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AN ASYMPTOTIC FORMULA FOR SOLUTIONS OF NONOSCILLATORY HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We establish a Hartman type asymptotic formula for nonoscillatory solutions of the half-linear second order differential equation

$$\big(r(t)\Phi(y')\big)' + c(t)\Phi(y) = 0\,, \quad \Phi(y) := |y|^{p-2}y\,,\ p > 1\,.$$

1. Introduction

In this paper we prove an asymptotic formula for nonoscillatory solutions of the half-linear second order differential equation

$$(r(t)\Phi(y'))' + c(t)\Phi(y) = 0\,, \quad \Phi(y) := |y|^{p-2}y\,,\ p > 1\,, \tag{1}$$

where r, c are continuous functions and $r(t) > 0$ for large t. Here, equation (1) is viewed as a perturbation of the nonoscillatory equation of the same form

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0 \tag{2}$$

with a continuous function $\tilde{c}$. It is shown that if the integral

$$\int_t^\infty [c(s) - \tilde{c}(s)]h^p(s)\,ds\,,$$

where h is the so-called principal solution of (2), see [9, 13], is convergent and tends to zero sufficiently rapidly as $t \to \infty$, then solutions of (1) and (2) have similar asymptotic behavior as $t \to \infty$. In the linear case $p = 2$, our criterion reduces to [10, Theorem 9.1].

The problem to find exact asymptotic formulas for solutions of the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0$$

is treated in the literature as Hartman-Wintner problem and attracted considerable attention in recent years. We refer to the papers [3, 15, 16] and the references given

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therein. It is a subject of the present investigation to elaborate a similar asymptotic theory for half-linear equation (1).

2. Preliminaries

The linear Sturmian separation theorem extends verbatim to (1), so this equation can be classified as oscillatory or nonoscillatory similarly as in the linear case. We refer to [1, Chap. 3], [5], or to [6] for more details concerning essentials of the half-linear oscillation theory. In our asymptotic formula the so-called *principal* and *nonprincipal* solutions of (2) appear. Nonoscillation of (1) implies the existence of a solution of the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0\,, \quad q = \frac{p}{p-1}\,, \tag{3}$$

(related to (1) by the substitution $w = r\Phi(y'/y)$) which is defined on some interval $[T,\infty)$. Among all solutions of (3) there exists the *minimal one* $\tilde{w}$, minimal in the sense that for any other solution w of (3) we have $w(t) > \tilde{w}(t)$ for large t. The *principal solution* $\tilde{x}$ of (1) is then the solution which "generates" the minimal solution $\tilde{w}$ via the Riccati substitution $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$, i.e., it is given by the formula

$$\tilde{y}(t) = C\exp\Big\{\int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\,ds\Big\},$$

where $\Phi^{-1}(y) = |y|^{q-2}y$ is the inverse function of Φ and C is a nonzero real constant. The *nonprincipal solution* of (1) is any solution linearly independent of the principal one.

We have the following inequality for minimal solutions of (3), see [6, Theorem 4.4.2].

Lemma 1. *Let $c(t) \ge \tilde{c}(t)$ for large t, i.e.,* (1) *is the Sturmian majorant of* (2)*, and suppose that* (1) *is nonoscillatory. Then the minimal solutions w of* (3) *and $\tilde{w}$ of the Riccati equation associated with* (2) *satisfy the inequality $w(t) \ge \tilde{w}(t)$ for large t.*

At the end of this section we recall the linear asymptotic criterion given in [10, Theorem 9.1].

Proposition 1. *Consider the pair of linear differential equations*

$$(r(t)u')' + c(t)u = 0 \tag{4}$$

and

$$(r(t)x')' + \tilde{c}(t)x = 0\,, \tag{5}$$

and let x_0, x_1 be the principal and nonprincipal solutions of (5)*, respectively. Suppose that*

$$\int^\infty (c(t) - \tilde{c}(t))x_0^2(t)\,dt \quad \textit{converges}$$

and

$$\int^\infty \frac{\Gamma(t)}{r(t)x_0^2(t)}\,dt < \infty\,, \quad \Gamma(t) := \sup_{t\le s<\infty}\left|\int_s^\infty \big(c(\tau)-\tilde c(\tau)\big)x_0^2(\tau)\,d\tau\right|.$$

Then (4) *possesses a pair of solutions* u_i, $i=0,1$, *such that their logarithmic derivatives satisfy*

$$\frac{u_i'(t)}{u_i(t)} = \frac{x_i'(t)}{x_i(t)} + o\bigg(\frac{1}{r(t)x_0(t)x_1(t)}\bigg)\,, \quad i=0,1\,, \tag{6}$$

as $t\to\infty$.

3. Main result

In this section we present the main result of the paper. We establish an asymptotic formula for solutions of (1).

Theorem 1. *Let* h *be the positive principal solution of* (2) *such that*

$$\liminf_{t\to\infty}|G(t)|>0\,, \qquad G(t):=r(t)h(t)\Phi\big(h'(t)\big) \tag{7}$$

and

$$\int^\infty R^{-1}(t)\,dt=\infty\,, \qquad R(t):=r(t)h^2(t)|h'(t)|^{p-2}\,. \tag{8}$$

Further suppose that $c(t)\ge\tilde c(t)$ *for large* t, *equation* (1) *is nonoscillatory, the integral*

$$\int^\infty \big(c(t)-\tilde c(t)\big)h^p(t)\,dt<\infty\,, \tag{9}$$

and that the function $\Gamma(t)=\int_t^\infty(c(s)-\tilde c(s))h^p(s)\,ds$ *satisfies*

$$\int^\infty \Gamma(t)R^{-1}(t)\,dt<\infty\,.$$

Then (1) *possesses a pair of solutions* y_i, $i=0,1$, *such that their logarithmic derivatives satisfy*

$$\frac{y_0'(t)}{y_0(t)}=\frac{h'(t)}{h(t)}+o\bigg(\frac{1}{R(t)\int^t R^{-1}(s)\,ds}\bigg)\,, \tag{10}$$

$$\frac{y_1'(t)}{y_1(t)}=\frac{h'(t)}{h(t)}+\frac{2}{pR(t)\int^t R^{-1}(s)\,ds}+o\bigg(\frac{1}{R(t)\int^t R^{-1}(s)\,ds}\bigg) \tag{11}$$

as $t\to\infty$.

Proof. Denote $v=h^p(w-w_h)$, where $w_h=r\Phi(h'/h)$ and w is any solution of (3) which is extensible up to ∞. Then by a direct computation (see, e.g. [7]) one can find that v solves the so-called modified Riccati equation

$$v'+\big(c(t)-\tilde c(t)\big)h^p(t)+(p-1)r^{1-q}(t)h^{-q}(t)H(t,v)=0\,, \tag{12}$$

where

$$H(t,v)=\big|v+G(t)\big|^q-q\Phi^{-1}\big(G(t)\big)v-|G(t)|^q\,.$$

Using the fact that $c(t) \geq \tilde{c}(t)$ we have by Lemma 1 that $w(t) \geq w_h(t)$ for large t, i.e., $v(t) \geq 0$ and since $H(t,v) \geq 0$ (see, e.g. [8]), the function v is nonincreasing, hence there exists a nonnegative limit $\lim_{t\to\infty} v(t) =: v_\infty$.

Conditions (7) and (8) imply that $v_\infty = 0$. Indeed, suppose that $v_\infty > 0$. The function H can be written in the form

$$H(t,v) = \big|G(t)\big|^q F(z)\,, \quad F(z) := |1+z|^q - qz - 1\,, \quad z = v/G\,.$$

Condition (7) implies that $z = \frac{v(t)}{G(t)}$ attains values in a compact interval, say $[A,B] \subset \mathbb{R}$, for large t. Consider the function

$$\tilde{F}(z) := \begin{cases} \frac{F(z)}{z^2}\,, & z \neq 0\,, \\ \frac{q(q-1)}{2}\,, & z = 0\,. \end{cases}$$

This function is positive and continuous in $\mathbb{R}$ (actually, it is a C^2 function), hence there exists $m > 0$ such that $\tilde{F}(z) \geq m$ for $z \in [A,B]$, i.e., $F(z) \geq mz^2$. Hence we have for large t (suppressing the argument t)

$$v' + (c - \tilde{c})h^p + (p-1)r^{1-q}h^{-q}|G|^q m\Big(\frac{v}{G}\Big)^2 \leq 0\,,$$

i.e., substituting for G from (7)

$$v' + (c - \tilde{c})h^p + m(p-1)\frac{v^2}{R} \leq 0\,,$$

where R is given by (8). Integrating the last inequality from T to t, letting $t \to \infty$, and using the fact that (9) holds, we get

$$\begin{aligned} m(p-1)v_\infty^2 \int_T^\infty \frac{dt}{R(t)} &\leq m(p-1)\int_T^\infty \frac{v^2(t)}{R(t)}\,dt \\ &\leq v(T) - \int_T^\infty \big(c(t) - \tilde{c}(t)\big)h^p(t)\,dt < \infty\,, \end{aligned}$$

a contradiction with (8). Hence $v_\infty = 0$, i.e. $z(\infty) = 0$, so we may apply the second order Taylor formula to F at the center $z = 0$. Since $F(0) = F'(0) = 0$, we have

$$F(z) = \frac{q(q-1)}{2} z^2\big[1 + o(1)\big]\,, \quad \text{as } \ z \to 0$$

and hence

$$H\big(t, v(t)\big) = \frac{q(q-1)}{2}\big|G(t)\big|^{q-2} v^2(t)\big(1 + o(1)\big) \quad \text{as } \ t \to \infty\,.$$

This means that we may write (12) in the form (since $(p-1)(q-1) = 1$)

$$v' + \big(c(t) - \tilde{c}(t)\big)h^p(t) + \frac{q(1+o(1))}{2}\frac{v^2}{R(t)} = 0 \tag{13}$$

as can be verified by a direct computation.

The last equation is the Riccati equation associated with the linear differential equation (related to this equation by the Riccati substitution $v = \frac{2R}{q(1+o(1))}\frac{u'}{u}$)

$$\left(\frac{2R(t)}{q\big(1+o(1)\big)}u'\right)' + \big(c(t)-\tilde{c}(t)\big)h^p(t)u = 0\,. \tag{14}$$

Now, we consider equation (14) as a perturbation of the one-term equation

$$\left(\frac{2R(t)}{q\big(1+o(1)\big)}x'\right)' = 0 \tag{15}$$

which plays the role of unperturbed equation (5) in Proposition 1. Taking into account that (8) holds, i.e. also $\int^\infty R^{-1}(t)(1+o(1))\,dt = \infty$,

$$x_0(t) = 1\,, \quad x_1(t) = \int^t R^{-1}(s)(1+o(1))\,ds$$

are the principal and nonprincipal solutions of (15), respectively. Now, applying Proposition 1 to the pair of equations (15) and (14) implies that (14) has a pair of solutions u_0, u_1 with logarithmic derivatives

$$\begin{aligned}\frac{u_0'(t)}{u_0(t)} &= o\left(\frac{q\big(1+o(1)\big)}{2R(t)\int^t R^{-1}(s)\big(1+o(1)\big)\,ds}\right)\\ \frac{u_1'(t)}{u_1(t)} &= \frac{R^{-1}(t)\big(1+o(1)\big)}{\int^t R^{-1}(s)\big(1+o(1)\big)\,ds} + o\left(\frac{q\big(1+o(1)\big)}{2R(t)\int^t R^{-1}(s)\big(1+o(1)\big)\,ds}\right)\end{aligned}$$

which means that (13) possesses a pair of solutions

$$\begin{aligned}v_0 &= \frac{2R}{q\big(1+o(1)\big)}\,\frac{u_0'}{u_0} = o\left(\frac{1}{\int^t R^{-1}(s)\,ds}\right),\\ v_1 &= \frac{2R}{q\big(1+o(1)\big)}\,\frac{u_1'}{u_1} = \frac{2}{q\int^t R^{-1}(s)\big(1+o(1)\big)\,ds} + o\left(\frac{1}{\int^t R^{-1}(s)\,ds}\right)\\ &= \frac{2}{q\int^t R^{-1}(s)\,ds}\big(1+o(1)\big)\end{aligned}$$

as $t\to\infty$. Here we have used that

$$o\left(\int^t R^{-1}(s)\big(1+o(1)\big)\,ds\right) = o\left(\int^t R^{-1}(s)\,ds\right) \quad \text{as} \quad t\to\infty$$

which follows from that fact that under (8)

$$\lim_{t\to\infty}\frac{\int^t R^{-1}(s)\big(1+o(1)\big)\,ds}{\int^t R^{-1}(s)\,ds} = 1\,.$$

Now, substituting $w = h^{-p}v + w_h$, where w is a solution of (3), we have for $t\to\infty$

$$w_0(t) = \frac{r(t)\Phi(y_0'(t))}{\Phi(y_0(t))} = h^{-p}(t)v_0(t) + w_h(t) = w_h(t)\left[1 + o\left(\frac{1}{\int^t R^{-1}}\right)\frac{1}{G(t)}\right].$$

Hence

$$\frac{y_0'(t)}{y_0(t)} = \frac{h'(t)}{h(t)}\left[1 + o\left(\frac{1}{\int^t R^{-1}}\right)\frac{1}{G(t)}\right]^{q-1} = \frac{h'(t)}{h(t)} + o\left(\frac{1}{R(t)\int^t R^{-1}}\right).$$

Similarly

$$w_1(t) = \frac{r(t)\Phi(y_1'(t))}{\Phi(y_1(t))} = h^{-p}(t)v_1(t) + w_h(t) = w_h(t)\left[1 + \frac{2(1+o(1))}{qG(t)\int^t R^{-1}}\right],$$

i.e.,

$$\frac{y_1'(t)}{y_1(t)} = \frac{h'(t)}{h(t)}\left[1 + \frac{2\big(1+o(1)\big)}{qG(t)\int^t R^{-1}}\right]^{q-1} = \frac{h'(t)}{h(t)} + \frac{2}{pR(t)\int^t R^{-1}} + o\left(\frac{1}{R(t)\int^t R^{-1}}\right).$$

The proof is complete. □

Remark 1. (i) In the linear case $p = 2$, Theorem 1 reduces essentially to Proposition 1. Indeed, the nonprincipal solution x_1 of (5) can be expressed by the D'Alembert formula as

$$x_1(t) = x_0(t)\int^t \frac{ds}{r(s)x_0^2(s)},$$

i.e.

$$\frac{x_1'(t)}{x_1(t)} = \frac{x_0'(t)}{x_0(t)} + \frac{1}{r(t)x_0^2(t)\int^t r^{-1}(s)x_0^{-2}(s)\,ds},$$

and we see that (10), (11) are the same as (6) in case $p = 2$. The difference is in the assumption $c(t) \geq \tilde{c}(t)$ and in additional assumption (7), this restriction is discussed in the next part of this remark.

(ii) In our main result we suppose (7) and (8). Condition (8) is closely related to the integral characterization of the principal solution of half-linear differential equations, we refer to [2] and [4] for discussion concerning this assumption. In the linear case $p = 2$ it is just the definition of the principal solution. Condition (7) is technical, we needed it (together with the assumption $c(t) \geq \tilde{c}(t)$ for large t) to prove an asymptotic formula for the function v in (12). The subject of the present investigation is to find a similar asymptotic formula in case when $\lim_{t\to\infty} G(t) = 0$.

(iii) Under some additional assumptions on the functions $r, c, \tilde{c}$ (regular monotonicity), asymptotic formulas for solutions of (1) can be formulated in terms of slowly and regularly varying functions, we refer to [11, 12, 14] for details.

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