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A NOTE ON ANOTHER CONSTRUCTION OF GRAPHS WITH $4n + 6$ VERTICES AND CYCLIC AUTOMORPHISM GROUP OF ORDER $4n$

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Abstract. The problem of finding minimal vertex number of graphs with a given automorphism group is addressed in this article for the case of cyclic groups. This problem was considered earlier by other authors. We give a construction of an undirected graph having $4n + 6$ vertices and automorphism group cyclic of order $4n$, $n \geq 1$. As a special case we get graphs with $2^k + 6$ vertices and cyclic automorphism groups of order 2^k. It can revive interest in related problems.

1. Introduction

This article addresses a problem in graph representation theory of finite groups - finding undirected graphs with a given full automorphism group and minimal number of vertices. All graphs in this article are undirected and simple.

It is known that finite graphs universally represent finite groups: for any finite group G there is a finite graph Γ such that $\operatorname{Aut}(\Gamma) \simeq G$, see Frucht [8]. It was proved by Babai [2] constructively that for any finite group G (except cyclic groups of order 3, 4 or 5) there is a graph Γ such that $\operatorname{Aut}(\Gamma) \simeq G$ and $|V(\Gamma)| \leq 2|G|$ (there are 2 G-orbits having $|G|$ vertices each). For certain group types such as symmetric groups Σ_n, dihedral groups D_{2n} and elementary abelian 2-groups $(\mathbb{Z}/2\mathbb{Z})^n$ graphs with smaller number of vertices (respectively, n, n and $2n$) are obvious.

In the recent decades the problem of finding $\mu(G) = \min\limits_{\Gamma:\ \operatorname{Aut}(\Gamma) \simeq G} |V(\Gamma)|$ for specific groups G does not seem to have been very popular although minimal graphs and directed graphs for most finite groups have not been found. See Babai [3] for an exposition of this area.

There are 10-vertex graphs having automorphism group $\mathbb{Z}/4\mathbb{Z}$, this fact is mentioned in Bouwer and Frucht [5] and Babai [2]. There are 12 such 10-vertex graph isomorphism types, see [6].

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In this paper we reminisce about the bound $\mu(G) = \min_{\Gamma:\mathrm{Aut}\,(\Gamma)\simeq G} |V(\Gamma)| \leq 2|G|$ not being sharp for $G \simeq \mathbb{Z}/4n\mathbb{Z}$, for any natural $n \geq 1$. Namely, for any $n \geq 1$ there is an undirected graph Γ on $4n+6$ vertices such that $\mathrm{Aut}\,(\Gamma) \simeq \mathbb{Z}/4n\mathbb{Z}$. The number of orbits is 3.

Graphs with abelian automorphism groups have been investigated in Arlinghaus [1]. In Harary [9] there is a claim (referring to Merriwether) that if G is a cyclic group of order 2^k, $k \geq 2$, then the minimal number of graph vertices is $2^k + 6$. In this paper we exhibit such graphs with the number of vertices $4n+6$, $n \geq 1$, and give an explicit construction. The construction works for graphs with any $n \geq 1$, but if $n = 2^k$, $k \geq 3$, we get graphs for which the number of vertices is smaller than the Babai's bound.

We use standard notations of graph theory, see Diestel [7]. Adjacency of vertices i and j is denoted by $i \sim j$ (edge (i,j)). For a graph $\Gamma = (V,E)$ the subgraph induced by $X \subseteq V$ is denoted by $\Gamma[X]$: $\Gamma[X] = \Gamma - \overline{X}$. The set $\{1,2,\ldots,n\}$ is denoted by V_n. The undirected cycle on n vertices is denoted by C_n. The cycle notation is used for permutations. Given a function $f\colon A \to B$ and a subset $C \subseteq A$ we denote the restriction of f to C by $f|_C$.

2. Main results

2.1. The graph Γ_n.

Definition 2.1. Let $n \geq 1$, $n \in \mathbb{N}$, $m = 4n$. Let $V(\Gamma_n) = V_{m+6} = \{1,2,\ldots,m+6\}$ and edges be given by the following adjacency description. We define 8 types of edges.

(1) $i \sim i+1$ for all $i \in V_{m-1}$ and $1 \sim m$.
(It implies that $\Gamma_n[1,2,\ldots,m] \simeq C_m$.)

(2) $m+1 \sim i$ with $i \in V_m$ iff $i \equiv 1$ or $2 (\mathrm{mod}\ 4)$.

(3) $m+2 \sim i$ with $i \in V_m$ iff $i \equiv 2$ or $3 (\mathrm{mod}\ 4)$.

(4) $m+3 \sim i$ with $i \in V_m$ iff $i \equiv 3$ or $0 (\mathrm{mod}\ 4)$.

(5) $m+4 \sim i$ with $i \in V_m$ iff $i \equiv 0$ or $1 (\mathrm{mod}\ 4)$.

(6) $m+5 \sim i$ with $i \in V_m$ iff $i \equiv 1 (\mathrm{mod}\ 2)$.

(7) $m+6 \sim i$ with $i \in V_m$ iff $i \equiv 0 (\mathrm{mod}\ 2)$.

(8) $m+1 \sim m+5 \sim m+3$, $m+2 \sim m+6 \sim m+4$.

Definition 2.2. Denote $O_1 = \{1,2,\ldots,m\}$, $O_2 = \{m+1, m+2, m+3, m+4\}$, $O_3 = \{m+5, m+6\}$. Note that O_i are the $\mathrm{Aut}\,(\Gamma_n)$-orbits.

2.2. The special case $n = 1$.

A graph with automorphism group $\mathbb{Z}/4\mathbb{Z}$ and minimal number of vertices (10) and edges (18) was exhibited in Bouwer and Frucht [5], p.58. Γ_1 (which is not isomorphic to the Bouwer-Frucht graph) is shown in Fig. 1. It can be thought as embedded in the 3D space. It is planar but a plane embedding is not given here. $\mathrm{Aut}\,(\Gamma_1) \simeq \mathbb{Z}/4\mathbb{Z}$ is generated by the vertex permutation $g = (1,2,3,4)(5,6,7,8)(9,10)$.

Subgraphs $\Gamma_1[1,2,3,4,5,7,9]$ and $\Gamma_1[1,2,3,4,6,8,10]$ which can be thought as being drawn above and below the orbit $\{1,2,3,4\}$ are interchanged by g.

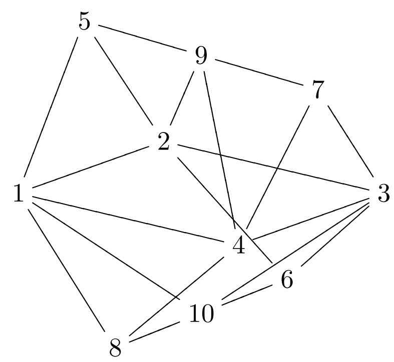

Fig. 1. – Γ_1

2.3. **Automorphism group of Γ_n.**

Proposition 2.3. *Let $n \geq 1$, $n \in \mathbb{N}$, $m = 4n$. Let Γ_n be defined as above. For any n,* $\mathrm{Aut}\,(\Gamma_n) \simeq \mathbb{Z}/m\mathbb{Z}$.

Proof. We will show that $\mathrm{Aut}\,(\Gamma_n) = \langle g \rangle$, where $g = (1,2,\ldots,m)(m+1,m+2,m+3,m+4)(m+5,m+6)$.

Inclusion $\langle g \rangle \leq \mathrm{Aut}\,(\Gamma_n)$ is proved by showing that g maps an edge of each type to an edge.

Let us prove the inclusion $\mathrm{Aut}\,(\Gamma_n) \leq \langle g \rangle$. Let $f \in \mathrm{Aut}\,(\Gamma_n)$. We will show that $f = g^\alpha$ for some α. There are two subcases $n \neq 2$ and $n = 2$.

For any $n \geq 1$ the vertices $m+5$ and $m+6$ are the only vertices having eccentricity 3, so they must form an orbit.

Let $n \neq 2$. Suppose $f(1) = k$. Since $n \neq 2$, we have that $\deg(1) = 5$, $\deg(v) = \frac{m}{2}+1 \neq 5$ for any $v \in O_2$, therefore $f(1) \in O_1$. Moreover, f stabilizes setwise both O_1 and O_2. Consider the f-image of the edge $(1, m+5)$. $(f(1), f(m+5)) = (k, f(m+5))$ must be an edge, therefore

(1) if $k \equiv 1 \pmod 2$, then $f(m+5) = m+5$,

(2) if $k \equiv 0 \pmod 2$, then $f(m+5) = m+6$.

It follows that $f|_{O_3} = g^{k-1}$.

Consider the f-image of $\Gamma_n[1, 2, m+1, m+5]$, denote its isomorphism type by Γ_0, see Fig. 5.

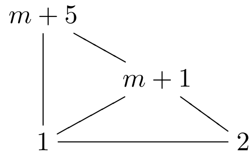

Fig. 5. – $\Gamma_0 \simeq \Gamma_n[1, 2, m+1, m+5]$

Vertex 2 must be mapped to a $\Gamma_n[O_1]$-neighbour of k. For any $k \in O_1$ there are two triangles containing the vertex k and a vertex adjacent to k in $\Gamma_n[O_1]$. Taking into account that $f(m+5) \in O_3$ we check that there is only one suitable induced Γ_n-subgraph – containing k, another vertex in O_1 adjacent to k and a vertex in O_3 – which is isomorphic to $\Gamma_n[1, 2, m+1, m+5]$.

It follows that in each case we must have $f(2) \equiv k+1 (\mathrm{mod}\ m)$. By similar arguments for all $j \in \{1, 2, \ldots, m\}$ it is proved that $f(j) \equiv (k-1)+j (\mathrm{mod}\ m)$, thus $f|_{O_1} = g^{k-1}$.

Finally we describe $f|_{O_2}$. It can also be found considering Γ_n-subgraphs isomorphic to Γ_0, but we will use edge inspection. Consider the f-images of the edges $(1, m+1)$ and $(1, m+4)$. Vertex pairs $(f(1), f(m+1)) = (k, f(m+1))$ and $(f(1), f(m+4))$ must be edges, therefore we can deduce images of all O_2 vertices.

If $n \neq 2$ and $f(1) = k$, then $f = g^{k-1}$, therefore $f \in \langle g \rangle$.

In the special case $n = 2$ we also consider f-images of $\Gamma_1[1, 2, 9, 13]$ and find suitable Γ_1-subgraphs isomorphic to Γ_0. It is shown similarly to the above argument that f can be expressed as a power of g and hence $f \in \langle g \rangle$. □

2.4. **Abelian 2-groups.**

It is known that $\mu(\mathbb{Z}/2^k\mathbb{Z}) = 2^k + 6$, it was proved in [1]. We note that it can be proved using the following steps. First notice that Γ with $\mathrm{Aut}\,(\Gamma) \simeq \mathbb{Z}/2^k\mathbb{Z}$ must have a least one orbit of size 2^k, thus $|V(\Gamma)| \geq 2^k$. Eliminate possibilities $2^k \leq |V(\Gamma)| < 2^k + 6$ by considering orbits of size 1, 2 or 4, which can be removed, or which cause $\mathrm{Aut}\,(\Gamma)$ to contain a dihedral subgroup $D_{2 \cdot 2^k}$.

We also give an implication – a bound for $\mu(G)$ if G is an abelian 2-group.

Proposition 2.4. *Let G be an abelian 2-group: $G \simeq \prod_{i=1}^{k} (\mathbb{Z}/2^i\mathbb{Z})^{n_i}$, $n_i \in \mathbb{N} \cap \{0\}$. Then $\mu(G) \leq 2n_1 + \sum_{i=2}^{k} n_i(2^i + 6)$.*

Proof. Denote $(\mathbb{Z}/2^i\mathbb{Z})^{n_i}$ by G_i, $G \simeq \prod_{i=1}^{k} G_i$. We can construct a sequence of graphs $\Delta_{i,n}$, $i \in \mathbb{N}$, $n \in \mathbb{N}$, inductively using complements and unions as follows. For $i > 1$ define $\Delta_{i,1} = \Gamma_{2^{i-2}}$ and define $\Delta_{1,1} = K_2$. Define inductively $\Delta_{i,n}$:

$\Delta_{i,n} = \overline{\Delta}_{i,n-1} \cup \Delta_{i,1}$. Since $\overline{\Delta}_{i,n-1} \not\simeq \Delta_{i,1}$ and $\overline{\Delta}_{i,j}$ is connected for all constructed graphs, we have inductively that $\mathrm{Aut}\,(\Delta_{i,n}) \simeq \mathrm{Aut}\,(\Delta_{i,n-1}) \times (\mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2^i\mathbb{Z})^n$.

Define $\Gamma = \bigcup_{i=1}^{k} \Delta_{i,n_i}$. For different values of i the Δ_{i,n_i} are nonisomorphic therefore $\mathrm{Aut}\,(\Gamma) \simeq \prod_{i=1}^{k} G_i \simeq G$. Thus $\mu(G) \leq |V(\Gamma)| = \sum_{i=1}^{k} |V(\Delta_{i,n_i})| = 2n_1 + \sum_{i=2}^{k} n_i(2^i + 6)$. □

2.5. Other graphs and developments.

We briefly describe without proofs graphs $\Gamma_{m,n}$ having $m^n + m$ vertices and cyclic automorphism group of order m^n, $m \geq 6$, $n \geq 2$. Existence of such graphs is mentioned in [9], see also [1]. We use the construction of graphs with $2m$ vertices having cyclic automorphism group of order m ($m \geq 6$) given in [11]. Let $V(\Gamma_{m,n}) = W \cup W'$, where $W = \{0, 1, \ldots, m^n - 1\}$, $W' = \{0', 1', \ldots, (m-1)'\}$. The edges of $\Gamma_{m,n}$ are defined as follows: 1) $\Gamma_{m,n}[W]$ and $\Gamma_{m,n}[W']$ are natural cycles of order m^n and m, respectively, with edges $(i, i+1)$, 2) for any vertex $i' \in W'$ there are $3m^{n-1}$ edges of type $(i', jm + i (\mathrm{mod}\ m^n))$, $(i', jm + i + 1 (\mathrm{mod}\ m^n))$ and $(i', jm + i - 2 (\mathrm{mod}\ m^n))$, $0 \leq i' \leq m - 1$, $0 \leq j \leq m^{n-1} - 1$. It can be checked that $\mathrm{Aut}\,(\Gamma_{m,n}) \simeq \mathbb{Z}/m^n\mathbb{Z}$, there are 2 orbits – W and W'.

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