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# INFINITESIMAL CR AUTOMORPHISMS FOR A CLASS OF POLYNOMIAL MODELS

Martin Kolář and Francine Meylan

Abstract. In this paper we study infinitesimal CR automorphisms of Levi degenerate hypersurfaces. We illustrate the recent general results of [18], [17], [15], on a class of concrete examples, polynomial models in $\mathbb{C}^3$ of the form $\mathsf{Im}\ w = \mathsf{Re}\,(P(z)\overline{Q(z)})$, where $P$ and $Q$ are weighted homogeneous holomorphic polynomials in $z = (z_1, z_2)$. We classify such models according to their Lie algebra of infinitesimal CR automorphisms. We also give the first example of a non monomial model which admits a nonlinear rigid automorphism.

## 1. Introduction

The study of possible complexity of automorphisms of CR manifolds has a long history. The classical case of Levi nondegenerate hypersurfaces was studied by Poincaré, Cartan, Tanaka, Chern and Moser, Vitushkin and many others. Most results on symmetries in this class are negative, indicating that interesting symmetries are very rare. In particular, Beloshapka and Kruzhilin showed that if the hypersurface is not locally spherical, then its symmetries are linear in Chern-Moser normal coordinates.

Similar results were obtained for finite type hypersurfaces in $\mathbb{C}^2$. In particular, all such hypersurfaces which admit a nonlinear symmetry are biholomophically equivalent to the model $Im\ w = |z|^k$.

Several recent results indicate that the situation is more interesting for finite type hypersurfaces in higher dimensions, and also for infinite type hypersurfaces in $\mathbb{C}^2$ (see e.g. [14], [18]).

On the one hand, the simple example of a finite type hypersurface in $\mathbb{C}^3$

$$\mathsf{Im}\ w = \mathsf{Re}\ z_1 \bar{z}_2^l \tag{1.1}$$

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which admits an infinitesimal automorphism of the form

$$Y = iz_2^l \partial_{z_1} \,,$$

where $l$ is an arbitrary integer, shows that infinitesimal automorphisms may have coefficients of arbitrary degree. On the other hand, as shown by the results of [18], the weighted degree of such coefficients is controlled by the Catlin multitype of the manifold. Note that when $l = 1$, we recover a Levi nondegenerate hyperquadric: in this case, the real dimension of the Lie algebra of the infinitesimal CR automorphisms is known to be 15.

Let $P$ and $Q$ be weighted homogeneous holomorphic polynomials in $z = (z_1, z_2) \in \mathbb{C}^2$, and let $M$ given by

$$M = \{(z, w) \in \mathbb{C}^3 : \mathsf{Im}\, w = P\overline{Q} + Q\overline{P}\} \tag{1.2}$$

be holomorphically non degenerate hypersurface. We will consider the following problem, generalizing in a natural way the above example:

- *Characterize in terms of $P$ and $Q$ the Lie algebra of infinitesimal CR automorphisms of $M$.*

Note that in dimensions higher than three the hypersurface given by (1.2) is necessarily holomorphically degenerate.

As a main result of this paper, we completely classify such models in terms of $P$ and $Q$ according to their Lie algebra aut$(M, 0)$ of infinitesimal CR automorphisms at 0 (see Theorems 4.1 and 4.2).

As a special case of (1.2) one obtains monomial hypersurfaces. We will call a model of the form (1.2) monomial, if there exist coordinates in which both $P$ and $Q$ are monomials (see Section 3 for the precise definition). The model (1.1) is an example of such a hypersurface. In fact, all the previously known examples of hypersurfaces which admit a nonlinear rigid infinitesimal CR automorphism are based on monomials ([2], [17]).

In Section 4 we will also construct the first example of a non monomial model which admits a nonlinear rigid automorphism (see Theorem 4.3).

The paper is organized as follows. Section 2 gives a self contained summary of the results obtained in [18], [17], [15]. Some of them (but not all) will be used in Section 4 to prove the main results of the paper. Section 3 contains several auxilliary results, which for each particular graded component of aut$(M, 0)$ characterize manifolds for which this component is nonzero. In case of rotations, i.e. the $\mathfrak{g}_0$ part, containing infinitesimal CR automorphisms of weight 0, such description is provided separately for real, imaginary and nilpotent rotations. Section 4 contains the statements and proofs of the main results of the paper.

## 2. Preliminaries and previous results

Recall that a hypersurface $M \subset \mathbb{C}^3$ is of finite Catlin multitype $(m_1, m_2)$ at $p \in M$ if there exist local holomorphic coordinates centered at $p$ such that $M$ is given by

$$\mathsf{Im}\, w = P_C(z, \bar{z}) + o_{\Lambda_M}(1) \,, \tag{2.1}$$

where $P_C$ is a weighted homogeneous polynomial of weighted degree one with respect to the weights $\Lambda_M = (\mu_1, \mu_2) = (\frac{1}{m_1}, \frac{1}{m_2})$ and $o_{\Lambda_M}(1)$ denotes terms of weighted degree bigger than one. Moreover, by the definition of multitype, $\Lambda_M$ is the lexicographically smallest weight with this property. The algebraic hypersurface $M_H$, defined by

$$\mathsf{Im}\, w = P_C(z, \bar{z})\,, \tag{2.2}$$

is called the model hypersurface, or shortly model.

Recall that the model $M_H$ is holomorphically nondegenerate at $p \in M_H$ if there is no germ at $p$ of a holomorphic vector field $X$ (complex) tangent to $M_H$. We refer the reader to [15] for more details.

We now summarize the main results from [18], [17], [15]. Some of them will be used in Section 4 to prove the main results of this paper.

Let us consider a holomorphically nondegenerate model given by (2.2).

It was proved in [18], that the Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \mathrm{aut}(M_H, 0)$ of $M_H$ admits a weighted decomposition, which we write in the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^{2} \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_n \oplus \mathfrak{g}_1\,, \tag{2.3}$$

where the vector fields in $\mathfrak{g}_c$ commute with $W = \partial_w$ and the nonzero vector fields in $\mathfrak{g}_n$ do not commute with $W$. Their weights in both cases lie in the interval $(0, 1)$. Note that by a result of [18], vector fields in $\mathfrak{g}_j$ with $j < 0$ are regular and vector fields in $\mathfrak{g}_0$ are linear.

The following theorem, which deals with $\mathfrak{g}_n$, was obtained in [15].

**Theorem 2.1.** *Let $P_C(z, \bar{z})$ be a weighted homogeneous polynomial of degree 1 with respect to the multitype weights, such that the hypersurface*

$$M_H := \{\mathsf{Im}\, w = P_C(z, \bar{z})\}\,, \quad (z, w) \in \mathbb{C}^2 \times \mathbb{C}\,, \tag{2.4}$$

*is holomorphically nondegenerate. Let $\mathfrak{g}_n$ in* (2.3) *satisfy*

$$\dim \mathfrak{g}_n > 0\,. \tag{2.5}$$

*Then $M_H$ is biholomorphically equivalent to*

$$\mathsf{Im}\ w = \mathsf{Re}\, z_1 \bar{z}_2^l \tag{2.6}$$

*or*

$$\mathsf{Im}\ w = |z_1|^2 \pm |z_2|^{2l}\,. \tag{2.7}$$

The next result deals with the component $\mathfrak{g}_c$ ([17]).

**Definition 2.2.** Let $Y$ be a weighted homogeneous vector field. A pair of finite sequences of holomorphic weighted homogeneous polynomials $\{U^1, \dots, U^n\}$ and

$\{V^1, \dots, V^n\}$ is called a symmetric pair of $Y$-chains if

$$Y(U^n) = 0\,, \quad Y(U^j) = c_j U^{j+1}\,, \quad j = 1, \dots, n-1\,, \tag{2.8}$$

$$Y(V^n) = 0\,, \quad Y(V^j) = d_j V^{j+1}\,, \quad j = 1, \dots, n-1\,, \tag{2.9}$$

where $c_j$, $d_j$ are non zero complex constants, which satisfy

$$c_j = -\bar{d}_{n-j}\,. \tag{2.10}$$

If the two sequences are identical we say that $\{U^1, \dots, U^n\}$ is a symmetric $Y$-chain.

The following theorem shows that in general the elements of $\mathfrak{g}_c$ arise from symmetric pairs of chains.

**Theorem 2.3.** *Let $M_H$ be a holomorphically nondegenerate hypersurface given by* (2.4)*, which admits a nontrivial $Y \in \mathfrak{g}_c$. Then $P_C$ can be decomposed in the following way*

$$P_C = \sum_{j=1}^{M} T_j\,, \tag{2.11}$$

*where each $T_j$ is given by*

$$T_j = \mathsf{Re}\Big(\sum_{k=1}^{N_j} U_j^k \overline{V_j^{N_j-k+1}}\Big)\,, \tag{2.12}$$

*where $\{U_j^1, \dots, U_j^{N_j}\}$ and $\{V_j^1, \dots, V_j^{N_j}\}$ are a symmetric pair of $Y$-chains.*

*Conversely, if $Y$ and $P_C$ satisfy* (2.8)–(2.12)*, then $Y \in \mathfrak{g}_c$.*

Note that $Y$ is uniquely and explicitly determined by $P$ (see [17]). Hence for a given hypersurface this result also provides a constructive tool to determine $\mathfrak{g}_c$, and shows that

$$\dim \mathfrak{g}_c \leq 1\,. \tag{2.13}$$

The description of the remaining component $\mathfrak{g}_1$ was obtained in Theorem 4.7 of [18].

**Definition 2.4.** We say that $P_C$ is balanced if it can be written as

$$P_C(z, \bar{z}) = \sum_{|\alpha|_\Lambda = |\bar{\alpha}|_\Lambda = 1} A_{\alpha, \bar{\alpha}} z^\alpha \bar{z}^{\bar{\alpha}}\,, \tag{2.14}$$

for some nonzero pair of real numbers $\Lambda = (\lambda_1, \lambda_2)$, where

$$|\alpha|_\Lambda := \lambda_1 \alpha_1 + \lambda_2 \alpha_2\,.$$

The associated hypersurface $M_H$ is called a balanced hypersurface.

Note that $P_C$ is balanced if and only if the linear vector field

$$Y = \lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2}$$

is a complex reproducing field in the terminology of [18], i.e., $Y(P_C) = P_C$.

**Theorem 2.5.** *The component* $\mathfrak{g}_1$ *satisfies*

$$\dim \mathfrak{g}_1 = 1 \tag{2.15}$$

*if and only if in suitable multitype coordinates* $M_H$ *is a balanced hypersurface. Otherwise,* $\dim \mathfrak{g}_1 = 0$.

## 3. Auxiliary lemmata

In this section we prove some necessary lemmata, needed to obtain the main results. We will assume throughout the section that $M$ is holomorphically nondegenerate.

**Definition 3.1.** The model of the form (1.2) is called monomial, if there exist multitype coordinates in which both $P$ and $Q$ are monomials.

**Lemma 3.2.** *Let* $M$ *be given by* (1.2)*. If* $M$ *is monomial, then it is a balanced hypersurface.*

**Proof.** Let $P\bar{Q} = z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}$. Then

$$X = \sum_{j=1}^{2} \lambda_j z_j \partial_{z_j}$$

is a complex reproducing field provided that $\lambda_1, \lambda_2$ satisfy

$$\begin{aligned} \lambda_1\alpha_1 + \lambda_2\alpha_2 &= 1\,, \\ \lambda_1\beta_1 + \lambda_2\beta_2 &= 1\,, \end{aligned} \tag{3.1}$$

which gives a unique solution, since $M$ is holomorphically nondegenerate and hence $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ are linearly independent. □

In the sequel we will use the following terminology. We will say $X \in \mathfrak{g}_0$ is a real (respectively imaginary) rotation if it is diagonal in normal form with purely real (respectively imaginary) coefficients. We will say that $X \in \mathfrak{g}_0$ is a nilpotent rotation if the diagonal terms vanish in normal form. Note that by a result of [18], the diagonal part and the nilpotent part of a rotation are also rotations, therefore it is enough to study diagonal and nilpotent rotations.

**Lemma 3.3.** *Let* $M$ *be a Levi degenerate model given by* (1.2)*. If* $P$ *and* $Q$ *are monomials, then* $X \in \mathfrak{g}_0$ *is diagonal in normal form if and only if it is diagonal in the given coordinates.*

**Proof.** The proof is immediate, applying a rotation in general form to a holomorphically nondegenerate monomial model. □

**Lemma 3.4.** *Let* $M$ *be given by* (1.2)*. Then* $M$ *admits an imaginary rotation if and only if* $M$ *is a balanced hypersurface. Moreover,* $X$ *is an imaginary rotation if and only if* $ciX$*, for some* $c \in \mathbb{R}$*, is a complex reproducing field.*

**Proof.** Let $M$ be balanced and $R$ be the associated complex reproducing field given by

$$R = \sum_{j=1}^{2} \lambda_j z_j \partial_{z_j} \,.$$

Then $M$ admits an imaginary rotation

$$X = iR = i\sum_{j=1}^{2} \lambda_j z_j \partial_{z_j} \,.$$

Conversely, let $M$ admit an imaginary rotation given in normal form by

$$X = i\sum_{j=1}^{2} \lambda_j z_j \partial_{z_j}$$

with $\lambda_j$ real. If $P$ and $Q$ are monomials in such coordinates, then $M$ is balanced by Lemma 3.2.

If $P$ and $Q$ are not both monomials, let us consider an arbitrary monomial $z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}$ in the expansion of $P\bar{Q}$. We obtain

$$\lambda_1(\alpha_1 - \beta_1) + \lambda_2(\alpha_2 - \beta_2) = 0\,.$$

Take another monomial in the expansion of $\mathsf{Re}\, P\bar{Q}$, which we write in the form

$$z_1^{\alpha_1-\gamma} z_2^{\alpha_1+\gamma\frac{\mu_1}{\mu_2}} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}\,.$$

We obtain

$$\lambda_1(\alpha_1 - \gamma - \beta_1) + \lambda_2(\alpha_2 + \gamma\frac{\mu_1}{\mu_2} - \beta_2) = 0\,.$$

Hence

$$\lambda_1 - \lambda_2\frac{\mu_1}{\mu_2} = 0\,,$$

which implies that $\frac{\lambda_1}{\lambda_2} = \frac{\mu_1}{\mu_2}$. It follows that $ciX$, for suitable $c \in \mathbb{R}$, is a complex reproducing field. □

**Lemma 3.5.** *Let $M$ be given by* (1.2). *$M$ admits a real rotation if and only if $M$ is monomial.*

**Proof.** Let

$$X = \lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2} \tag{3.2}$$

be a real rotation. For any monomial $z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}$ in the expansion of $P\bar{Q}$ we have

$$\lambda_1(\alpha_1 + \beta_1) + \lambda_2(\alpha_2 + \beta_2) = 0\,. \tag{3.3}$$

Since

$$\mu_1(\alpha_1 + \beta_1) + \mu_2(\alpha_2 + \beta_2) = 1\,, \tag{3.4}$$

and $(\lambda_1, \lambda_2)$ and $(\mu_1, \mu_2)$ are linearly independent, we obtain a unique solution for $\alpha_1 + \beta_1$ and $\alpha_2 + \beta_2$. Hence the total degrees in $z_1$ and $z_2$ in $P\bar{Q}$ are constant, and both $P$ and $Q$ have to be monomials. The converse follows immediately from (3.3). □

**Lemma 3.6.** *Let $M$ be given by* (1.2)*. If $M$ admits a tubular symmetry, then it is biholomorphically equivalent to*

$$\mathsf{Im}\, w = \mathsf{Re}\, z_1 \overline{z_2}^{\,l}\,, \tag{3.5}$$

*or*

$$\mathsf{Im}\, w = \mathsf{Re}\, z_1 z_2^l \overline{z_2^l}\,. \tag{3.6}$$

*Moreover, the real dimension of the tubular symmetries for* (3.6) *is one.*

**Proof.** After a change of multitype coordinates (possibly introducing pluriharmonic terms), we can assume that $X = \partial_{z_1}$ is a tubular symmetry. We have

$$P_{z_1}\overline{Q} + \overline{P}Q_{z_1} + \overline{P_{z_1}}Q + P\overline{Q_{z_1}} + H_{z_1} + \overline{H_{z_1}} = 0\,,$$

where $H$ is a holomorphic polynomial. $H = 0$ leads to (3.6) by assuming that the maximum degree in $z_1$ is realized in $P$. $H \neq 0$ leads to (3.5). It is immediate to verify by direct computation that (3.6) does not admit any other tubular symmetry. □

**Lemma 3.7.** *Let $M$ be given by* (1.2)*. If $M$ admits a nilpotent rotation, then it is biholomorphically equivalent to*

$$\mathsf{Im}\, w = \mathsf{Re}\, i z_1^{k+1}\overline{z_1^k z_2}\,. \tag{3.7}$$

*Moreover, the real dimension of the nilpotent rotations is one.*

**Proof.** After a linear change of coordinates, we can assume that $X = z_1\partial_{z_2}$ is a nilpotent rotation, i.e. $\mathsf{Re}\, X(P\overline{Q} + \overline{P}Q) = 0$. We have

$$z_1P_{z_2}\overline{Q} + \overline{P}z_1Q_{z_2} + \overline{z_1P_{z_2}}Q + P\overline{z_1Q_{z_2}} = 0\,.$$

Without any loss of generality, let the maximum degree in $z_2$ be realized in $P$. It follows that either $X(Q) = 0$ or $P_{z_2} = cQ_{z_2}$. This leads to (3.7). We verify by direct computation that (3.7) does not admit any other rotation which is not diagonal in the given coordinates. □

Let us remark that the model given by (1.2) is closely related to the hyperquadric $H$ of mixed signature $\mathsf{Im}\, w' = \mathsf{Re}\, z_1'\bar{z}_2'$. In order to formulate the next result, let us consider the mapping between the two hypersurfaces,

$$f_{PQ} : \mathbb{C}^3 \to \mathbb{C}^3$$

given by

$$z_1' = P(z_1, z_2)\,, \quad z_2' = Q(z_1, z_2)\,, \quad w' = w\,,$$

and let us denote by $J(P, Q)$ the Jacobian of this mapping, i.e. the determinant of the matrix

$$\begin{pmatrix} \partial_{z_1}P & \partial_{z_2}P \\ \partial_{z_1}Q & \partial_{z_2}Q \end{pmatrix}. \tag{3.8}$$

**Lemma 3.8.** *Let $M$ be given by* (1.2)*. Then $M$ admits a generalized rotation if and only if $J(P, Q)$ divides (in the space of homogeneous polynomials) $QQ_{z_j}$, $j = 1, 2$.*

**Proof.** Let $X$ be a generalized rotation. Let first the weighted degree of $P$ be the same as the weighted degree of $Q$. Applying $X$ to $\mathsf{Im}\, w - P\overline{Q} - Q\overline{P}$, we obtain

$$X(P)\overline{Q} + X(Q)\overline{P} = 0\,, \tag{3.9}$$

which implies that

$$Q = \alpha P\,. \tag{3.10}$$

But this is impossible since $M$ is holomorphically non degenerate. Suppose now that the weighted degree of $P$ is strictly less than the weighted degree of $Q$. Applying $X$ to $\mathsf{Im}\, w - P\overline{Q} - Q\overline{P}$, we obtain

$$X(Q) = 0 \qquad X(P) = icQ\,, \tag{3.11}$$

where $c \in \mathbb{R} \setminus \{0\}$. Putting

$$X = a_1 \partial_{z_1} + a_2 \partial_{z_2}\,, \tag{3.12}$$

where $a_1$, $a_2$ are holomorphic functions, we get, using (3.11)

$$a_1 Q_{z_1} + a_2 Q_{z_2} = 0\,. \tag{3.13}$$

It implies that

$$X = h(z_1, z_2)(Q_{z_2}, -Q_{z_1})$$

for some meromorphic function $h$. Using the second equation $X(P) = icQ$, we obtain

$$h(z_1, z_2) J(P, Q) = icQ(z_1, z_2)\,.$$

Hence

$$X = ic \frac{Q}{J(P, Q)} (Q_{z_2}, -Q_{z_1})\,,$$

which leads to the claim. The converse is immediate. This achieves the proof of the lemma. □

## 4. The main results

In this section we will prove the following two main theorems. Since we know that $\dim \mathfrak{g}_c$ is either 0 or 1, we consider the two cases separately.

**Theorem 4.1.** *Let $M$ be a Levi degenerate model given by* (1.2). *Suppose that* $\dim \mathfrak{g}_c = 1$. *Then the possible dimensions of* $\mathfrak{g} = \mathrm{aut}(M, 0)$ *are* $\{10, 7, 6, 3\}$. *Moreover,*

- $\dim \mathfrak{g} = 10$ *if and only if $M$ is biholomorphically equivalent to* (1.1),
- $\dim \mathfrak{g} = 7$ *if and only if $M$ is biholomorphically equivalent to* (3.6),
- $\dim \mathfrak{g} = 6$ *if and only if $M$ is monomial and not equivalent to* (1.1) *or* (3.6),
- $\dim \mathfrak{g} = 3$ *if and only if $M$ is not monomial.*

Let us note that verifying whether $\dim \mathfrak{g}_c = 1$ using Lemma 3.8 is immediate for monomial models.

**Theorem 4.2.** *Let $M$ be a Levi degenerate model given by* (1.2). *Suppose that* $\dim \mathfrak{g}_c = 0$. *Then the possible dimensions of* $\mathfrak{g} = \mathrm{aut}(M, 0)$ *are* $\{6, 5, 4, 2\}$. *Moreover,*

- $\dim \mathfrak{g} = 6$ *if and only if $M$ is biholomorphically equivalent to* (3.7),

- $\dim \mathfrak{g} = 5$ *if and only if $M$ is monomial and not biholomorphically equivalent to* (3.7)*,*
- $\dim \mathfrak{g} = 4$ *if and only if $M$ is balanced and not monomial,*
- $\dim \mathfrak{g} = 2$ *for all the other models.*

The following example shows that nonmomial models admitting a generalized rotation indeed exist.

**Theorem 4.3.** *There exists a non-monomial model $M$ such that* $\dim \mathfrak{g}_c > 0$.

**Proof.** Take

$$P(z_1, z_2) = iz_1^2 z_2^3(z_1 - z_2)$$
$$Q(z_1, z_2) = 3z_1^3 z_2^5(z_1 - z_2) \tag{4.1}$$

and

$$X = z_1 z_2^2(5z_1 - 6z_2)\partial_{z_1} - z_2^3(4z_1 - 3z_2)\partial_{z_2} \, . \tag{4.2}$$

It is easy to check that $X(P) = iQ$ and $X(Q) = 0$, and therefore $X$ is a generalized rotation for $M$ given by $\mathsf{Im}\, w = \mathsf{Re}\, P\bar{Q}$. Clearly, this hypersurface is not monomial, since $P$ and $Q$ vanish along three complex lines. □

In order to prove the two main theorems, we will need the following result.

**Theorem 4.4.** *Let $M$ be a Levi degenerate model given by* (1.2)*. Then* $1 \leq \dim \mathfrak{g}_0 \leq 4$*. Moreover,*

- $\dim \mathfrak{g}_0 = 4$ *if and only if $M$ biholomorphically equivalent to* (3.7)*,*
- $\dim \mathfrak{g}_0 = 3$ *if and only if $M$ is monomial and not equivalent to* (3.7)*,*
- $\dim \mathfrak{g}_0 = 2$ *if and only if $M$ is balanced and not monomial.*

**Proof.** First we note that the real dimension of imaginary rotations is at most one, since otherwise there will be two complex reproducing fields, contradicting holomorphic degeneracy, using Lemma 3.4. Recall that the real dimension of real rotations is also at most one, because of the existence of the Euler field. By Lemma 3.7, the only model which admits a nilpotent rotation is (3.7), which is monomial, hence it also admits a real and imaginary rotation. It follows that $\dim \mathfrak{g}_0 = 4$ for (3.7). It remains to consider diagonalizable rotations. By Lemma 3.2, admitting a real rotation is equivalent to being monomial, and by Lemma 3.4, admitting an imaginary rotation is equivalent to being balanced. It follows, using Lemma 3.4, that for monomial models, with the exception of (3.7), we have $\dim \mathfrak{g}_0 = 3$. On the other hand, by Lemma 3.5, $M$ admits only an imaginary roation, i.e. $\dim \mathfrak{g}_0 = 2$, if and only if $M$ is balanced and not monomial. That finishes the proof. □

**Proof of Theorem 4.1.** First note that $\dim \mathfrak{g}_n = 0$, except for $M$ given by (1.1), by Theorem 2.1. Also, as proved in [16], if $M$ is biholomorphically equivalent to (1.1), then $\dim \mathfrak{g} = 10$. Let us further assume that $M$ is not biholomorphic to (1.1), hence $\dim \mathfrak{g}_n = 0$. If $\dim \mathfrak{g}_t \neq 0$, then by Lemma 3.6, $M$ is biholomorphically equivalent to (3.6), which is a monomial model and therefore balanced. Hence by Lemma 3.8, $\dim \mathfrak{g}_0 = 3$. Further, in this case $\dim \mathfrak{g}_1 = 1$ and $\dim \mathfrak{g} = 7$. If $\dim \mathfrak{g}_t = 0$ and $M$ is monomial, then again $\dim \mathfrak{g}_0 = 3$ and $\dim \mathfrak{g}_1 = 1$. Hence $\dim \mathfrak{g} = 6$. Next, assume that $M$ is not monomial. Observe that if a nonmonomial model has $\dim \mathfrak{g}_c > 0$, then it is not balanced. Indeed, looking at the proof of Lemma 3.4, we observe that the reproducing field is the $(z_1, z_2)$ part of the Euler field. Therefore, the commutator of an element of $\mathfrak{g}_c$ and $\mathfrak{g}_1$ can not vanish, and is of weighted degree bigger than one, which gives a contradiction with (2.3).

It follows that if $M$ is not monomial, then under the assumptions of the Theorem, $\dim \mathfrak{g} = 3$. That finishes the proof. □

The proof of Theorem 4.2 is completely analogous.

It would be interesting to find an explicit list of hypersurfaces given by (1.2) admitting nontrivial $\mathfrak{g}_c$.

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Department of Mathematics and Statistics,
Masaryk University,
Kotlářská 2, 611 37 Brno, Czech Republic
*E-mail*: mkolar@math.muni.cz

Department of Mathematics,
University of Fribourg,
Chemin du Musée 23,
CH 1700 Fribourg
*E-mail*: francine.meylan@unifr.ch