ARCHIVUM MATHEMATICUM (BRNO)
Tomus 55 (2019), 261–264

ON THE γ-EQUIVALENCE OF SEMIHOLONOMIC JETS

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Abstract. It is well known that the concept of holonomic r-jet can be geometrically characterized in terms of the contact of individual curves. However, this is not true for the semiholonomic r-jets, [5], [8]. In the present paper, we discuss systematically the semiholonomic case.

In [5], the second author introduced the concept of equivalence with respect to curves, or γ-equivalence, of semiholonomic r-jets, when he studied the contact of spaces with higher order connection according to C. Ehresmann, [4]. In the present paper, we study the general form of this problem. In Section 5, we describe how to discuss the γ-equivalence of two arbitrary semiholonomic r-jets. We use an original concept of k-sesquiholonomic r-jet, that generalizes an idea by P. Libermann, [8].

Unless otherwise specified, we use the terminology and notation from the book [6].

1. Semiholonomic r-jets

The r-th semiholonomic prolongation $\overline{J}^r Y \to M$ of a fibered manifold $p\colon Y \to M$ is defined as follows. By induction, we have constructed a projection $\pi^{r-2}_{r-1}\colon \overline{J}^{r-1} Y \to \overline{J}^{r-2} Y$. The elements of $\overline{J}^r Y$ are 1-jets $j^1_x s$ of the local sections $s\colon M \to \overline{J}^{r-1} Y$ satisfying

$$s(x) = j^1_x(\pi^{r-2}_{r-1} \circ s)\,. \tag{1}$$

If x^i, y^p are some local fiber coordinates on Y and $y^p_i, \dots, y^p_{i_1 \dots i_{r-1}}$ are the induced local coordinates on $\overline{J}^{r-1} Y$ arbitrary in all subscripts, then the induced local coordinates on $\overline{J}^r Y \to \overline{J}^{r-1} Y$ are

$$y^p_{i_1 \dots i_r}(j^1_x s) = \frac{\partial y^p_{i_1 \dots i_{r-1}}(s)}{\partial x^{i_r}}(x)\,. \tag{2}$$

Hence even $y^p_{i_1 \dots i_r}$ are arbitrary in all subscripts. The r-th holonomic prolongation $J^r Y$ is a subbundle of $\overline{J}^r Y$, whose all coordinates are symmetric in all subscripts. The space $\overline{J}^r(M, N)$ of semiholonomic r-jets of M into N is defined as the r-th

2010 *Mathematics Subject Classification*: primary 58A20; secondary 58A32.
Key words and phrases: jet bundle, semiholonomic r-jet.
The second author was supported by GA ČR under the grant GA17-01171F.
Received April 2, 2019. Editor J. Slovák.
DOI: 10.5817/AM2019-4-261

semiholonomic prolongation of the product fibered manifold $Y = M \times N \to M$. We write α or β for the source or target projection.

The semiholonomic r-jets are endowed with the restriction of the composition of nonholonomic r-jets by Ehresmann, [3]. If $X = j^1_x f \in \overline{J}^r(M,N)$ and $Y = j^1_y g \in \overline{J}^r(N,Q)$, $y = \beta X$, then

$$Y \circ X = j^1_x\big(g\big(\beta f(u)\big) \circ f(u)\big) \in \overline{J}^r(M,Q)\,, \quad u \in M\,, \tag{3}$$

with the composition of semiholonomic $(r-1)$-jets on the right hand side. The composition of holonomic r-jets coincides with the classical one.

2. The equivalence with respect to curves

In [5], the second author introduced a special version of this idea, when he investigated the contact of spaces with higher order connection in the sense of Ehresmann, [4]. In the general situation, we define

Definition 1. Two r-jets $B, C \in \overline{J}^r_x(M,N)_y$ are equivalent by curves, or γ-equivalent, if

$$B \circ A = C \circ A \quad \text{for all} \quad A = j^r_0\gamma\,,\ \gamma\colon \mathbb{R} \to M,\ \gamma(0) = x\,. \tag{4}$$

We shall write $B \sim_\gamma C$.

It is well known that two holonomic r-jets $B, C \in J^r_x(M,N)_y$ are equivalent by curves, if and only if $B = C$. But in the semiholonomic case, the situation is different.

In the simpliest case $B, C \in \overline{J}^2_0(\mathbb{R}^m, \mathbb{R}^n)_0$, $B = (y^p_i, y^p_{ij})$, $C = (z^p_i, z^p_{ij})$ and $A = (a^i_1, a^i_2) \in J^2_0(\mathbb{R}, \mathbb{R}^m)_0$, $B \circ A = C \circ A$ for all A means

$$(y^p_i a^i_1, y^p_{ij} a^i_1 a^j_1 + y^p_i a^i_2) = (z^p_i a^i_1, z^p_{ij} a^i_1 a^j_1 + z^p_i a^i_2)\,, \tag{5}$$

i.e. $y^p_i = z^p_i$ and $y^p_{(ij)} = z^p_{(ij)}$. This proves that B, $C \in \overline{J}^2_x(M,N)_y$ are γ-equivalent if and only if the symmetrizations of B and C in $J^2_x(M,N)_y$ coincide.

3. The k-sesquiholonomic r-jets

According to P. Libermann, [8], $\overline{J}^r(M,N)$ is a pullback of $TN \otimes \otimes^r T^*M$ over $\overline{J}^{r-1}(M,N)$. (We remark that the fact $J^r(M,N)$ is a pullback of $TN \otimes S^r T^*M$ over $J^{r-1}(M,N)$ was deduced in [6].) Further, she defined the r-th sesquiholonomic prolongation $\check{J}^r(M,N) \subset \overline{J}^r(M,N)$ by

$$X \in \check{J}^r(M,N) \quad \text{means} \quad \pi^{r-1}_r X \in J^{r-1}(M,N)\,. \tag{6}$$

So, $\check{J}^r(M,N)$ is the pullback of $TN \otimes \otimes^r T^*M$ over $J^{r-1}(M,N)$. Further, the tensor symmetrization $TN \otimes \otimes^r T^*M \to TN \otimes S^r T^*M$ induces a map $\rho_r : \check{J}^r(M,N) \to J^r(M,N)$, see also [2]. Analogously to (5), one verifies that $B, C \in \check{J}^r_x(M,N)_y$ are γ-equivalent, if and only if $\pi^{r-1}_r B = \pi^{r-1}_r C$ and

$$\rho_r(B) = \rho_r(C) \in J^r_x(M,N)_y\,. \tag{7}$$

We generalize the concept of sesquiholonomic r-jet as follows.

Definition 2. A jet $X \in \overline{J}^r(M,N)$ is called k-sesquiholonomic, $k < r$, if $\pi^k_r X \in J^k(M,N)$.

We shall write $X \in \check{J}^{r,k}(M,N)$. So sesquiholonomic in the sense of Libermann means $(r-1)$-sesquiholonomic under our approach.

Proposition 1. *B, $C \in \check{J}^r_x(M,N)_y$ are γ-equivalent, if and only if $\pi^{r-1}_r B = \pi^{r-1}_r C$ and $\rho_r(B) = \rho_r(C)$.*

Proof. The highest order coordinates of B or C are $y^p_{i_1\dots i_r}$ or $z^p_{i_1\dots i_r}$, respectively. One finds easily that $B \sim_\gamma C$ means

$$z^p_{i_1\dots i_r} a^{i_1}_1 \dots a^{i_r}_1 = y^p_{i_1\dots i_r} a^{i_1}_1 \dots a^{i_r}_1 \,. \tag{8}$$

This is the coordinate form of $\rho_r(B) = \rho_r(C)$. □

4. The case $r = 3$

It is useful to discuss this special case separately.

Proposition 2. *B, $C \in \overline{J}^3_0(M,N)_0$ are γ-equivalent, if and only if $\pi^2_3 B = \pi^2_3 C \in J^2_0(M,N)_0$ and $\rho_3(B) = \rho_3(C) \in J^3_0(M,N)_0$.*

Proof. Let $B = (y^p_i, y^p_{ij}, y^p_{ijk})$, $C = (z^p_i, z^p_{ij}, z^p_{ijk})$ and $A = (a^i_1, a^i_2, a^i_3) \in J^3_0(\mathbb{R}, \mathbb{R}^m)_0$. From (3), we deduce for $C \circ A = B \circ A$

$$z^p_i a^i_1 = y^p_i a^i_1 \,, \tag{9}$$

$$z^p_{ij} a^i_1 a^j_1 + z^p_i a^i_2 = y^p_{ij} a^i_1 a^j_1 + y^p_i a^i_2, \tag{10}$$

$$z^p_{ijk} a^i_1 a^j_1 a^k_1 + z^p_{ij}(a^i_2 a^j_1 + a^i_1 a^j_2) + z^p_{ij} a^i_2 a^j_1 + z^p_i a^i_3 = \{y\} \,, \tag{11}$$

where $\{y\}$ in (11) means that all $z's$ on the left hand side are replaced by the corresponding $y's$. Since a^i_1 are arbitrary quantities, (9)–(11) imply

$$z^p_i = y^p_i \,, \quad z^p_{(ij)} = y^p_{(ij)} \,, \quad z^p_{(ijk)} = y^p_{(ijk)} \,. \tag{12}$$

Further, for $a^i_2 = 1$, $a^j_1 = 1$ and other $a's$ equal to zero, we obtain from (11) the additional conditions

$$z^p_{(ij)} + z^p_{ij} = y^p_{(ij)} + y^p_{ij} \,, \tag{13}$$

what implies $z^p_{ij} = y^p_{ij}$. This proves our assertion. □

We remark that the coordinate formula for the composition $Y \circ X$ of two arbitrary semiholonomic r-jets is deduced in [1]. However, in our case the coordinate form of $X = A$ is very special. So we find more suitable the direct use of (3) than the specialization of the general formula from [1].

5. The general situation

Consider two k-sesquiholonomic r-jets B, $C \in \check{J}^{r,k}(M,N)$. If $B \sim_\gamma C$, then $\pi^s_r B \sim_\gamma \pi^s_r C$ for all $s \geq k$. In the case $s = k$, these jets are holonomic and $\pi^k_r B \sim_\gamma \pi^k_r C$ is equivalent to $\pi^k_r B = \pi^k_r C$. Then $\pi^{k+1}_r B, \pi^{k+1}_r C \in \check{J}^{k+1}(M,N)$ and $\pi^{k+1}_r B \sim_\gamma \pi^{k+1}_r C$ is equivalent to $\rho_{k+1}(\pi^{k+1}_r B) = \rho_{k+1}(\pi^{k+1}_r C)$. On the other hand, if the last equation is not satisfied, we do not have $\pi^{k+1}_r B \sim_\gamma \pi^{k+1}_r C$. Hence even $B \sim_\gamma C$ cannot be true.

The situation $r = 3$ is specific in that sense, that we can deduce $\pi^2_3 B = \pi^2_3 C$ directly from (11) with no additional conditions on B and C.

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