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ON HYPONORMAL OPERATORS IN KREIN SPACES

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Abstract. In this paper the hyponormal operators on Krein spaces are introduced. We state conditions for the hyponormality of bounded operators focusing, in particular, on those operators T for which there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ of the Krein space $\mathcal{K}$ with $\mathcal{K}^+$ and $\mathcal{K}^-$ invariant under T.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *hyponormal* if $T^*T - TT^*$ is a positive operator. This class of operators was introduced in 1950 by P.R. Halmos in [8] and its properties have been extensively studied [1, 5, 10, 11, 12, 13].

If T is a bounded operator acting on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ by means of the rule $T(x_1+x_2) = T_1x_1 + T_2x_2$, with T_i a hyponormal operator on the Hilbert space $\mathcal{H}_i$, $i = 1, 2$ and $x_1 + x_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have that T is hyponormal in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. However if T is hyponormal in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then $T_1 = TP_1$ and $T_2 = TP_2$ are hyponormal in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively if and only if $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant under T, see, for instance [3, §9]. In 1962 J.G. Stampfli established important results about hyponomal operators and their invariant subspaces [12]. This fact is a motivation to study hyponormal operators in these sum-spaces, in particular, in Krein spaces.

A vector space $\mathcal{K}$ over the complex numbers $\mathbb{C}$ with a sesquilinear form $[\cdot, \cdot]\colon \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ is a *Krein space* if $\mathcal{K}$ can be decomposed as $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ where $(\mathcal{K}^+, [\cdot, \cdot])$, $(\mathcal{K}^-, -[\cdot, \cdot])$ are Hilbert spaces [2, 4]. The representation $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$, where $(\mathcal{K}^+, [\cdot, \cdot])$ and $(\mathcal{K}^-, -[\cdot, \cdot])$ are Hilbert spaces is called a *fundamental decomposition* of the Krein space $\mathcal{K}$ and it is not unique. Since every $x \in \mathcal{K}$ can be written as $x = x^+ + x^-$ with $x^+ \in \mathcal{K}^+$ and $x^- \in \mathcal{K}^-$ it follows that the operator $J : \mathcal{K} \to \mathcal{K}$ given by $Jx = x^+ - x^-$ is well defined and it is the so-called *fundamental symmetry associated to the given fundamental decomposition.* This operator allows to introduce on $\mathcal{K}$ a positive definite inner product through the formula $[x, y]_J = [Jx, y]$ and also a norm in $\mathcal{K}$ associated with each fundamental decomposition by means of the formula $||x||_J = \sqrt{[x, Jx]}$. It was proved (see [2, 4]) that even though the norms induced by different fundamental decompositions are different themselves, they are

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equivalent, so they induced the same topology. This topology is called the *strong topology on* $\mathcal{K}$, for details see [4]. From now on all the topological concepts used in this paper are related to this topology.

An operator T acting on the Krein space $\mathcal{K}$ is said to be *fundamentally reducible* if there is a fundamental decomposition of $\mathcal{K} = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$ sucht that $\mathcal{K}_1^+$ and $\mathcal{K}_1^-$ are invariant under T. This property has been largely studied [2, 4, 6, 7, 9] and plays an important role due to the relation with their invariant subspaces. Since the fundamental decomposition of $\mathcal{K}$ could not be the same decomposition by which the operator is fundamental reducible, it is very important to specify the fundamental decomposition we are working with in each situation. It is worth mentioning that even though fundamental symmetries generate equivalent norms the related adjoint operators are different. Various conditions for an operator to be fundamentally reducible have been given, see for example [4, 6].

The paper is devoted to the study of hyponormal operators on Krein spaces and it is structured as follows. Section 2 is based on the references [2, 4]. We will consider some basic aspects related to the Krein spaces and bounded operators that are fundamentally reducibles. In Section 3 we present the main results. We introduce the hyponormal operators on Krein spaces and study some of their basic properties. The Proposition 3.13 states that the fundamentally reducible operators by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ that are hyponormal both in the Krein space and in the associated Hilbert space have the form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ where U is hyponormal in $\mathcal{K}^+$ and V is normal in $\mathcal{K}^-$. We also show that $T \in \mathcal{B}(\mathcal{K})$ is hyponormal in a Krein space $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ and fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ if and only if it is hyponormal in the associated Hilbert space $(\mathcal{K}, [\cdot,\cdot]_J)$, $T\mathcal{K}^- \subset \mathcal{K}^-$ and $T|_{\mathcal{K}^-}$ is a normal operator (Theorem 3.14). Finally, we prove that if T is a bounded, hyponormal operator which is fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-$ with $\mathcal{K}_T^\pm \neq \mathcal{K}^\pm$ then there exists an associated Hilbert space $(\mathcal{K}{=}\mathcal{K}_T^+ \oplus \mathcal{K}_T^-, \langle\cdot,\cdot\rangle)$ where T is hyponormal and $T|_{\mathcal{K}_T^-}$ is normal (Theorem 3.16).

2. Preliminaries

The purpose of this section is to fix the notations and to recall the basic elements of the Krein spaces theory. For more details on Krein spaces, see, for instance, [2] and [4].

Throughout this paper, $(\mathcal{K}, [\cdot,\cdot])$ denotes a Krein space with fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ and fundamental symmetry J given by

$$J(k^+ + k^-) = k^+ - k^-, \qquad k^+ + k^- \in \mathcal{K}_+ \oplus \mathcal{K}_-\,, \tag{2.1}$$

the J-inner product

$$\begin{aligned} [h^+ + h^-, k^+ + k^-]_J &:= [h^+ + h^-, J(k^+ + k^-)] \\ &= [h^+, k^+] - [h^-, k^-]\,, \qquad h^\pm, k^\pm \in \mathcal{K}_\pm\,, \end{aligned} \tag{2.2}$$

turns $(\mathcal{K}, [\cdot,\cdot]_J)$ into a Hilbert space with the topology induced by the J-norm

$$\|k\|_J := \sqrt{[k\,,k]_J} = \sqrt{[k\,,Jk]}\,, \qquad k \in \mathcal{K}\,, \tag{2.3}$$

thus $\mathcal{K}_+ \oplus \mathcal{K}_-$ becomes the orthogonal sum of Hilbert spaces. Note that $J^2 = 1$ by (2.1). The Hilbert space $(\mathcal{K}, [\cdot,\cdot]_J)$ is used to study linear operators acting on the Krein space $(\mathcal{K}, [\cdot,\cdot])$. Topological concepts such as continuity, closedness of operators, spectral theory and so on, refer to the topology induced by the J-norm given in (2.3). Therefore, we may apply the same definitions as in the *operator theory of Hilbert spaces*. For instance, the adjoint of an operator T in Krein spaces $\big(T^{[*]}\big)$ satisfies $[T(x), y] = [x, T^{[*]}(y)]$, but we must take into account that T also has an adjoint operator in the Hilbert space $(\mathcal{K}, [\cdot,\cdot]_J)$, denoted by T^{*J}, where J is the fundamental symmetry in $\mathcal{K}$. The relation between T^{*J} and $T^{[*]}$ is $T^{[*]} = JT^{*J}J$. Moreover, let $\mathcal{K}$ and $\mathcal{K}'$ be Krein spaces with fundamental symmetries $J_{\mathcal{K}}$ and $J_{\mathcal{K}'}$ respectively. If $T \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ then $T^{[*]} = J_{\mathcal{K}} T^{*J_{\mathcal{K}}} J_{\mathcal{K}'}$. An operator $T \in \mathcal{B}(\mathcal{K})$ is said to be *self-adjoint* if $T = T^{[*]}$, and *J-self-adjoint* if $T = T^{*J}$, it is said to be *normal* if $TT^{[*]} = T^{[*]}T$, it is called *positive* if $[Tk, k] \geq 0$ for all $k \in \mathcal{K}$. Equivalently, since $[Tk, k] = [JTk, k]_J$, we have that T is positive if JT is positive on the Hilbert space $(\mathcal{K}, [\cdot,\cdot]_J)$. The fundamental projections

$$P_+ := \tfrac{1}{2}(\mathrm{Id} + J)\,, \qquad P_- := \tfrac{1}{2}(\mathrm{Id} - J) \tag{2.4}$$

act on $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ by $P_+(k^+ + k^-) = k^+$ and $P_-(k^+ + k^-) = k^-$. Equation (2.4) implies immediately that $P_\pm$ and J commute. Moreover, P_+ and P_- are orthogonal projections, i.e. $P_\pm^2 = P_\pm = P_\pm^*$, regardless of whether we consider $[\cdot,\cdot]$ or $[\cdot,\cdot]_J$ on $\mathcal{K}$. For more details on Krein spaces, see [2, 4].

Definition 2.1. Let V be a closed subspace of the Krein space $(\mathcal{K}, [\cdot,\cdot])$. The subspaces

$$V^{[\perp]} = \{x \in \mathcal{K} : [x, y] = 0 \quad \text{for all } y \in V\} \tag{2.5}$$

$$V^{\perp} = \{x \in \mathcal{K} : [x, y]_J = 0 \quad \text{for all } y \in V\} \tag{2.6}$$

are called the *orthogonal complement of* V and the *J-orthogonal complement of* V respectively.

Definition 2.2. A closed subspace V of $\mathcal{K}$ such that $V \cap V^{[\perp]} = \{0\}$ and $V + V^{[\perp]} = \mathcal{K}$, where $V^{[\perp]}$ is given in (2.5), is called *projectively complete.*

The following result is important in the development of this paper its proof can be found in [4, 6].

Proposition 2.3. *Let $\mathcal{K}$ be a Krein space and J be the fundamental symmetry associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and $T \in \mathcal{B}(\mathcal{K})$. The following statements are equivalent*

(i) $TJ = JT$

(ii) $T^{[*]}J = JT^{[*]}$

(iii) $T^{*J} = T^{[*]}$

(iv) $TP_+ = P_+T$ *and* $TP_- = P_-T$

(v) *$\mathcal{K}^+$ and $\mathcal{K}^-$ are invariant under T.*

Definition 2.4. Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space. A bounded operator T is said to be *fundamentally reducible* if there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ such that $\mathcal{K}^+$ and $\mathcal{K}^-$ are invariant under T.

Let $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ be a Krein space and $T \in \mathcal{B}(\mathcal{K})$ be a positive operator. Then denote by

$$\mu_+(T) = \inf_{x\in\mathcal{K}^{++}} \frac{[Tx,x]}{[x,x]}\,; \quad \mu_-(T) = \sup_{x\in\mathcal{K}^{--}} \frac{[Tx,x]}{[x,x]}\,, \tag{2.7}$$

where

$$\mathcal{K}^{++} = \{x \in \mathcal{K} \colon [x,x] > 0\}\,, \quad \mathcal{K}^{--} = \{x \in \mathcal{K} \colon [x,x] < 0\}\,. \tag{2.8}$$

The numbers $\mu_+(T)$, $\mu_-(T)$ are finite and for every $x \in \mathcal{K}$ (see [4, Lemma II.6.1, Theorem II.6.2])

$$[Tx,x] \geq \mu[x,x] \quad \text{for each} \quad \mu \in [\mu_-(T), \mu_+(T)]\,. \tag{2.9}$$

Remark 2.5. Observe that an operator can be fundamentally reducible by a given fundamental decomposition and not being fundamentally reducible by another one. Let $\mathcal{K} = \mathbb{C}^2$ be a Krein space with indefinite inner product $[(z,w),(u,v)] = z\overline{u} - w\overline{v}$. Observe that $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^- = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$, where

$$\mathcal{K}^+ = \{\lambda(2,1) \colon \lambda \in \mathbb{C}\}\,, \qquad \mathcal{K}^- = \{\lambda(1,2) \colon \lambda \in \mathbb{C}\} \tag{2.10}$$

$$\mathcal{K}_1^+ = \{\lambda(1,0) \colon \lambda \in \mathbb{C}\}\,, \qquad \mathcal{K}_1^- = \{\lambda(0,1) \colon \lambda \in \mathbb{C}\}\,. \tag{2.11}$$

The fundamental symmetry J associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ is given by the rule

$$J(z,w) = \frac{2z-w}{3}(2,1) + \frac{z-2w}{3}(1,2)\,, \quad z,\, w \in \mathbb{C}\,, \tag{2.12}$$

and the fundamental symmetry J_1 associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$ is given by the rule

$$J_1(z,w) = (z,-w) = z(1,0) - w(0,1)\,. \tag{2.13}$$

On $\mathbb{C}^2$ consider the operator $S \colon \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$S(z,w) = (z,-3w)\,, \quad z,\, w \in \mathbb{C}\,. \tag{2.14}$$

For every $z,\, w \in \mathbb{C}$

$$J_1S(z,w) = J_1(z,-3w) = (z,3w) = SJ_1(z,w),$$

$$JS(z,w) = J(z,-3w) = \Big(\frac{2z+3w}{3}(2,1) + \frac{z+6w}{3}(1,2)\Big)$$

$$SJ(z,w) = \frac{2z-w}{3}(2,-3) + \frac{z-2w}{3}(1,-6) = \frac{22z-23w}{9}(2,1) + \frac{34w-29z}{9}(1,2).$$

Therefore, S commutes with the fundamental symmetry associated to the fundamental decomposition (2.11), but it does not commute with the fundamental symmetry associated to the fundamental decomposition (2.10). By the (i) and (v) parts of the Proposition 2.3 we obtain the desired result.

3. Hyponormal operators on Krein spaces

In this section our main results are stated and proved. First of all we introduce the hyponormal operators and J-hyponormal operators in Krein spaces.

Definition 3.1. An operator $T \in \mathcal{B}(\mathcal{K})$ is said to be *hyponormal* if for every $x \in \mathcal{K}$

$$[(T^{[*]}T - TT^{[*]})x, x] \geq 0\,. \tag{3.1}$$

Definition 3.2. An operator $T \in \mathcal{B}(\mathcal{K})$ is said to be *J-hyponormal* if for every $x \in \mathcal{K}$

$$[(T^{*J}T - TT^{*J})x, x]_J \geq 0\,. \tag{3.2}$$

Example 3.3. Let $\mathcal{K} = \mathbb{C}^2$ be a Krein space with indefinite inner product $[(z, w), (u, v)] = z\overline{u} - w\overline{v}$. Observe that $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$, where

$$\mathcal{K}^+ = \{\lambda(2,1) \colon \lambda \in \mathbb{C}\}\,, \qquad \mathcal{K}^- = \{\lambda(1,2) \colon \lambda \in \mathbb{C}\}\,.$$

The fundamental symmetry J associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ is given in (2.12). On $\mathbb{C}^2$ consider the operator $S \colon \mathbb{C}^2 \to \mathbb{C}^2$ given in Remark 2.5. This operator is self-adjoint and hence it is hyponormal. On the other hand, since $S = S^{[*]} = JS^{*J}J$, we get by Remark 2.5 that

$$S^{*J}(z, w) = JSJ(z, w) = \frac{22z - 23w}{9}(2, 1) + \frac{29z - 34w}{9}(1, 2)\,.$$

Therefore, by direct calculations one gets $[(S^{*J}S - SS^{*J})(1, 0), (1, 0)]_J < 0$. Thus, S is not a J-hyponormal operator.

The next result is proposed in [3, 12] and its proof is a slight modification of Example 9 in [3, §9].

Proposition 3.4. *Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space with fundamental symmetry J associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and let $T \in \mathcal{B}(\mathcal{K})$ be such that $TM \subset M$ for some closed subspace M of $\mathcal{K}$. If T is J-hyponormal then $T|_M$ is J-hyponormal in the Hilbert space $(M, [\cdot, \cdot]_J)$.*

Proof. Let $y \in M$. For any $x \in M$ we have $[Tx, y]_J = [TP_M x, y]_J = [Rx, y]_J = [x, R^{*J}y]_J$, where $R = TP_M$ and P_M is the orthogonal projection from $\mathcal{K}$ onto M. i.e. $[x, (T^{*J} - R^{*J})y]_J = 0$ for each $x \in M$. It follows that $(T^{*J} - R^{*J})y \in M^\perp$. Furthermore, for every $y \in M$:

$$[(T^{*J} - R^{*J})y, R^{*J}y]_J = [TP_M T^{*J}y, y]_J - [TP_M P_M T^{*J}y, y]_J = 0\,.$$

By the Pythagorean Theorem, for every $y \in M$

$$\|T^{*J}y\|_J^2 = \|(T^{*J} - R^{*J})y + R^{*J}y\|_J^2 = \|(T^{*J} - R^{*J})y\|_J^2 + \|R^{*J}y\|_J^2\,.$$

Now, being T a J-hyponormal operator one gets for every $y \in M$

$$\|R^{*J}y\|_J^2 \leq \|T^{*J}y\|_J^2 \leq \|Ty\|_J^2 = \|TP_M y\|_J^2 = \|Ry\|_J^2\,.$$

i.e. the operator $R = T|_M$ is hyponormal in the Hilbert space $(M, [\cdot, \cdot]_J)$. □

Proposition 3.5. *Let* $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ *be a Krein space with fundamental symmetry* J *associated to the fundamental decomposition* $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$, $M \subset \mathcal{K}$ *be a closed subspace with respect to* $[\cdot,\cdot]_J$ *and* $T \in \mathcal{B}(\mathcal{K})$. *If* T *is* J*-hyponormal with* $T|_M$ *normal and* $TM \subset M$, *then* M *reduces* T.

Proof. We now proceed analogously to the proof of Proposition 3.4. Let $y \in M$. Then for any $x \in M$ we have $[Tx, y]_J = [TP_M x, y]_J = [Rx, y]_J = [x, R^{*J} y]_J$, where $R = TP_M$. That is, $[x, (T^{*J} - R^{*J})y] = 0$ for each $x \in M$. Therefore it follows that $(T^{*J} - R^{*J})y \in M^\perp$ for all $y \in M$. Furthermore, for every $y \in M$

$$[(T^{*J} - R^{*J})y, R^{*J}y]_J = [TP_M T^{*J} y, y]_J - [TP_M P_M T^{*J} y, y] = 0 .$$

Since R is J-normal one gets that by Pythagorean Theorem for every $y \in M$

$$\begin{aligned}\|T^{*J}y\|_J^2 &= \|(T^{*J} - R^{*J})y\|_J^2 + \|R^{*J}y\|_J^2 = \|(T^{*J} - R^{*J})y\|_J^2 + \|Ry\|_J^2 \\ &= \|(T^{*J} - R^{*J})y\|_J^2 + \|TP_M y\|_J^2 = \|(T^{*J} - R^{*J})y\|_J^2 + \|Ty\|_J^2 .\end{aligned}$$

This implies that $\|(T^{*J} - R^{*J})y\|_J^2 = \|T^{*J}y\|_J^2 - \|Ty\|_J^2$ for each $y \in M$, however, by Proposition 3.4 the operator $T|_M$ is hyponormal in the Hilbert space $(M, [\cdot,\cdot]_J)$, equivalently,

$$\|Ty\|_J^2 - \|T^{*J}y\|_J^2 = [Ty, Ty]_J - [T^{*J}y, T^{*J}y]_J = [(T^{*J}T - TT^{*J})y, y]_J \geq 0 .$$

Therefore, it follows that $\|(T^{*J} - R^{*J})y\|_J^2 = 0$ for each $y \in M$, i.e. for every $x = y_M + y_{M^\perp} \in \mathcal{K} = M \oplus M^\perp$ we have

$$T^{*J} P_M x = T^{*J} y_M = R^{*J} y_M = (TP_M)^{*J} y_M = P_M T^{*J} y_M = P_M T^{*J} P_M x .$$

Thus, $T^{*J} P_M = P_M T^{*J} P_M$. By Theorem 2 in [3, §9] we conclude that $T^{*J} M \subset M$. Since $TM \subset M$, again Theorem 2 in [3, §9] we also have $T^{*J} M^\perp \subset M^\perp$ and $TM^\perp = (T^{*J})^{*J} M^\perp \subset M^\perp$. According to this, we have that M reduces T. □

The next result is an immediate consequence of Definition 3.1.

Proposition 3.6. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space. Then* T *and* $T^{[*]} \in \mathcal{B}(\mathcal{K})$ *are hyponormal if and only if* T *is a normal operator.*

Definition 3.7. Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space. A pair of bounded operators T, $V \in \mathcal{B}(\mathcal{K})$ are said to be *unitarily equivalent* if there exists a unitary operator U on $\mathcal{K}$ such that $V = UTU^{[*]}$.

It is well-known that in the Hilbert case the hyponormality is preserved by unitary equivalence of operators, see e.g. [11, 12]. Since the proof only depends on the existence of the unitary operator that relate them, we have that in the Krein space case this also holds.

Proposition 3.8. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space and* T, $V \in \mathcal{B}(\mathcal{K})$ *be unitary equivalent operators. If* T *is hyponormal then* V *is also hyponormal.*

If T is a hyponormal operator then by Definition 3.1 the bounded operator $T^{[*]}T - TT^{[*]}$ is positive on the Krein space $\mathcal{K}$. Letting $(x, x)_1 = [(T^{[*]}T - TT^{[*]})x, x]$ in Theorem II.6.2 in [4] the following result holds.

Proposition 3.9. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space and* $T \in \mathcal{B}(\mathcal{K})$. *If* T *is hyponormal in* $\mathcal{K}$ *then for every* $x \in \mathcal{K}$

$$[(T^{[*]}T - TT^{[*]})x, x] \geq \mu_+(T^{[*]}T - TT^{[*]})[x, x]\,. \tag{3.3}$$

Proposition 3.10. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space. If the linear operator* $T \in \mathcal{B}(\mathcal{K})$ *is hyponormal then* $0 \in [\mu_-(T^{[*]}T - TT^{[*]}), \mu_+(T^{[*]}T - TT^{[*]})]$, *where* $\mu_\pm(T)$ *are given in* (2.7).

Proof. Let T be a hyponormal operator. Then by (2.8) for every $x \in \mathcal{K}^{++}$ and $y \in \mathcal{K}^{--}$ we have that

$$\frac{[(T^{[*]}T - TT^{[*]})x, x]}{[x, x]} \geq 0 \geq \frac{[(T^{[*]}T - TT^{[*]})y, y]}{[y, y]}\,.$$

Therefore $0 \in [\mu_-(T^{[*]}T - TT^{[*]}), \mu_+(T^{[*]}T - TT^{[*]})]$. □

The following result is a slight modification of Lemma 1 in [12].

Proposition 3.11. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space and* $T, \mathcal{W} \in \mathcal{B}(\mathcal{K})$ *be such that* $\mathcal{W} = \mathcal{W}^{[*]}$ *and* $T\mathcal{W} = \mathcal{W}T$. *Then* T *is hyponormal in* $\mathcal{K}$ *if and only if for every* $\lambda \in \mathbb{C}$ *the operator* $T - \lambda\mathcal{W}$ *is hypornormal .*

Proof. The equivalence follows since for every $\lambda \in \mathbb{C}$ one has that $(T - \lambda\mathcal{W})^{[*]}(T - \lambda\mathcal{W}) - (T - \lambda\mathcal{W})(T - \lambda\mathcal{W})^{[*]} = T^{[*]}T - TT^{[*]}$. □

3.1. **Fundamentally reducible hyponormal operators.** As it was mentioned in the Introduction a bounded operator T on the Krein space $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^+$ is said to be *fundamentally reducible* if there is a fundamental decomposition of $\mathcal{K} = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$ such that $\mathcal{K}_1^+$ and $\mathcal{K}_1^-$ are invariant under T. By proposition 2.3 one gets that $TJ_1 = J_1T$, where J_1 is the fundamental symmetry associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$. In this case, we have two fundamental symmetries J and J_1 associated to $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and $\mathcal{K} = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$ respectively. It is well-known that the corresponding J-norms $\|\cdot\|_J$ and $\|\cdot\|_{J_1}$ are equivalent, see [2, Theorem 7.19], hence if T is bounded with respect to the norm $\|\cdot\|_J$ also is bounded with respect to the norm $\|\cdot\|_{J_1}$ and by Proposition 2.3 we have $T^{*J_1} = T^{[*]} = JT^{*J}J$. Thus, T^{*J_1} and T^{*J} are unitarily equivalent by Definition 3.7. On the other hand, there are bounded operators that are hyponormal but not J-hyponormal. In Example 3.3 we have seen such kind of operators, in fact, the operator considered there, it is not fundamentally reducible with respect to the given fundamental decomposition of the Krein space. Notice that $(2, 1)$ belongs to $\mathcal{K}^+$ but $S(2, 1) = (4, -6) = \frac{14}{3}(2, 1) - \frac{16}{3}(1, 2)$.

In what follows we consider hyponormal operators on Krein spaces that are fundamentally reducibles. Now, a natural question arises here: *If* T *is a hyponormal operator on a Krein space* $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ *and it is fundamentally reducible by the fundamental decomposition* $\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-$, *then* T *is* J*-hyponormal, where* J *is the fundamental symmetry associated to* $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ *or is it* J_T*-hyponormal, where* J_T *is the fundamental symmetry associated to* $\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-$ *?*

First of all, we give some answers to this question assuming that T is fundamentally reducible by the given fundamental decomposition of $\mathcal{K}$.

As by-product of Proposition 3.4 any J-hypornomal operator which is fundamentally reducible by the given fundamental decomposition is sum of J-hyponormal operators on $\mathcal{K}^+$ and $\mathcal{K}^-$.

Proposition 3.12. *Let* $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ *be a Krein space with fundamental symmetry* J *associated to the fundamental decomposition* $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ *and* $T \in \mathcal{B}(\mathcal{K})$ *be a fundamentally reducible operator by the fundamental decomposition* $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$. *Then* T *is* J*-hyponormal if and only if* $T = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} p$, *where* U, V *are hyponormal in* $(\mathcal{K}^+, [\cdot,\cdot])$ *and* $(\mathcal{K}^-, -[\cdot,\cdot])$ *respectively.*

Proof. By Proposition 2.3 one gets that $\mathcal{K}^\pm$ are invariant under T and $T^{[*]}$ (By item (iii) in Proposition 2.3 we also have $T^{[*]} = T^{*J}$). Let $U = TP_+$ and $V = TP_-$, where $P_\pm$ are the orthogonal projection from $\mathcal{K}$ onto $\mathcal{K}^\pm$ respectively. Now, the statement holds since

$$[(T^{*J}T - TT^{*J})x, x]_J = [(U^{*J}U - UU^{*J})x_+, x_+] - [(V^{*J}V - VV^{*J})x_-, x_-]$$

for each $x = x_+ + x_- \in \mathcal{K}$. □

Proposition 3.13. *Let* $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ *be a Krein space with fundamental symmetry* J *associated to the fundamental decomposition* $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ *and* $T \in \mathcal{B}(\mathcal{K})$ *be a fundamentally reducible hyponormal operator. Then* T *is* J*-hyponormal if and only if* $T|_{\mathcal{K}^-}$ *is normal.*

Proof. Let $T \in \mathcal{B}(\mathcal{K})$ be a hyponormal operator in $(\mathcal{K}, [\cdot,\cdot])$ such that $T\mathcal{K}^\pm \subset \mathcal{K}^\pm$. Observe that $T^{*J} = T^{[*]}$ by Proposition 2.3. If T is also J-hyponormal operator then by Proposition 3.4 we have that TP_+ is hyponormal in the Hilbert space $(\mathcal{K}^+, [\cdot,\cdot])$ and TP_- is hyponormal in the Hilbert space $(\mathcal{K}^-, -[\cdot,\cdot])$, i.e., for every $x^- \in \mathcal{K}^-$ we have that

$$\begin{aligned} 0 &\le [(T^{[*]}T - TT^{[*]})x^-, x^-] = [Tx^-, Tx^-] - [T^{[*]}x^-, T^{[*]}x^-] \\ &= [P_-TP_-x^-, P_-TP_-x^-] - [P_-T^{[*]}P_-x^-, P_-T^{[*]}P_-x^-] \\ &= [((P_-TP_-)^{[*]}(P_-TP_-) - (P_-TP_-)(P_-TP_-)^{[*]})x^-, x^-] \\ &= -[((P_-TP_-)(P_-TP_-)^{[*]} - (P_-TP_-)^{[*]}(P_-TP_-))x^-, x^-]\,. \end{aligned}$$

Therefore $TP_- = P_-TP_-$ and $P_-T^{*J}P_- = P_-T^{[*]}P_- = (P_-TP_-)^{[*]} = (TP_-)^{[*]}$ are hyponormal in $(\mathcal{K}^-, -[\cdot,\cdot])$ and hence by Proposition 3.6 we conclude that $T|_{\mathcal{K}^-} = TP_- = P_-TP_-$ is normal.

Conversely, if $T|_{\mathcal{K}^-} = TP_- = P_-TP_-$ is a normal operator, then by Proposition 2.3 one gets

$$(T^{*J}T - TT^{*J})P_-x = ((P_-TP_-)(P_-TP_-)^{[*]} - (P_-TP_-)^{[*]}(P_-TP_-))P_-x = 0\,.$$

Thus, by direct calculations one has for every $x \in \mathcal{K}$

$$\begin{aligned} [(T^{*J}T - TT^{*J})x, x]_J &= [(T^{*J}T - TT^{*J})P_+x, P_+x] - [(T^{*J}T - TT^{*J})P_-x, P_-x] \\ &= [(T^{[*]}T - TT^{[*]})P_+x, P_+x] \ge 0\,, \end{aligned}$$

i.e. T is a J-hyponormal operator in $(\mathcal{K}, [\cdot,\cdot]_J)$. □

Proposition 3.13 says that the hyponormality of a fundamentally reducible operator T is hereditary from the Krein space to the associated Hilbert space if T has the form $T = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, where U is a hyponormal operator in $(\mathcal{K}^+, [\cdot,\cdot])$ and V is normal in $(\mathcal{K}^-, -[\cdot,\cdot])$. On the other hand, since $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^-, -[\cdot,\cdot])$ are Hilbert spaces and $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$, Proposition 3.12 can be considered for any T on a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $TP_1 = P_1T$ and $P_2T = TP_2$, where P_i are orthogonal projection on $\mathcal{H}_i$ respectively $i = 1, 2$.

The next result establishes a characterization of hyponormal operators in $\mathcal{K}$ that are fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$.

Theorem 3.14. *Let $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J associated to the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and $T \in \mathcal{B}(\mathcal{K})$. Then T is hyponormal and fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ if and only if T is J-hyponormal, $T|_{\mathcal{K}^-}$ is normal and $T\mathcal{K}^- \subset \mathcal{K}^-$.*

Proof. Let T be a hyponormal operator and fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$. Then it follows that $T\mathcal{K}^- \subset \mathcal{K}^-$, $T^{[*]} = T^{*J}$ and $[(T^{[*]}T - TT^{[*]})x_-, x_-] \geq 0$ for each x_- in $\mathcal{K}^-$. On the other hand, the operator $(T^{[*]}T - TT^{[*]})$ is also fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$. This implies that $[(T^{[*]}T - TT^{[*]})P_-x, P_-x] \leq 0$ for each $x \in \mathcal{K}$. Therefore, $T|_{\mathcal{K}^-}$ is normal and by Proposition 3.13 we have that T is J-hyponormal.

Conversely, let T be J-hyponormal, $T|_{\mathcal{K}^-}$ normal and $T\mathcal{K}^- \subset \mathcal{K}^-$. Then by Proposition 3.5 we have that $\mathcal{K}^-$ reduces T. That is $T\mathcal{K}^- \subset \mathcal{K}^-$ and $T\mathcal{K}^+ \subset \mathcal{K}^+$ since $\mathcal{K}^+ = (\mathcal{K}^-)^\perp$. In consequence T is fundamentally reducible by the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$. According to Proposition 3.12, we have that TP_+ is hyponormal in $(\mathcal{K}^+, [\cdot,\cdot])$ and $TP_- = T|_{\mathcal{K}^-}$ is normal in $(\mathcal{K}^-, -[\cdot,\cdot])$. Therefore, T is hyponormal in $\mathcal{K}$ since $[(T^{*J}T - TT^{*J})P_+x, P_+x]_J = [(T^{[*]}T - TT^{[*]})x, x]$. $\square$

From now on, we assume hyponormal operators on the Krein space $\mathcal{K}$ with fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ that are fundamentally reducibles by the fundamental decomposition $\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-$ where $\mathcal{K}_T^\pm \neq \mathcal{K}^\pm$.

Lemma 3.15. *Let $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ be a Krein space with fundamental decomposition J associated to $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and $T \in \mathcal{B}(\mathcal{K})$ be a fundamentally reducible operator. Then there exist a Hilbert space $(\mathcal{K}_T, \langle\cdot,\cdot\rangle)$ associated to $\mathcal{K}$ which is isomorphic to $(\mathcal{K}, [\cdot,\cdot]_J)$ such that $T^{[*]} = T^*$, where T^* is the adjoint operator of T with respect to the inner product $\langle\cdot,\cdot\rangle$.*

Proof. If T is fundamentally reducible on $\mathcal{K}$, then there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-$ such that $\mathcal{K}_T^\pm$ are invariant under T. Let J_T be the fundamental symmetry associated to $\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-$, hence $TJ_T = J_TT$. Letting $\mathcal{K}_T = \overline{\mathcal{K}}^{\|\cdot\|}$, where $\|\cdot\|$ is the norm generated by the scalar inner product $\langle x, y\rangle = [J_Tx, y]$. Since the norms $\|\cdot\|$ and $\|\cdot\|_J$ are equivalent, it follows that

$\mathcal{K}_T = (\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-, \langle\cdot,\cdot\rangle)$. Now, for every $x, y \in \mathcal{K}$

$$\langle Tx, y\rangle = [J_T Tx, y] = [TJ_T x, y] = [J_T x, T^{[*]}y] = \langle x, T^{[*]}y\rangle .$$

This gives $T^* = T^{[*]}$. On the other hand, notice for every x, $y \in \mathcal{K}$ that

$$\langle Jx, y\rangle = [J_T Jx, y] = [x, JJ_T y] = [J_T x, J_T JJ_T y] = \langle x,_T JJ_T y\rangle .$$

Thus $J^* = J_T JJ_T$. It is clear that the operator JJ_T is well defined, positive and invertible on $(\mathcal{K}, [\cdot,\cdot]_J)$. In effect, since $J_T J_T = \mathrm{Id}$, one gets

$$[JJ_T x, x]_J = [JJJ_T x, x] = [J_T x, x] = \langle x, x\rangle \geq 0$$
$$JJ_T(J_T J) = JJ_T^2 J = JJ = \mathrm{Id}_{\mathcal{K}}$$

for each $x \in \mathcal{K}$. Let $U = \sqrt{JJ_T}\colon \mathcal{K} \subseteq \mathcal{K}_T \to \mathcal{K}$. Then $U^2 = JJ_T$ and for every x, $y \in \mathcal{K} \subseteq \mathcal{K}_T$

$$\|Ux\|_J^2 = [Ux, Ux]_J = [U^2 x, x]_J = [JJ_T x, x]_J = [x, x]_{J_J} = \|x\|_{J_T}^2 .$$

This implies that U is an isometry from $\mathcal{K} \subseteq \mathcal{K}_T$ into $(\mathcal{K}, [\cdot,\cdot]_J)$ and hence it has unitary extension $\hat{U}$ on $\mathcal{K}_T$. □

Theorem 3.16. *Let $(\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, [\cdot,\cdot])$ be a Krein space with fundamental decomposition J associated to $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$. If $T \in \mathcal{B}(\mathcal{K})$ is a hyponormal and fundamental reducible operator in the Krein space $\mathcal{K}$ then there exists an associated Hilbert space $(\mathcal{K}_T, \langle\cdot,\cdot\rangle)$ that is isomorphic to $(\mathcal{K}, [\cdot,\cdot]_J)$ in which T is hyponormal and $T|_{\mathcal{K}_T^-}$ is normal.*

Proof. Since T is fundamental reducible on the Krein space by Lemma 3.15 there exists a Hilbert space $\mathcal{K}_T = (\mathcal{K} = \mathcal{K}_T^+ \oplus \mathcal{K}_T^-, \langle\cdot,\cdot\rangle)$ associated to $\mathcal{K}$ isomorphic to the Hilbert space $(\mathcal{K}, [\cdot,\cdot]_J)$ where $T|_{\mathcal{K}_T^\pm} \subset \mathcal{K}_T^\pm$ and $T^{[*]} = T^*$, where T^* is the adjoint operator of T with respect to the inner product $\langle\cdot,\cdot\rangle$. Therefore, by Theorem 3.14 we have that T is hyponormal in $(\mathcal{K}_T, \langle\cdot,\cdot\rangle)$ and $T|_{\mathcal{K}_T^-}$ is normal. □

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