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PARALLEL AND TOTALLY GEODESIC HYPERSURFACES OF SOLVABLE LIE GROUPS

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Abstract. In this paper we consider special examples of homogeneous spaces of arbitrary odd dimension which are given in [5] and [16]. We obtain the complete classification and explicitly describe parallel and totally geodesic hypersurfaces of these spaces in both Riemannian and Lorentzian cases.

1. Introduction

Parallel submanifolds are the first important class of submanifolds to study [13]. They play an important role in geometry and general relativity and the study of these submanifolds helps us to enrich our knowledge of the geometry of the ambient spaces.

A submanifold is called parallel if its second fundamental form is covariantly constant and it is called totally geodesic if its second fundamental form vanishes identically. Hence, the extrinsic invariants of parallel submanifolds do not vary from point to point and these submanifold can be considered as a natural extension of totally geodesic submanifolds.

Parallel and totally geodesic surfaces in four dimensional Lorentzian space forms and in pseudo-Riemannian space forms with an arbitrary index and dimension have been classified respectively in [10] and [11]. Also the classification of parallel and totally geodesic hypersurfaces in real space forms of any dimension can be found in [17] and [19].

A natural generalization of spaces of constant curvature are homogeneous spaces. Thus it is interesting to choose these spaces as ambient spaces and classify their parallel and totally geodesic hypersurfaces. Up to our knowledge, this study has been done for the homogeneous spaces with dimension less than 6. In fact the complete classification of parallel and totally geodesic surfaces in all three dimensional Riemannian and Lorentzian homogeneous spaces is given in [4, 7, 8, 14, 15]. Also, parallel hypersurfaces of four dimensional oscillator groups and totally geodesic hypersurfaces of four dimensional generalized symmetric spaces are classified in [9] and [12], respectively. Moreover, the complete classification of parallel and totally

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geodesic hypersurfaces of two-step homogeneous nilmanifolds of dimension five is given in [18].

In the present paper, we deal with the problem of classifying parallel and totally geodesic hypersurfaces for a class of solvable Lie groups of arbitrary odd dimension. These Lie groups consist of all matrices of the form

$$\begin{pmatrix} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{1}$$

where $(x_0, x_1, \dots, x_n, u_1, \dots, u_n) \in \mathbf{R}^{2n+1}$, $u_0 = -(u_1 + \cdots + u_n)$ and n is any integer $n \geq 1$. Following the works [5, 6, 16] to which we may refer for more details, in [1] we investigated some geometrical properties of these spaces with dimension five in both Riemannian and Lorentzian cases. Then in [3] we generalized this study for an arbitrary odd dimension and in [2] we investigated the Randers metrics of Douglas type on these spaces. Our aim in the present paper is to give the complete classification and explicitly describe parallel and totally geodesic hypersurfaces of these spaces in both Riemannian and Lorentzian cases. Moreover we describe some results of this classification which are related to the number of these hypersurfaces.

2. Curvature properties of the class of solvable Lie groups G_n

Let us denote this class of solvable Lie groups by G_n and consider the following left-invariant vector fields on G_n,

$$X_i = e^{u_i} \frac{\partial}{\partial x_i}\,, \quad i = 0, 1, \dots, n, \quad U_\alpha = \frac{\partial}{\partial u_\alpha} \ \alpha = 1, \dots, n\,.$$

Following [2], we can equip these spaces by the left-invariant Riemannian metric

$$g = \sum_{i=0}^{n} e^{-2u_i} (dx_i)^2 + \sum_{\alpha=1}^{n} (du_\alpha)^2\,,$$

and the left-invariant Lorentzian metric

$$\widehat{g} = -e^{-2u_0} (dx_0)^2 + \sum_{i=1}^{n} e^{-2u_i} (dx_i)^2 + \sum_{\alpha=1}^{n} (du_\alpha)^2\,.$$

Then the set $\{X_0, \dots, X_n, U_1, \dots, U_n\}$ with respect to the inner product $\langle\ ,\ \rangle$ which is induced by the Riemannian metric g (Lorentzian metric $\widehat{g}$) is an orthonormal (pseudo-orthonormal) frame field for the Lie algebra $\mathcal{G}_n$ of G_n and we have

$$[X_0, U_\alpha] = X_0\,, \quad [X_\alpha, U_\beta] = -\delta_{\alpha\beta} X_\alpha \quad \text{and} \quad [X_i, X_j] = [U_\alpha, U_\beta] = 0\,.$$

where $\alpha, \beta = 1, \dots, n$ and $i, j = 0, 1, \dots, n$. Also the non-zero Levi-Civita connection components in the Riemannian case are given by

$$\nabla_{X_0} U_\alpha = X_0\,,\ \nabla_{X_0} X_0 = -(\sum_{\alpha=1}^{n} U_\alpha)\,,\ \nabla_{X_i} U_\alpha = -\delta_{i\alpha} X_i, \nabla_{X_i} X_i = \delta_{\alpha i} U_\alpha\,, \tag{2}$$

and in the Lorentzian case are given by

$$(3)\qquad \nabla_{X_0}U_\alpha = X_0\,,\ \nabla_{X_0}X_0 = \sum_{\alpha=1}^{n} U_\alpha\,,\ \nabla_{X_i}U_\alpha = -\delta_{i\alpha}X_i\,,\ \nabla_{X_i}X_i = \delta_{i\alpha}U_\alpha\,,$$

where α, $i = 1,\dots,n$. If we adopt the following sign conventions for the curvature tensor field R,

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y] \quad \text{and} \quad R_{XYZW} = \langle R(X,Y)Z,W\rangle\,,$$

where X, Y, Z and W are left-invariant vector fields on G_n, then the non-zero curvature components in the Riemannian case are

$$R_{X_0U_iX_0U_j} = -R_{X_iU_iU_iX_i} = R_{X_0X_iX_iX_0} = -1\,, \quad i,j = 1,\dots,n$$

and the ones obtained by these components using the symmetries of the curvature tensor. Also the non-zero curvature components in the Lorentzian case are

$$R_{X_0U_iX_0U_j} = R_{X_iU_iU_iX_i} = R_{X_0X_iX_iX_0} = 1\,, \quad i,j = 1,\dots,n$$

and the ones implied by them using the symmetries of the curvature tensor.

3. Parallel and totally geodesic hypersurfaces of G_n

Let $F\colon M^{2n} \to N^{2n+1}$ be an isometric immersion of pseudo-Riemannian manifolds $(M,\langle\ ,\ \rangle)$ and $(N,\langle\ ,\ \rangle)$. Denote by ∇^M and ∇ the Levi-Civita connections of M and N and by ξ a normal vector field on the hypersurface M with $\langle\xi,\xi\rangle = \varepsilon$, where $\varepsilon = \{1,-1\}$. Let us define the shape operator S by $SX = -\nabla_X\xi$ and identify vector fields tangent to M with their images under dF. Then the formula of Gauss is given by

$$(4)\qquad \nabla_X Y = \nabla^M_X Y + h(X,Y)\xi\,,$$

where X and Y are vector fields tangent to M and h is the second fundamental form which is defined by $h(X,Y) = \varepsilon\langle SX,Y\rangle$. If R is the curvature tensor of the ambient space N, then the equation of Codazzi can be described by

$$(5)\qquad \langle R(X,Y)Z,\xi\rangle = \varepsilon\big((\nabla^M h)(Y,X,Z) - (\nabla^M h)(X,Y,Z)\big)\,,$$

where X, Y, Z and W are vector fields tangent to M and $(\nabla^M h)$ is defined by

$$(\nabla^M h)(X,Y,Z) = X\big(h(Y,Z)\big) - h(\nabla^M_X Y,Z) - h(Y,\nabla^M_X Z)\,.$$

The hypersurface M is said to be a totally geodesic hypersurface in N, if $h = 0$ and it is said to be a parallel hypersurface in N, if $\nabla^M h = 0$.
In order to classify parallel hypersurfaces of the class of solvable Lie groups G_n, we prove the following result.

Theorem 3.1. *Let $F\colon M^{2n} \to G_n$ be a parallel hypersurface of the class of solvable Riemannian Lie groups (G_n,g) (Lorentzian Lie groups $(G_n,\widehat{g})$). Also let ξ be a unit (ε-unit) normal vector field on M and $\{X_0,\dots,U_n\}$ be an (pseudo-)orthonormal frame field on G_n. Then ξ has one of the following forms*
Case (a): $\xi = \pm X_0$,
Case (b): $\xi = \pm X_r$, *where* $r \in \{1,\dots,n\}$,
Case (c): $\xi = \pm U_r$, *where* $r \in \{1,\dots,n\}$.

Proof. First we suppose that, M is a parallel hypersurface in G_n and assume that $\xi = \sum_{i=0}^{n} K_i X_i + \sum_{i=1}^{n} K_{n+i} U_i$, where $K_i \colon U \subseteq M \to \mathbf{R}$ are some functions. Then the following vector fields, with respect to the Riemannian metric g (Lorentzian metric $\widehat{g}$) are tangent to the hypersurface:

(6)

$$
\begin{array}{lll}
\textbf{Riemannian} & \textbf{Lorentzian} & \\
X_{i0} = K_i X_0 - K_0 X_i, & X_{i0} = K_i X_0 + K_0 X_i, & i = 1, \dots, n, \\
X_{i1} = K_i X_1 - K_1 X_i, & X_{i1} = K_i X_1 - K_1 X_i, & i = 2, \dots, n, \\
\vdots & \vdots & \vdots \\
X_{i(n-1)} = K_i X_{n-1} - K_{n-1} X_i, & X_{i(n-1)} = K_i X_{n-1} - K_{n-1} X_i, & i = n, \\
Y_{t0} = K_{n+t} X_0 - K_0 U_t, & Y_{t0} = K_{n+t} X_0 + K_0 U_t, & t = 1, \dots, n, \\
Y_{t1} = K_{n+t} X_1 - K_1 U_t, & Y_{t1} = K_{n+t} X_1 - K_1 U_t, & t = 1, \dots, n, \\
\vdots & \vdots & \vdots \\
Y_{tn} = K_{n+t} X_n - K_n U_t, & Y_{tn} = K_{n+t} X_n - K_n U_t, & t = 1, \cdots, n, \\
Z_{j1} = K_{p+j} U_1 - K_p U_{1+j}, & Z_{j1} = K_{p+j} U_1 - K_p U_{1+j}, & j = \overset{p=n+1}{1}, \dots, n-1, \\
Z_{j2} = K_{q+j} U_2 - K_q U_{2+j}, & Z_{j2} = K_{q+j} U_2 - K_q U_{2+j}, & j = \overset{q=n+2}{1}, \dots, n-2, \\
\vdots & \vdots & \vdots \\
Z_{j(n-1)} = K_{2n} U_{n-1} - K_{n-1} U_n, & Z_{j(n-1)} = K_{2n} U_{n-1} - K_{n-1} U_n, & j = 1.
\end{array}
$$

Since M is parallel in G_n, we have $\nabla^M h = 0$. Thus by the equation (5) we have

$$
\big\langle R(X_{ik}, Y_{tl}) Z_{jm}, \xi \big\rangle = 0\,, \tag{7}
$$

where X_{ik}, Y_{tl} and Z_{jm} are among the vector fields which are given in the system (6). Here we apply (7) to obtain the acceptable forms of ξ for the Riemannian and Lorentzian cases as follows

In the Riemannian case

We will consider the following two cases, namely $K_0 = 0$ and $K_0 \neq 0$.

Case 1: $K_0 \neq 0$. In this case from $0 = \langle R(X_{10}, X_{j0}) X_{10}, \xi \rangle = K_0^3 K_j$ where $j = 2, \dots, n$ we have $K_2 = \cdots = K_n = 0$. Thus by $0 = \langle R(X_{10}, X_{20}) X_{21}, \xi \rangle = -K_0^2 (K_1^2 + K_2^2)$ we obtain that $K_0^2 K_1^2 = 0$ which gives us $K_1 = 0$. Also since the condition

$$
0 = \big\langle R(X_{i0}, Y_{t0}) X_{i0}, \xi \big\rangle = K_i^2 K_0 \Big(\sum_{i=1}^{n} K_{n+i} + K_{n+i} \Big) + 2 K_0^3 K_{n+t}\,,
$$

where $i = t = 1, \dots, n$ is equivalent to $2K_0^3 K_{n+t} = 0$, we have $K_{n+1} = \cdots = K_{2n} = 0$. Thus the condition $\langle \xi, \xi \rangle = 1$ gives us $\xi = \pm X_0$.

Case 2: $K_0 = 0$. In this case we will consider the following two subcases $K_1 = 0$ and $K_1 \neq 0$.

Case 2.1: $K_1 \neq 0$. In this case from $0 = \langle R(X_{i1}, Y_{t1})X_{i1}, \xi\rangle = K_1^3 K_{n+t}$ and $0 = \langle R(X_{i1}, Y_{11})Y_{11}, \xi\rangle = K_i K_1 (K_{n+1}^2 + K_i^2)$, where $i = t = 2, \ldots, n$, we obtain that $K_2 = \cdots = K_n = K_{n+2} = \ldots = K_{2n} = 0$. Also by considering these solutions and using

$$0 = \langle R(Y_{10}, Y_{11})Y_{10}, \xi\rangle = K_{n+1}^2 \Big(K_1 K_{n+1} + K_1 \sum_{i=1}^{n} K_{n+i} \Big),$$

we have $2K_1 K_{n+1}^3 = 0$ which gives us $K_{n+1} = 0$. Thus by $\langle \xi, \xi\rangle = 1$ we have $\xi = \pm X_1$.

Case 2.2: $K_1 = 0$. In this case we will consider the following two subcases $K_2 \neq 0$ and $K_2 = 0$.

Case 2.2.1: $K_2 \neq 0$. In this case from $0 = \langle R(X_{i2}, Y_{t2})X_{i2}, \xi\rangle = K_2^3 K_{n+i}$ and $0 = \langle R(X_{i2}, Y_{22})Y_{22}, \xi\rangle = K_1 K_2 (K_{n+2}^2 + K_2^2)$, where $i = t = 3, \ldots, n$, we obtain that $K_3 = \cdots = K_n = K_{n+3} = \cdots = K_{2n} = 0$. Also from

$$0 = \langle R(Y_{t0}, Y_{t2})Y_{t0}, \xi\rangle = K_2 K_{n+t}^2 \Big(K_{n+t} + \sum_{i=1}^{n} K_{n+i} \Big), \quad t = 1, 2,$$

we obtain that $K_{n+1} = K_{n+2} = 0$. Thus by $\langle \xi, \xi\rangle = 1$ we have $\xi = \pm X_2$.

Case 2.2.2: $K_2 = 0$. In this case we will distinguish between the cases $K_3 \neq 0$ and $K_3 = 0$.

Case 2.2.2.1: $K_3 \neq 0$. In this case from $0 = \langle R(X_{i3}, Y_{t3})X_{i3}, \xi\rangle = -K_3^3 K_{n+t}$ and $0 = \langle R(X_{i3}, Y_{33})Y_{33}, \xi\rangle = K_i K_3 (K_{n+3}^2 + K_3^2)$, where $i = t = 4, \ldots, n$ we obtain that $K_4 = \cdots = K_n = K_{n+4} = \ldots = K_{2n} = 0$. Also from

$$0 = \langle R(Y_{t0}, Y_{t3})Y_{t0}, \xi\rangle = K_3 K_{n+t}^2 \Big(K_{n+t} + \sum_{i=1}^{n} K_{n+i} \Big),$$

where $t = 1, 2, 3$ we have $K_{n+1} = K_{n+2} = K_{n+3} = 0$. Thus $\langle \xi, \xi\rangle = 1$ gives us $\xi = \pm X_3$.

Case 2.2.2.2: $K_3 = 0$. In this case we will consider the two subcases $K_4 \neq 0$ and $K_4 = 0$.

By a similar argument from the cases, **case** $\overbrace{2.\ldots.2}^{4\ \text{times}}$**.1**, …, **case** $\overbrace{2.\ldots.2}^{n\ \text{times}}$**.1**, respectively we obtain $\xi = \pm X_4, \ldots, \xi = \pm X_n$.

Case $\overbrace{2.\ldots.2}^{n\ \text{times}}$**.2:** $K_n = 0$. In this case we will distinguish between the cases $K_{n+1} \neq 0$ and $K_{n+1} = 0$.

Case $\overbrace{2.\ldots.2}^{n+1\ \text{times}}$**.1:** $K_{n+1} \neq 0$. In this case if we consider the condition $0 = \langle R(Y_{11}, Z_{j1})Y_{11}, \xi\rangle = -K_{n+1} K_{n+1+j} (K_1^2 + K_{n+1}^2)$, where $j = 1, \cdots, n-1$ (since in this case $K_1 = \ldots = K_n = 0$), then we obtain that $K_{n+2} = \ldots = K_{2n} = 0$. Thus by $\langle \xi, \xi\rangle = 1$ we have $\xi = \pm U_1$.

Case $\overbrace{2.\ldots.2}^{n+1\ \text{times}}$ **.2:** $K_{n+1}=0$. In this case we will consider the following two subcases $K_{n+2}=0$ and $K_{n+2}\neq 0$.

Case $\overbrace{2.\ldots.2}^{n+2\ \text{times}}$ **.1:** $K_{n+2}\neq 0$. In this case if we consider the condition $0=\langle R(Y_{22},Z_{j2})Y_{22},\xi\rangle=-K_{n+2}K_{n+2+j}(K_{n+2}^2+K_2^2)$, where $j=1,\ldots,n-2$ we obtain that $K_{n+3}=\ldots=K_{2n}=0$. Thus $\xi=\pm U_2$.

Case $\overbrace{2.\ldots.2}^{n+2\ \text{times}}$ **.2:** $K_{n+2}=0$. In this case we will consider the two subcases $K_{n+3}=0$ and $K_{n+3}\neq 0$.

By the same arguments from the cases, **case** $\overbrace{2.\ldots.2}^{n+3\ \text{times}}$ **.1**, ... ,**case** $\overbrace{2.\ldots.2}^{2n\ \text{times}}$ **.1**, respectively we get $\xi=\pm U_3,\ldots,\xi=\pm U_n$.

Case $\overbrace{2.\ldots.2}^{2n\ \text{times}}$**.2:** $K_{2n}=0$. In this case since $K_0=\ldots=K_{2n}=0$, we have $\xi=0$ which yields the contradiction $\langle\xi,\xi\rangle=0\neq 1$.

In the Lorentzian case

If we use X_{ik},Y_{tl} and Z_{jm} which are among the vector fields which are given in the second column of the system (6), then by a straightforward computation similar to the Riemannian case we have the result. □

By the Theorem 3.1 we can obtain a complete classification of parallel hypersurfaces of these homogeneous spaces in both Riemannian and Lorentzian cases as follows.

Theorem 3.2. *Let $F\colon M^{2n}\to G_n$ be a parallel hypersurface of the class of solvable Riemannian Lie groups (G_n,g) (Lorentzian Lie groups $(G_n,\widehat{g})$). Then there exist local coordinates $(w_1,\ldots,w_{2n})$ on M^{2n}, such that this immersion with respect to these coordinates, up to isometrics, is given by one of the following expressions:*

$$F(w_1,\ldots,w_{2n})=(0,e^{w_{n+1}}w_1,e^{w_{n+2}}w_2\ldots,e^{w_{2n}}w_n,w_{n+1},\ldots,w_{2n}),$$

$$F(w_1,\ldots,w_{2n})=(e^{-(\sum_{i=1}^n w_{n+i})}w_1,0,e^{w_{n+2}}w_2\ldots,e^{w_{2n}}w_n,w_{n+1},w_{n+2}\ldots,w_{2n}),$$

$$\vdots$$

$$F(w_1,\ldots,w_{2n})=(e^{-(\sum_{i=1}^n w_{n+i})}w_1,e^{w_{n+1}}w_2,\ldots,e^{w_{n+r-1}}w_r,0,\\ e^{w_n+r+1}w_{r+1},\ldots,e^{w_{2n}}w_n,w_{n+1},\ldots,w_{2n}),$$

$$\vdots$$

$$F(w_1,\ldots,w_{2n})=(e^{-(\sum_{i=1}^n w_{n+i})}w_1,e^{w_{n+1}}w_2,\ldots,e^{w_{2n-1}}w_n,0,w_{n+1},\ldots,w_{2n}),$$

$$F(w_1,\dots,w_{2n}) = \big(e^{-(\sum_{i=1}^{n} w_{n+i})}w_1, e^{w_{n+1}}w_2,\dots,e^{w_{2n}}w_{n+1}, w_{n+2},\dots,w_{n+r},$$
$$0, w_{n+r+1}\dots,w_{2n}\big)\,,$$

$$\vdots$$

$$F(w_1,\dots,w_{2n}) = \big(e^{-(\sum_{i=1}^{n} w_{n+i})}w_1, e^{w_{n+1}}w_2,\dots,e^{w_{2n}}w_{n+1}, w_{n+2}\dots,w_{2n},0\big)\,.$$

Conversely, all these hypersurfaces are parallel.

Proof. Assume that M is a parallel hypersurface in G_n. Then in both Riemannian and Lorentzian cases, ξ has one of the forms which are given in the cases (a), (b) and (c) of the Theorem 3.1. Let us start with the case (a), i.e. $\xi = \pm X_0$. Then the following vector fields span the tangent space to M at each point

$$Y_1 = X_1\,,\quad \cdots\,,\ Y_n = X_n\,,\ Y_{n+1} = U_1\,,\quad \cdots\,,\ Y_{2n} = U_n\,. \tag{8}$$

Also, by using the equations (2) and (8), we see that the non-zero connection components are

$$\nabla_{Y_i}Y_i = Y_{n+i}\,,\quad \nabla_{Y_i}Y_{n+i} = -Y_i\,,\quad i = 1,\dots,n\,. \tag{9}$$

Then by (9) and the Gauss formula (4), the second fundamental form is determined by $h(Y_k, Y_l) = 0$, where $k, l \in \{1,\dots,2n\}$. Thus $\nabla^M h = 0$ and the hypersurface is parallel. In order to obtain this hypersurface we put $\partial w_i = Y_i$, where $i = 1,\dots,2n$ and denote by $F\colon M^{2n} \to G_n\colon (w_1,\dots,w_{2n}) \mapsto (F_1(w_1,\dots,w_4),\dots,F_{2n+1}(w_1,\dots, w_{2n}))$ the immersion of the hypersurface. Thus by (8) we obtain

$$\begin{aligned}
&(\partial_{w_1}F_1,\dots,\partial_{w_1}F_{2n+1}) = (0, e^{w_{n+1}}, 0,\dots,0)\,,\\
&\quad\vdots\\
&(\partial_{w_n}F_1,\dots,\partial_{w_n}F_{2n+1}) = (\overbrace{0,\dots,0}^{n\ \text{times}}, e^{w_{2n}}, 0,\dots,0)\,,\\
&(\partial_{w_{n+1}}F_1,\dots,\partial_{w_{n+1}}F_{2n+1}) = (\overbrace{0,\dots,0}^{n+1\ \text{times}}, 1, \overbrace{0,\dots,0}^{n-1\ \text{times}})\,,\\
&\quad\vdots\\
&(\partial_{w_{2n}}F_1,\dots,\partial_{w_{2n}}F_{2n+1}) = (\overbrace{0,\dots,0}^{2n\ \text{times}}, 1)\,.
\end{aligned} \tag{10}$$

Then the general solution of the system (10) is given by

$$F_1 = a_1, F_2 = e^{w_{n+1}}w_1 + a_2,\ \dots, F_{n+1} = e^{w_{2n}}w_n + a_{n+1}\,,$$
$$F_{n+2} = w_{n+1} + a_{n+2},\dots,F_{2n+1} = w_{2n} + a_{2n+1}\,,$$

where $a_1,\dots,a_{2n+1}$ are real constants and give us the immersion which is isometric with the first immersion of the theorem.

Let us consider the case (b), i.e. $\xi = \pm X_r$, where $r \in \{1, \ldots, n\}$. Then the following vector fields span the tangent space to M at each point.

$$\begin{aligned}&Y_0 = X_0\,,\; Y_1 = X_1\,,\quad Y_2 = X_2\,,\;\ldots, Y_{r-1} = X_{r-1}\,,\\ &Y_{r+1} = X_{r+1}\,,\;\ldots\; Y_n = X_n\,,\quad Y_{n+1} = U_1\,,\;\ldots, Y_{2n} = U_n\,.\end{aligned} \tag{11}$$

From the equations (2) and (11) we obtain

$$\nabla_{Y_0} Y_0 = -\Big(\sum_{i=1}^{n} Y_{n+i}\Big)\,,\;\; \nabla_{Y_0} Y_{n+j} = Y_0\,,\;\; \nabla_{Y_i} Y_{n+i} = -Y_i\,,\;\; \nabla_{Y_i} Y_i = Y_{n+i}\,, \tag{12}$$

where $i \neq r$, $i = j = 1, \ldots, n$ and the remaining connection components are zero. Therefore from (12) and the Gauss formula (4), the second fundamental form is given by $h(Y_k, Y_l) = 0$, where $k, l \in \{0, \ldots, r-1, r+1, \ldots, 2n\}$. Then the hypersurface is parallel and if we put $\partial w_{i+1} = Y_i$ where $i = 0, \ldots, r-1$ and put $\partial w_i = Y_i$ where $i = r+1, \ldots, 2n$, then we obtain

$$\begin{aligned}
&(\partial_{w_1} F_1, \ldots, \partial_{w_1} F_{2n+1}) = \big(e^{-(\sum_{i=1}^{n} w_{n+i})}, 0, \ldots, 0\big)\,,\\
&(\partial_{w_2} F_1, \ldots, \partial_{w_1} F_{2n+1}) = \big(0, e^{w_{n+1}}, \overbrace{0, \ldots, 0}^{2n-1 \text{ times}}\;\big)\,,\\
&\quad\vdots\\
&(\partial_{w_{r-1}} F_1, \ldots, \partial_{w_{r-1}} F_{2n+1}) = \big(\;\overbrace{0, \ldots, 0}^{r-2 \text{ times}}, e^{w_{n+r-2}}, \;\overbrace{0, \ldots, 0}^{2n-r+2 \text{ times}}\;\big)\,,\\
&(\partial_{w_r} F_1, \cdots, \partial_{w_r} F_{2n+1}) = \big(\;\overbrace{0, \ldots, 0}^{r-1 \text{ times}}, e^{w_{n+r-1}}, \;\overbrace{0, \ldots, 0}^{2n+1-r \text{ times}}\;\big)\,,\\
&\partial_{w_{r+1}} F_1, \ldots, \partial_{w_{r+1}} F_{2n+1}) = \big(\;\overbrace{0, \ldots, 0}^{r+1 \text{ times}}, e^{w_{n+r+1}}, \;\overbrace{0, \ldots, 0}^{2n-r-1 \text{ times}}\;\big)\,,\\
&\quad\vdots\\
&(\partial_{w_n} F_1, \ldots, \partial_{w_n} F_{2n+1}) = \big(\;\overbrace{0, \ldots, 0}^{n \text{ times}}, e^{w_{2n}}, \overbrace{0, \ldots, 0}^{n \text{ times}}\big)\,,\\
&(\partial_{w_{n+1}} F_1, \ldots, \partial_{w_{n+1}} F_{2n+1}) = \big(\;\overbrace{0, \ldots, 0}^{n+1 \text{ times}}, 1, \overbrace{0, \ldots, 0}^{n-1 \text{ times}}\;\big)\,,\\
&\quad\vdots\\
&(\partial_{w_{2n}} F_1, \ldots, \partial_{w_{2n}} F_{2n+1}) = \big(\;\overbrace{0, \ldots, 0}^{2n \text{ times}}, 1\big)\,.
\end{aligned}$$

From these equations we obtain the following solutions for $F_1, \ldots, F_{2n+1}$

$$\begin{aligned}
&F_1 = e^{-(\sum_{i=1}^n w_{n+i})} w_1 + b_1\,, && F_2 = e^{w_{n+1}} w_2 + b_2, \ldots\,,\\
&F_{r-1} = e^{w_{n+r-2}} w_{r-1} + b_{r-1}\,, && F_r = e^{w_{n+r-1}} w_r + b_r, F_{r+1} = b_{r+1}\,,\\
&F_{r+2} = e^{w_{n+r+1}} w_{r+1} + b_{r+2}, \ldots\,, && F_{n+1} = e^{w_{2n}} w_n + b_n\,,\\
&F_{n+2} = w_{n+1} + b_{n+2}, \ldots\,, && F_{2n+1} = w_{2n} + b_{2n+1}\,,
\end{aligned}$$

where $b_1, \ldots, b_{2n+1}$ are real constants and give us the immersions which are isometric with the immersions given in the cases (2), ... ,(n+1) of the theorem.

Finally we consider the case (c), where $\xi = \pm U_r$, with $r \in \{1, \ldots, n\}$. Then the following vector fields span the tangent space to M at each point

$$\begin{aligned}
&Y_0 = X_0,\ Y_1 = X_1, \ldots, Y_n = X_n,\ Y_{n+1} = U_1, \ldots, Y_{n+r-1} = U_{r-1}\,,\\
&Y_{n+r+1} = U_{r+1}, \ldots, Y_{2n} = U_n\,,
\end{aligned} \tag{13}$$

By a direct computation, using (2) and (13), we obtain the following non-zero connection components

$$\begin{aligned}
&\nabla_{Y_0} Y_0 = -\Big(\sum_{i=1, i\neq r}^{n} Y_{n+i}\Big) - \xi\,, && \nabla_{Y_0} Y_{n+i} = Y_0\,,\ \nabla_{Y_i} Y_{n+i} = -Y_i\,,\\
&\nabla_{Y_i} Y_i = Y_{n+i}\,, && \nabla_{Y_r} Y_r = \xi\,,\ r \neq i,\ i = 1, \ldots, n\,.
\end{aligned} \tag{14}$$

Thus from (14) and the Gauss formula (4) we can see that the second fundamental form is determined by $h(Y_k, Y_l) = C$, where $k, l \in \{0, \ldots, n+r-1, n+r+1, \ldots, 2n\}$ and C is a real constant. Hence, the hypersurface is parallel and if we put $\partial_{w_{i+1}} = Y_i$ where $i = 0, \ldots, n+r-1$ and put $\partial_{w_i} = Y_i$ where $i = n+r+1, \ldots, n$, then by some computations similar to the cases (a) and (b) we obtain

$$\begin{aligned}
&F_1 = e^{-(\sum_{i=1}^n w_{n+i})} w_1 + c_1\,, && F_2 = e^{w_{n+1}} w_2 + c_2, \ldots\,,\\
&F_{n+1} = e^{w_{2n}} w_{n+1} + c_{n+1}\,, && F_{n+2} = w_{n+2} + c_{n+2}, \ldots\,,\\
&F_{n+r} = w_{n+r} + c_{n+r}\,, && F_{n+r+1} = c_{n+n+1}\,,\\
&F_{n+r+2} = w_{n+r+1} + c_{n+r+2}\,, \ldots\,, && F_{2n+1} = w_{2n} + c_{2n}\,.
\end{aligned}$$

where $c_1, \ldots, c_n$ are real constants and give us the immersions which are isometric with the immersions given in the cases $(n+2), \ldots, (2n+1)$ of the theorem.
The converse of theorem can be obtained by a straightforward computation. A similar argument holds for the Lorentzian case. □

Since every totally geodesic hypersurface is parallel, Theorem 3.2 gives us the following result.

Theorem 3.3. *Let $F\colon M^{2n} \to G_n$ be a totally geodesic hypersurface of the class of solvable Riemannian Lie groups (G_n, g) (Lorentzian Lie groups $(G_n, \widehat{g})$). Then there exist local coordinates $(w_1, \ldots, w_{2n})$ on M^{2n} such that this immersion with respect to these coordinates, up to isometries, is given by one of the following* *expressions:*

$$F(w_1,\cdots,w_{2n}) = (0, e^{w_{n+1}}w_1, e^{w_{n+2}}w_2\ldots, e^{w_{2n}}w_n, w_{n+1},\ldots,w_{2n}),$$

$$F(w_1,\ldots,w_{2n}) = \big(e^{-(\sum_{i=1}^{n} w_{n+i})}w_1, 0, e^{w_{n+2}}w_2\ldots, e^{w_{2n}}w_n, w_{n+1}, w_{n+2},\ldots,w_{2n}\big),$$

$$\vdots$$

$$F(w_1,\ldots,w_{2n}) = \big(e^{-(\sum_{i=1}^{n} w_{n+i})}w_1, e^{w_{n+1}}w_2,\ldots, e^{w_{n+r-1}}w_r, 0, e^{w_n+r+1}w_{r+1},\ldots, e^{w_{2n}}w_n, w_{n+1},\ldots,w_{2n}\big),$$

$$\vdots$$

$$F(w_1,\ldots,w_{2n}) = \big(e^{-(\sum_{i=1}^{n} w_{n+i})}w_1, e^{w_{n+1}}w_2,\ldots, e^{w_{2n-1}}w_n, 0, w_{n+1},\ldots,w_{2n}\big),$$

Conversely, these hypersurfaces are totally geodesic.

Proof. Assume that M is a totally geodesic hypersurface in G_n. Then it is sufficient to choose the hypersurfaces which are obtained in the Theorem 3.2 such that for them the second fundamental form vanishes identically. Since in the case that $\xi = \pm U_r$, where $r \in \{1,\ldots,n\}$ we obtain that $h(Y_r, Y_r) = 1 \neq 0$, where Y_r is given in (13). Then the acceptable immersions are the ones which are given in the cases $(1),\ldots,(n+1)$ of the Theorem 3.2. The converse can be verified by a straightforward computation. The Lorentzian case can be proved by a similar argument. □

As a consequence of Theorems 3.2 and 3.3 we have the following result.

Corollary 3.4. *Let (G_n, g) $((G_n, \widehat{g}))$ be the class of solvable Riemannian (Lorentzian) Lie groups. If we denote by $\dim G_n$ the dimension of G_n, then up to isometries we obtain the following results.*

(I) *These spaces always admit an odd number of parallel hypersurfaces which is equal to the $\dim G_n$.*

(II) *These spaces can admit an even or odd number of totally geodesic hypersurfaces which is equal to $\frac{\dim G_n+1}{2}$.*

Proof. Assume that $F\colon M^{2n} \to G_n$ is an isometric immersion of the class of solvable Lie groups. Then up to isometries parallel hypersurfaces can be expressed by $2n+1 = \dim G_n$ cases which are given in the Theorem 3.2. Also from the Theorem 3.3 it follows that $n+1 = \frac{\dim G_n+1}{2}$ of them are totally geodesic. These give us the results given in (I) and (II). □

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