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ENERGY GAPS FOR EXPONENTIAL YANG-MILLS FIELDS

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Abstract. In this paper, some inequalities of Simons type for exponential Yang-Mills fields over compact Riemannian manifolds are established, and the energy gaps are obtained.

1. Introduction

Let M be an m-dimensional Riemannian manifold, G an r_0-dimensional Lie group, E a Riemannian vector bundle over M with structure group G, $\mathfrak{g}_E \subseteq \operatorname{End}(E)$ the adjoint vector bundle, whose fiber type is $\mathfrak{g}$, the Lie algebra of G. We denote the space of $\mathfrak{g}_E$-valued p-forms by $\Omega^p(\mathfrak{g}_E)$. Let ∇ be a connection on E, then, the curvature $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ is defined by $R^\nabla_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ for tangent vector fields X, Y on M.

Extend the connection ∇ into an exterior differential operator $\mathrm{d}^\nabla\colon \Omega^p(\mathfrak{g}_E) \to \Omega^{p+1}(\mathfrak{g}_E)$ as follows: for each $\omega \in \Omega^p(M)$ and $\sigma \in \Omega^0(\mathfrak{g}_E)$, let

$$\mathrm{d}^\nabla(\omega\otimes\sigma) = \mathrm{d}\omega\otimes\sigma + (-1)^p\omega\wedge\nabla\sigma\,,$$

and extend to all members of $\Omega^p(\mathfrak{g}_E)$ by linearity.

When G is a subgroup of $\mathrm{O}(r_0)$, the Killing form in $\mathfrak{g}$ is negatively defined, and hence induces an inner product in $\mathfrak{g}_E$. This inner product and the Riemannian metric of M define an inner product $\langle\cdot,\cdot\rangle$ in $\Omega^p(\mathfrak{g}_E)$. The exterior differential operator $\mathrm{d}^\nabla\colon \Omega^p(\mathfrak{g}_E) \to \Omega^{p+1}(\mathfrak{g}_E)$ has a formal adjoint operator $\delta^\nabla\colon \Omega^{p+1}(\mathfrak{g}_E) \to \Omega^p(\mathfrak{g}_E)$ with respect to the L^2-inner product $(\varphi,\psi) = \int_M\langle\varphi,\psi\rangle$. Take a local orthonormal frame field $\{e_1,\dots,e_m\}$ on M. Then, for any $\varphi\in\Omega^p(E)$ and any local tangent vector fields $X_0, X_1,\dots,X_p$ to M, we have

$$\left(\mathrm{d}^\nabla\varphi\right)_{X_0,X_1,\dots,X_p} = \sum_{k=0}^{p}(-1)^k\left(\nabla_{X_k}\varphi\right)_{X_0,\dots,\hat{X}_k,\dots,X_p}\,,$$

$$\left(\delta^\nabla\varphi\right)_{X_1,\dots,X_{p-1}} = \sum_{k=1}^{m}\left(\nabla_{e_k}\varphi\right)_{e_k,X_1,\dots,X_{p-1}}\,.$$

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The Laplacian acting on $\Omega^p(\mathfrak{g}_E)$ is defined by $\Delta^\nabla = \mathrm{d}^\nabla \circ \delta^\nabla + \delta^\nabla \circ \mathrm{d}^\nabla : \Omega^p(\mathfrak{g}_E) \to \Omega^p(\mathfrak{g}_E)$. If $\varphi \in \Omega^p(\mathfrak{g}_E)$ satisfies $\Delta^\nabla \varphi = 0$, we call it a harmonic p-form with values in $\mathfrak{g}_E$.

Let $\mathcal{C}_E$ be the collection of all metric connections on E, and fix a connection $\nabla_0 \in \mathcal{C}_E$. Then, any connection $\nabla \in \mathcal{C}_E$ can be expressed as $\nabla = \nabla_0 + A$, where $A \in \Omega^1(\mathfrak{g}_E)$. The Yang-Mills functional is defined as: For $\nabla \in \mathcal{C}_E$,

$$\mathcal{S}(\nabla) = \frac{1}{2}\int_M |R^\nabla|^2 . \tag{1}$$

A connection $\nabla \in \mathcal{C}_E$ is called is a a Yang-Mills connection, if it is a critical point of the Yang-Mills functional, and the associated curvature tensor is called a Yang-Mills field.

The Euler-Lagrange equation of the Yang-Mills functional $\mathcal{S}(\cdot)$ can be written as

$$\delta^\nabla R^\nabla = 0 . \tag{2}$$

Hence, by Bianchi identity $\mathrm{d}^\nabla R^\nabla = 0$, a Yang-Mills field is a harmonic 2-form with values in $\mathfrak{g}_E$.

The following gap property for Yang-Mills fields is obtained in [2]:

Theorem 1. *Let R^∇ be a Yang-Mills field on $\mathbb{S}^m (m \geq 5)$ satisfying that*

$$\|R^\nabla\|^2_{L^\infty} \leq \frac{1}{2}\binom{m}{2} ,$$

then $R^\nabla \equiv 0$.

Denote the Riemannian curvature operator of M by R, the Ricci operator by Ric. Let $C = \mathrm{Ric} \wedge I + 2R$, where I is the identity transformation on TM, and define the Ricci-Riemannian curvature operator $\mathcal{C} \colon \Omega^2(\mathfrak{g}_E) \to \Omega^2(\mathfrak{g}_E)$ as follows: for $\varphi \in \Omega^2(\mathfrak{g}_E)$ and X, Y, $Z \in \Gamma(M)$,

$$\big(\mathcal{C}(\varphi)\big)_{X,Y} = \frac{1}{2}\sum \varphi_{e_j, C_{X,Y}(e_j)} . \tag{3}$$

Here,

$$(\mathrm{Ric} \wedge I)_{X,Y} = \mathrm{Ric}(X) \wedge Y + X \wedge \mathrm{Ric}(Y) , \tag{4}$$

and $X \wedge Y$ is identified as a skew-symmetric linear transformation by

$$(X \wedge Y)(Z) = \langle X, Z\rangle Y - \langle Y, Z\rangle X . \tag{5}$$

In the following, that $\mathcal{C} \geq \lambda$ means that $\langle \mathcal{C}(\varphi), \varphi\rangle \geq \lambda|\varphi|^2$ for each $\varphi \in \Omega^2(\mathfrak{g}_E)$.

In [13], an inequality of Simons type for Yang-Mills fields is obtained:

Theorem 2. *Let M^m $(m \geq 3)$ be a compact Riemannian manifold with $\mathcal{C} \geq \lambda$. Then, for each Yang-Mills field R^∇, we have*

$$\int_M |\nabla R^\nabla|^2 \leq \int_M \Big(\frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla| - \lambda\Big)|R^\nabla|^2 . \tag{6}$$

If $m \geq 5$, the equality holds if and only if $R^\nabla = 0$.

This inequality implies a gap property (see [13]):

Corollary 3. *Let M^m and λ be as in Theorem 2, $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ be a Yang-Mills field over M. If $m \geq 3$ and $\|R^\nabla\|^2_{L^\infty} < \frac{\lambda^2 m(m-1)}{16(m-2)^2}$, then we have $R^\nabla = 0$. If $m \geq 5$ and $\|R^\nabla\|^2_{L^\infty} \leq \frac{\lambda^2 m(m-1)}{16(m-2)^2}$, then we also have $R^\nabla = 0$.*

When $M = \mathbb{S}^m$, we have $\lambda = 2(m-2)$. Therefore Corollary 3 implies Theorem 1.

A p-Yang-Mills functional is defined by $\mathcal{S}_p(\nabla) = \frac{1}{p}\int_M |R^\nabla|^p$, the critical points of which are called p-Yang-Mills connections, and the associated curvature tensors are called p-Yang-Mills fields. The article [4] investegated the gaps of p-Yang-Mills fields of Euclidean and sphere submanifolds, and generalized the related results of [2].

Theorem 4 (See [4, 13])**.** *Let M^m be a submanifold of $\mathbb{R}^{m+k}$ or $\mathbb{S}^{m+k}$. If $\mathcal{C} \geq 2(m-2)$, and if R^∇ is a p-Yang-Mills field ($p \geq 2$) with $\|R^\nabla\|^2_{L^\infty} \leq \frac{1}{2}\binom{m}{2}$ ($m \geq 5$), then we have $R^\nabla \equiv 0$.*

Theorem 4 is also a generalization of Theorem 1.

An exponential Yang-Mills functional is defined by $\mathcal{S}_e(\nabla) = \int_M \exp\left(\frac{|R^\nabla|^2}{2}\right)$, an exponential Yang-Mills connection is a critical point of $\mathcal{S}_e$, and an exponential Yang-Mills field is the curvature R^∇ of an exponential Yang-Mills connection $\nabla \in \mathcal{C}_E$. The Euler-Lagrange equation of $\mathcal{S}_e(\cdot)$ is

$$\delta^\nabla\left[\exp\left(\frac{|R^\nabla|^2}{2}\right)R^\nabla\right] = 0\,. \tag{7}$$

Some L^2-energy gaps are obtained for four dimensional Yang-Mills fields, see for example [5, 6, 7, 11, 12] etc. The existence of $L^{m/2}$-energy gaps for Yang-Mills fields over m-dimensional compact or non-compact but complete Riemannian manifolds are verified independently under some non-positive curvature conditions in [15] and [9]. P.M.N. Feehan prove an existence of $L^{m/2}$-energy gaps over compact manifolds without any curvature assumptions in [8]. Recently, we estimate the L^p-energy gaps for $p \geq m/2$ over the unit sphere $\mathbb{S}^m$ and the $m/2$-energy gaps over the hyperbolic space $\mathbb{H}^m$ in [14].

In this paper, we establish some inequalities of Simons type for exponential Yang-Mills fields over compact Riemannian manifolds. Then, we use these inequalities to obtain some energy gaps.

2. Inequalities of Simons type for exponential Yang-Mills fields

Take a local orthornormal frame field $\{e_i\}_{i=1,\dots,m}$ on M. We adopt the convention of summation, and let indices i, j, k, l, u run in $\{1,\dots,m\}$.

For each $\varphi \in \Omega^2(\mathfrak{g}_E)$, let

$$\mathfrak{R}^\nabla(\varphi)_{X,Y} = \sum\left\{[R^\nabla_{e_j,X}, \varphi_{e_j,Y}] - [R^\nabla_{e_j,Y}, \varphi_{e_j,X}]\right\}. \tag{8}$$

Then, we have (see [2])

$$\Delta^\nabla\varphi = \nabla^*\nabla\varphi + \mathcal{C}(\varphi) + \mathfrak{R}^\nabla(\varphi)\,, \tag{9}$$

where, $\nabla^*\nabla = -\sum \nabla_{e_i}\nabla_{e_i} + \nabla_{D_{e_i}e_i}$ is the rough Laplacian (D is the Levi-Civita connection of M). Hence we have

$$\frac{1}{2}\Delta|\varphi|^2 = \left\langle \Delta^\nabla \varphi, \varphi \right\rangle - |\nabla\varphi|^2 - \langle \mathcal{C}(\varphi), \varphi\rangle - \left\langle \Re^\nabla(\varphi), \varphi \right\rangle . \tag{10}$$

By a straightforward calculation, we get

$$\begin{aligned}\Delta \exp\Big(\frac{|\varphi|^2}{2}\Big) &= -\exp\Big(\frac{|\varphi|^2}{2}\Big)|\varphi|^2|\nabla|\varphi||^2 \\ &\quad + \exp\Big(\frac{|\varphi|^2}{2}\Big)\left\langle \Delta^\nabla\varphi, \varphi\right\rangle - \exp\Big(\frac{|\varphi|^2}{2}\Big)|\nabla\varphi|^2 \\ &\quad - \exp\Big(\frac{|\varphi|^2}{2}\Big)\langle \mathcal{C}(\varphi), \varphi\rangle - \exp\Big(\frac{|\varphi|^2}{2}\Big)\left\langle \Re^\nabla(\varphi), \varphi\right\rangle .\end{aligned} \tag{11}$$

Integrating both sides of (11), we have

Lemma 5. *For each $\varphi \in \Omega^2(\mathfrak{g}_E)$, we have*

$$\begin{aligned}\int_M \exp\Big(\frac{|\varphi|^2}{2}\Big)|\nabla\varphi|^2 + \int_M \exp\Big(\frac{|\varphi|^2}{2}\Big)|\varphi|^2\,|\nabla|\varphi||^2 \\ = \int_M \exp\Big(\frac{|\varphi|^2}{2}\Big)\left\langle \Delta^\nabla\varphi, \varphi\right\rangle \\ - \int_M \exp\Big(\frac{|\varphi|^2}{2}\Big)\langle \mathcal{C}(\varphi), \varphi\rangle - \int_M \exp\Big(\frac{|\varphi|^2}{2}\Big)\left\langle \Re^\nabla(\varphi), \varphi\right\rangle .\end{aligned} \tag{12}$$

In [13], we establish the following inequality:

Lemma 6. *For $\varphi \in \Omega^2(\mathfrak{g}_E)$, let*

$$\rho(\varphi) = \sum \left\langle [\varphi_{e_i,e_j}, \varphi_{e_j,e_k}], \varphi_{e_k,e_i}\right\rangle . \tag{13}$$

Then, we have

$$|\rho(\varphi)| \le \frac{4(m-2)}{\sqrt{m(m-1)}}\,|\varphi|^3 . \tag{14}$$

If $m \ge 5$, the inequality is strict unless $\varphi = 0$.

Applying Lemma 6 to Lemma 5, we can obtain the following inequality of Simons type for exponential Yang-Mills fields:

Theorem 7. *Let M^m ($m \ge 3$) be a Riemannian m-manifold, and R^∇ be an exponential Yang-Mills field over M^m. If $\mathcal{C} \ge \lambda$, then we have*

$$\begin{aligned}\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^2\,\big|\nabla|R^\nabla|\big|^2 + \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|\nabla R^\nabla|^2 \\ \le \int_M \Big(\frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla| - \lambda\Big)\exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^2 .\end{aligned} \tag{15}$$

Proof. By Bianchi identity, we have

$$\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\Big\langle \Delta^\nabla R^\nabla, R^\nabla\Big\rangle = \int_M \Big\langle \delta^\nabla R^\nabla, \delta^\nabla\Big(\exp\Big(\frac{|R^\nabla|^2}{2}\Big)R^\nabla\Big)\Big\rangle .$$

Because R^∇ is an exponential Yang-Mills fields, we have

$$\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left\langle \Delta^\nabla R^\nabla, R^\nabla\right\rangle = 0 .$$

Hence by (12) we have

$$\begin{aligned}
&\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^2\left|\nabla|R^\nabla|\right|^2 + \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|\nabla R^\nabla|^2 \\
&\quad = -\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left\langle \mathcal{C}(R^\nabla), R^\nabla\right\rangle - \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left\langle \Re^\nabla(R^\nabla), R^\nabla\right\rangle \\
&\quad = -\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left\langle \mathcal{C}(R^\nabla), R^\nabla\right\rangle - \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\rho(R^\nabla) . \qquad (16)
\end{aligned}$$

If $\mathcal{C} \geq \lambda$, then we get

$$-\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left\langle \mathcal{C}(R^\nabla), R^\nabla\right\rangle \leq -\lambda \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^2 . \qquad (17)$$

For $m \geq 3$, from Lemma 6 we have

$$-\frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla|^3 \leq \pm\rho(R^\nabla) \leq \frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla|^3 .$$

So we have

$$-\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\rho(R^\nabla) \leq \frac{4(m-2)}{\sqrt{m(m-1)}}\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^3 . \qquad (18)$$

Hence from (17) and (18) we have (15). □

Corollary 8. *Let M^m $(m \geq 3)$ be a Riemannian n-manifold, and R^∇ be an exponential Yang-Mills field over M^m. If $\mathcal{C} \geq \lambda$, then we have*

$$\begin{aligned}
&\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left|\nabla R^\nabla\right|^2 + 4\int_M \left|\nabla \exp\Big(\frac{|R^\nabla|^2}{4}\Big)\right|^2 \\
&\qquad \leq \int_M \Big(\frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla| - \lambda\Big)\exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left|R^\nabla\right|^2 . \qquad (19)
\end{aligned}$$

Proof. Because

$$\begin{aligned}
&\int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^2\left|\nabla|R^\nabla|\right|^2 \\
&\quad = \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)\left|\nabla\frac{|R^\nabla|^2}{2}\right|^2 = 4\int_M \left|\nabla \exp\Big(\frac{|R^\nabla|^2}{4}\Big)\right|^2 ,
\end{aligned}$$

then, from (15) we have

$$(20)\quad \begin{aligned} &4\int_M \Big|\nabla \exp\Big(\frac{|R^\nabla|^2}{4}\Big)\Big|^2 + \int_M \exp\Big(\frac{|R^\nabla|^2}{2}\Big)|\nabla R^\nabla|^2 \\ &\le \int_M \Big(\frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla| - \lambda\Big)\exp\Big(\frac{|R^\nabla|^2}{2}\Big)|R^\nabla|^2 . \end{aligned}$$

By $|\nabla R^\nabla|^2 \ge \big|\nabla|R^\nabla|\big|^2$ and (20) we get (19). □

By Theorem 7, we have

Corollary 9. *Let M^m ($m \ge 3$) be a Riemannian m-manifold, and R^∇ be an exponential Yang-Mills field over M^m. Suppose that $\mathcal{C} \ge \lambda$. Then, if $\|R^\nabla\|^2_{L^\infty} \le \frac{m(m-1)\lambda^2}{16(m-2)^2}$, we have $\nabla R^\nabla = 0$. Especially, on $\mathbb{S}^m$, if $\|R^\nabla\|^2_{L^\infty} < \frac{1}{2}\binom{m}{2}$, we have $R^\nabla = 0$.*

3. Energy gaps for exponential Yang-Mills fields

Let M^m be an m-dimensional compact Riemannian manifold. We say that the q-Sobolev inequality holds on M^m with k_1, k_2 if for all $u \in C^\infty(M^m)$ we have

$$(21)\quad \|\nabla u\|_2^2 \ge k_1\|u\|_q^2 - k_2\|u\|_2^2 .$$

On the unit sphere $\mathbb{S}^m$, the following Sobolev inequality holds (see [1, 10]): for $2 \le q \le 2m/(m-2)$,

$$(22)\quad \|u\|_q^2 \le \frac{q-2}{m\omega_m^{1-2/q}}\|\nabla u\|_2^2 + \frac{1}{\omega_m^{1-2/q}}\|u\|_2^2 ,$$

where ω_m is the volume of the unit sphere $\mathbb{S}^m$. Hence we have

Lemma 10. *On $\mathbb{S}^m$, for $2 < q \le 2m/(m-2)$, the q-Sobolev inequality holds with $k_1 = \frac{m\omega_m^{1-2/q}}{q-2}$, $k_2 = \frac{m}{q-2}$.*

Denote

$$d_{a,m,r} = \min\Big\{k_1, \frac{k_1 a}{k_2}\Big\},$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

In [14], we prove the following

Lemma 11. *Let T be a tensor over a compact Riemannian manifold M^m where the $2q$-Sobolev inequality holds with k_1, k_2 for $2 < 2q \le \frac{2m}{m-2}$. Assume that there exist a positive constant a and a function f on M, such that*

$$(23)\quad \|\nabla|T|\|_2^2 \le -a\|T\|_2^2 + \|f|T|^2\|_1 .$$

If $\|f\|_r < d_{a,m,r}$, then we have $T = 0$, where $r = \frac{q}{q-1} \ge \frac{m}{2}$.

Theorem 12. *Let $M^m (m \ge 3)$ be a compact Riemannian manifold with $\mathcal{C} \ge \lambda > 0$, where $2q$-Sobolev inequality holds with k_1 and k_2 for $2 < 2q \le \frac{2m}{m-2}$. Suppose that R^∇ is an exponential Yang-Mills field over M. If $\big\|R^\nabla \exp\big(\frac{|R^\nabla|^2}{2}\big)\big\|_r < \frac{\sqrt{m(m-1)}}{4(m-2)}d_{\lambda,m,r}$, then we have $R^\nabla = 0$, where $r = \frac{q}{q-1} \ge \frac{m}{2}$.*

Proof. By (19) we have

$$\int_M \left|\nabla |R^\nabla|\right|^2 \leq \int_M \left(\frac{4(m-2)}{\sqrt{m(m-1)}}|R^\nabla| - \lambda\right) \exp\left(\frac{|R^\nabla|^2}{2}\right)|R^\nabla|^2 .$$

Let $u = |R^\nabla|$, then

$$\int_M |\nabla u|^2 \leq \int_M \left(\frac{4(m-2)}{\sqrt{m(m-1)}}u - \lambda\right) \exp\left(\frac{u^2}{2}\right)u^2 .$$

So, we have

$$\begin{aligned}\int_M |\nabla u|^2 &\leq \int_M \left(\frac{4(m-2)}{\sqrt{m(m-1)}}u \exp\left(\frac{u^2}{2}\right)u^2 - \lambda \exp\left(\frac{u^2}{2}\right)u^2\right)\\ &\leq \int_M \left(\frac{4(m-2)}{\sqrt{m(m-1)}}u \exp\left(\frac{u^2}{2}\right)u^2 - \lambda u^2\right)\\ &= \int_M \left(\frac{4(m-2)}{\sqrt{m(m-1)}}u \exp\left(\frac{u^2}{2}\right) - \lambda\right)u^2\end{aligned}$$

i.e. $\int_M |\nabla u|^2 \leq \int_M fu^2 - \lambda \int_M u^2$, where $f = \frac{4(m-2)}{\sqrt{m(m-1)}}u \exp\left(\frac{u^2}{2}\right)$. Then by Lemma 11 we can get the theorem. □

Corollary 13. *Suppose that R^∇ is an exponential Yang-Mills field over $\mathbb{S}^m$ $(m \geq 3)$. If*

$$\left\|R^\nabla \exp\left(\frac{|R^\nabla|^2}{2}\right)\right\|_r < \frac{\sqrt{m(m-1)}}{4(m-2)}\omega_m^{\frac{1}{r}} \min\left\{\frac{m(r-1)}{2}, 2(m-2)\right\}$$

then, we have $R^\nabla = 0$, where $r \geq \frac{m}{2}$.

Proof. On $\mathbb{S}^m$, $\lambda = 2(m-2)$, and the $2q$-Sobolev inequality holds for $2 < 2q \leq \frac{2m}{m-2}$ with $k_1 = \frac{n\omega_m^{1-2/2q}}{2q-2} = \frac{m(r-1)}{2}\omega_m^{1/r}$, $k_2 = \frac{m}{2q-2} = \frac{m(r-1)}{2}$. By a straightforward calculation, we get

$$d_{2(m-2),m,r} = \omega_m^{1/r} \min\left\{\frac{m(r-1)}{2}, 2(m-2)\right\} \tag{24}$$

and

$$d_{2(m-2),m,\infty} = 2(m-2) .$$

Then by Theorem 12, if

$$\left\|R^\nabla \exp\left(\frac{|R^\nabla|^2}{2}\right)\right\|_r < \frac{\sqrt{m(m-1)}}{4(m-2)} d_{2(m-2),m,r} ,$$

then we have $R^\nabla = 0$.

Especially, if

$$\left\|R^\nabla \exp\left(\frac{|R^\nabla|^2}{2}\right)\right\|_\infty < \frac{\sqrt{m(m-1)}}{4(m-2)} d_{2(m-2),m,\infty} = \frac{\sqrt{m(m-1)}}{2} ,$$

we have $R^\nabla = 0$. □

References

[1] Beckner, W., *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2) **138** (1993), 213–242.

[2] Bourguignon, J.P., Lawson, H.B., *Stability and isolation phenomena for Yang-Mills fields*, Comm. Math. Phys. **79** (2) (1981).

[3] Bourguignon, J.P., Lawson, H.B., Simons, J., *Stability and gap phenomena for yang-mills fields*, Proc. Natl. Acad. Sci. USA **76** (1979).

[4] Chen, Q., Zhou, Z.R., *On gap properties and instabilities of p-Yang-Mills fields*, Canad. J. Math. **59** (6) (2007), 1245–1259.

[5] Dodziuk, J., Min-Oo, *An L_2-isolation theorem for Yang-Mills fields over complete manifolds*, Compositio Math **47** (2) (1982), 165–169.

[6] Donaldson, S.K., Kronheimer, P.B., *The geometry of four-manifolds*, Proceedings of Oxford Mathematical Monographs, The Clarendon Press, Oxford University Pres, Oxford Science Publications, New York, 1990.

[7] Feehan, P.M.N., *Energy gap for Yang-Mills connections, I: Four-dimensional closed Riemannian manifold*, Adv. Math. **296** (2016), 55–84.

[8] Feehan, P.M.N., *Energy gap for Yang-Mills connections, II: Arbitrary closed Riemannian manifolds*, Adv. Math. **312** (2017), 547–587.

[9] Gerhardt, C., *An energy gap for Yang-Mills connections*, Comm. Math. Phys. **298** (2010), 515–522.

[10] Hebey, E., *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Math., vol. 1635, Springer-Verlag Berlin Heidelberg, 1996.

[11] Min-Oo, *An L_2-isolation theorem for Yang-Mills fields*, Compositio Math. **47** (2) (1982), 153–163.

[12] Shen, C.L., *The gap phenomena of Yang-Mills fields over the complete manifolds*, Math. Z. **180** (1982), 69–77.

[13] Zhou, Z.R., *Inequalities of Simons type and gaps for Yang-Mills fields*, Ann. Global Anal. Geom. **48** (3) (2015), 223–232.

[14] Zhou, Z.R., *Energy gaps for Yang-Mills fields*, J. Math. Anal. Appl. **439** (2016), 514–522.

[15] Zhou, Z.R., Chen Qun, *Global pinching lemmas and their applications to geometry of submanifolds, harmonic maps and Yang-Mills fields*, Adv. Math. **32** (1) (2003), 319–326.

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