilde{f}(\mathcal{U}(x))$ iff there is a $U \ni x$ such that $V \in \varphi^\circ(U)$ iff there is a $U \ni x$ such that $\varphi[U] \subseteq V$; by continuity this is iff $\varphi(x) \in V$).

4.5.3. Lemma. *For any topological spaces X and Y let $f\colon \mathfrak{O}(X) \multimap \mathfrak{O}(Y)$ be an a-map and let $\varphi\colon X \to Y$ be a mapping such that*

$$\widetilde{f}\big(\mathcal{U}(x)\big) = \mathcal{U}\big(\varphi(x)\big)\,.$$

Then φ is continuous and $f = \varphi^\circ$.

Proof. We have to prove that

$$V \in f(U) \quad \text{iff} \quad \varphi[U] \subseteq V\,.$$

Let $\varphi[U] \subseteq V$ and $x \in U$. Then $V \in \mathcal{U}(\varphi(x))$ and hence there is a $W_x \ni x$ such that $V \in f(W_x)$. Now $x \in U \cup W_x$ and hence $V \in f(U \cap W_x)$. Finally, $V \in \bigcap f(U \cap W_x) = f\big(\bigcup_x (U \cap W_x)\big) = f(U)$.

Conversely, let $V \in f(U)$ and $x \in U$. Then $V \in \mathcal{U}(\varphi(x))$ and hence $\varphi(x) \in U$; thus, $\varphi[U] \subseteq V$. □

4.5.4. Proposition. *For any topological space X and any sober space Y, $f\colon \mathfrak{O}(X) \multimap \mathfrak{O}(Y)$ is cc-prime iff for each completely prime $F \subseteq \mathfrak{O}(X)$, the filter $\widetilde{f}(F)$ is completely prime.*

Proof. Every completely prime filter in $\mathfrak{O}(Y)$ is of the form $\mathcal{U}(y)$, $y \in Y$. Thus, for each $x \in X$ we have a $y = \varphi(x)$ such that $\widetilde{f}(\mathcal{U}(x)) = \mathcal{U}(\varphi(x))$. By Lemma 4.5.3, thus chosen $\varphi\colon X \to Y$ is continuous, and $f = \varphi^\circ$ is cc-prime by 4.4. □

4.5.5. Lemma 4.3.3 also yields a counterpart of the well known fact on representation of continuous maps into sober spaces by frame homomorphisms.

Proposition. *Let X, Y be topological spaces and let Y be sober. Then the cc-prime a-maps $f\colon \mathfrak{O}(X) \multimap \mathfrak{O}(Y)$ are precisely the φ° with $\varphi\colon X \to Y$ continuous maps.*

Proof. Let $f\colon \mathfrak{O}(X) \multimap \mathfrak{O}(Y)$ be a cc-prime a-map. For $x \in X$ we have the completely prime $\mathcal{U}(x)$. By 4.5, $\widetilde{f}(\mathcal{U}(x))$ is completely prime, and hence $\mathcal{U}(y)$ for some $y \in Y$ (uniquely determined since our spaces are T_0). If we denote this y by $\varphi(x)$, we obtain $\widetilde{f}(\mathcal{U}(x)) = \mathcal{U}(\varphi(x))$ and the statement follows. □

4.6. Theorem. *The correspondences $h \mapsto h^\bullet$ and $f \mapsto f^{-1}\langle - \rangle$ constitute a dual equivalence between* **Frm** *resp.* α**Frm** *and the full subcategory of*

$$\mathcal{B}_{\mathbb{F}} \quad \textit{resp.} \quad \mathcal{B}(\alpha)_{\mathbb{F}}$$

generated by the objects (L, L) where L is a frame resp. α-frame.

Proof. We need to prove that an $f\colon L \to \mathsf{Flt}M$ is cc-prime resp. cα-prime iff $f\colon (L, L) \to \mathsf{Flt}(M, M)$ is a morphism in $\mathcal{B}$ resp. $\mathcal{B}(\alpha)$.

We have $\bigvee_{i \in J} b_i \in f(a)$ iff $\uparrow\bigvee_i b_i = \bigvee_i \uparrow b_i \subseteq f(a)$ iff $f(a) \le \bigvee_i \uparrow b_i$. Now $a \le \bigvee_i a_i$ with $f(a_i) \le \uparrow b_i$ iff we have there $b_i \in f(a_i)$. □

5. Spectra in the approximate setting

In this section we will relate our description of the dual of the category of frames to the familiar facts about the dual adjointness between frames and spaces.

5.1. By 4.4.2 we have the category of locales represented as

Loc : the subcategory of $\mathcal{A}_{\mathbb{F}}$ with frames for objects, and all the $f\colon L \multimap M$ that are collectionwise completely prime for morphisms.

5.2. Denote by $\mathsf{Flt}_{\mathsf{cp}}L$ the subset of $\mathsf{Flt}L$ constituted by the completely prime filters on L, and by $\tau(L)$ the set

$$\{\Sigma_a \mid a \in L\} \quad \text{where} \quad \Sigma_a = \{F \in \mathsf{Flt}_{\mathsf{cp}}L \mid a \in F\}.$$

Obviously

$$\Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b \quad \text{and} \quad \Sigma_{\bigvee_J a_i} = \bigcup_J \Sigma_{a_i}, \tag{5.2.1}$$

and hence $\tau(L)$ is a topology on $\mathsf{Flt}_{\mathsf{cp}}L$ and we have a space

$$\Sigma L = \left(\mathsf{Flt}_{\mathsf{cp}}L, \tau(L)\right).$$

Furthermore, for an $f\colon L \multimap M$ in **Loc** define

$$\Sigma f\colon \Sigma L \to \Sigma M$$

by setting $\Sigma f(F) = \widetilde{f}(F)$. This is correct: by 4.5 if F is in $\mathsf{Flt}_{\mathsf{cp}}L$ then $\widetilde{f}(F)$ is in $\mathsf{Flt}_{\mathsf{cp}}M$, and the map is continuous since we have

$$\Sigma f^{-1}[\Sigma_b] = \Sigma_{f^{-1}\langle b\rangle} \tag{5.2.2}$$

(indeed: recall that $f^{-1}\langle b\rangle = \bigvee\{a \in L \mid b \in f(a)\}$ and hence

$$\{F \mid \widetilde{f}(F) \in \Sigma_b\} = \{F \mid \exists a \in F,\ b \in f(a)\} = \{F \mid f^{-1}\langle b\rangle \in F\}).$$

From the formulas $\widetilde{\eta} = \mathrm{id}$ and $\widetilde{f \circ g} = \widetilde{\widetilde{f} \cdot g} = \widetilde{f} \cdot \widetilde{g}$ in 2.3 we immediately infer that we have obtained a functor

$$\Sigma\colon \mathbf{Loc} \to \mathbf{Top}.$$

5.3. Our next aim is to obtain a functor in the opposite direction. Denote by $\Omega X = \mathfrak{O}(X)$ the frame of open sets of a space X. For a continuous map $\varphi\colon X \to Y$ we have already defined $\varphi^\circ\colon \Omega X \to \mathsf{Flt}\,\Omega Y$ (Introduction, 4.4), and in 4.4 we have observed that, in the notation of 5.1, $\Omega(\varphi) = \varphi^\circ\colon \Omega X \multimap \Omega Y$ is a morphism in **Loc**. We see that we have $\Omega(\mathrm{id}) = \eta_{\Omega X}$, the identity $\Omega X \multimap \Omega X$ in **Loc**, and we easily check that $\Omega(fg) = \omega\widetilde{(f) \cdot \Omega}(g) = \Omega(f) \circ \Omega(g)$ in **Loc**. Thus, we have a functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Loc}.$$

5.4. The spectrum adjunction. Define

$$\lambda_M\colon \Omega\Sigma L \multimap L$$

by setting

$$\lambda_L(U) = \{a \mid U \subseteq \Sigma_a\}$$

(of course, U is one of the Σ_x's). We have

5.4.1. Lemma.

(1) $\lambda_L^{-1}\langle a\rangle = \Sigma_a$.

(2) $(\Omega\Sigma f)^{-1}\langle\Sigma_b\rangle = \Sigma_{f^{-1}\langle b\rangle}$.

(3) $\lambda = (\lambda_L)_L$ *is a natural transformation.*

Proof. (1) $\bigvee\{U \mid a \in \lambda_L(U)\} = \bigcup\{U \mid U \subseteq \Sigma_a\} = \Sigma_a$.

(2) By (PREIM) in Introduction, and by (5.2.2) we obtain

$$(\Omega\Sigma f)^{-1}\langle\Sigma_b\rangle = (\Sigma f^\circ)^{-1}\langle\Sigma b\rangle = \Sigma f^{-1}[\Sigma_b] = \Sigma_{f^{-1}\langle b\rangle}\,.$$

(3) First, $\lambda_L(\bigcup U_i) = \{a \mid \bigcup U_i \subseteq \Sigma_a\} = \{a \mid \forall i,\ U_i \subseteq \Sigma_a\} = \bigcap \lambda_L(U_i)$. If $\bigvee b_i \in \lambda_L(U)$ then $U \subseteq \Sigma_{\bigvee b_i} = \bigcup \Sigma_{b_i}$, and $U = \bigcup U_i$ where $U_i = U \cap \Sigma_{b_i}$ with $b_i \in \lambda(U_i)$.

To prove that $f \circ \lambda_L = \lambda_M \circ \Omega\Sigma f$ we will use the dual representations by $g^{-1}\langle -\rangle$. We have

$$(\lambda_M \circ \Omega\Sigma f)^{-1}\langle b\rangle = (\Omega\Sigma f)^{-1}\langle\lambda_M^{-1}\langle b\rangle\rangle = (\Omega\Sigma f)^{-1}\langle\Sigma_b\rangle = \Sigma_{f^{-1}\langle b\rangle}$$

by (1) and (2), and $(f \circ \lambda_L)^{-1}\langle b\rangle = \lambda_L^{-1}\langle f^{-1}\langle b\rangle\rangle = \Sigma_{f^{-1}\langle b\rangle}$ by (1). □

5.4.2. For a space X we have the familiar continuous map

$$\rho_X \colon X \to \Sigma\Omega X\,, \quad x \mapsto \mathcal{U}(x)\,,$$

(recall 1.3.1.) for which

$$(*) \qquad \rho_X^{-1}(\Sigma_U) = \{x \mid \mathcal{U}(x) \in \Sigma_U\} = \{x \mid U \in \mathcal{U}(x)\} = U\,.$$

Note that for T_0-spaces ρ_X is one-one, and it is onto iff X is sober (recall 1.3.2) so that $(*)$ makes it a homeomorphism.

See also 4.5.5.

Lemma. $\rho = (\rho_X)_X$ *is a natural transformation.*

Proof. We have

$$\begin{aligned}\Sigma\Omega\varphi\big(\rho_X(x)\big) &= \widetilde{\varphi^\circ}\big(\mathcal{U}(x)\big) = \bigcup\{\varphi^\circ(U) \mid x \in U\} = \\ &= \{V \mid \exists U, x \in U, \varphi[U] \subseteq V\} = \{V \mid \varphi(x) \in V\} = \rho_X\big(\varphi(x)\big)\,.\end{aligned}$$

□

5.4.3. Proposition. Σ *is right adjoint to* Ω*, with the adjunction units* λ *and* ρ*.*

Proof. In the composition

$$\Sigma L \xrightarrow{\ \rho_{\Sigma L}\ } \Sigma\Omega\Sigma L \xrightarrow{\ \Sigma\lambda_L\ } \Sigma L$$

we have $\Sigma\lambda_L(\rho_{\Sigma L}(F) = \widetilde{\lambda_L}(\mathcal{U}(F)) = \bigcup\{\lambda_L(\Sigma_a) \mid a \in F\} = \bigcup\{\{b \mid \Sigma_a \subseteq \Sigma_b\} \mid a \in F\} = F$.

To prove the identity resulting from the composition

$$\Omega(X) \overset{\Omega\rho_X}{\multimap} \Omega\Sigma\Omega(X) \overset{\lambda_{\Omega(X)}}{\multimap} \Omega(X)$$

we will use the dual representation by the preimages $g^{-1}\langle - \rangle$ similarly like in 5.4.1(3). We have

$$(\lambda_{\Omega X} \circ \Omega\rho_X)^{-1}\langle U\rangle = (\Omega\rho_X)^{-1}\langle \lambda_{\Omega X}^{-1}\langle U\rangle\rangle = (\rho_X^{\circ})^{-1}\langle \Sigma_U\rangle = \rho_X^{-1}[\Sigma_U] = U$$

by 5.4.1(1) and (PREIM) in Introduction. □

5.5. Remark. All that was proved in this section can be done, more generally, for the category of locales modified to $\mathbf{Loc}_\alpha$ with complete distributive lattices for objects and collectionwise α-prime $f\colon L \multimap M$ for morphisms. This is why we have formally introduced the extra symbol $\tau(L)$ for the topology $\{\Sigma_a \mid a \in L\}$ - in the more general context the topology is just *generated by* $\{\Sigma_a \mid a \in L\}$ – and why we have used the symbol U working with the λ (see 5.4) – in the more general context it is not necessarily one of the Σ_a.

It may be of interest that in the case of $\alpha = \omega_0$ the construction yields a fragment of Priestley duality ([14]) restricted to complete distributive lattices.

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Department of Mathematics and Statistics,
McMaster University, 1280 Main St. W, Hamilton,
Ontario L8S 4K1, Canada

Department of Applied Mathematics and ITI,
MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25
E-mail: pultr@kam.ms.mff.cuni.cz