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REPDIGITS IN GENERALIZED PELL SEQUENCES

Jhon J. Bravo and Jose L. Herrera

Abstract. For an integer $k \geq 2$, let $(P_n^{(k)})_n$ be the $k-$generalized Pell sequence which starts with $0, \dots, 0, 1$ (k terms) and each term afterwards is given by the linear recurrence $P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}$. In this paper, we find all k-generalized Pell numbers with only one distinct digit (the so-called repdigits). Some interesting estimations involving generalized Pell numbers, that we believe are of independent interest, are also deduced. This paper continues a previous work that searched for repdigits in the usual Pell sequence $(P_n^{(2)})_n$.

1. Introduction

Diophantine equations are one of the oldest subjects in number theory. The mathematical community dedicated to the study of Diophantine equations has been interested in problems involving linear recurrence sequences and with special attention in the Fibonacci sequence $F = (F_n)_{n=0}^{\infty}$, which is given by the recurrence $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$ as initial conditions. There is also a lot of interest in studying equations that contain the Pell sequence, which is as important as the Fibonacci sequence. The Pell sequence $P = (P_n)_{n=0}^{\infty}$ is given by $P_0 = 0$, $P_1 = 1$ and the linear recurrence $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$. For additional information on these sequences, including a numerous applications in a range of disciplines, see Koshy's book [11].

For an integer $k \geq 2$, we consider a generalization of the Pell sequence called the k-generalized Pell sequence or, for simplicity, the k-Pell sequence $P^{(k)} = (P_n^{(k)})_{n=-(k-2)}^{\infty}$ given by the following linear recurrence of higher order

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2\,,$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$.

We shall refer to $P_n^{(k)}$ as the *nth* k-Pell number. We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Pell sequence is obtained for $k = 2$, i.e., $P^{(2)} = P$.

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Below we present the values of these numbers for the first few values of k and $n \geq 1$.

First non–zero $k-$Pell numbers

k	Name	First non-zero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...
3	3-Pell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, ...
4	4-Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, ...
5	5-Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, ...
6	6-Pell	1, 2, 5, 13, 34, 89, 233, 609, 1592, 4162, 10881, 28447, 74371, ...
7	7-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1596, 4176, 10927, 28592, 74815, ...
8	8-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4180, 10941, 28638, 74960, ...
9	9-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10945, 28652, 75006, ...
10	10-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28656, 75020,...

The k-Pell numbers and their properties have been studied by some authors (see [3, 8, 9, 10]). In [9], Kiliç gave some relations involving Fibonacci and k-Pell numbers showing that the k-Pell numbers can be expressed as the summation of the Fibonacci numbers. The authors of [10] defined $P^{(k)}$ in matrix representation and showed that the sums of the k-Pell numbers could be derived directly using this representation.

The first interesting fact about the k-Pell sequence, showed by Kiliç in [9], is that the first $k+1$ non–zero terms in $P^{(k)}$ are Fibonacci numbers with odd index, namely

$$P_n^{(k)} = F_{2n-1} \quad \text{for all} \quad 1 \leq n \leq k+1\,, \tag{1}$$

while the next term is $P_{k+2}^{(k)} = F_{2k+3} - 1$. In adittion, it was also proved in [9] that if $k+2 \leq n \leq 2k+2$, then

$$P_n^{(k)} = F_{2n-1} - \sum_{j=1}^{n-k-1} F_{2j-1} F_{2(n-k-j)}\,.$$

Bravo, Herrera and Luca in [3] investigated the k-generalized Pell sequence and presented some alternative recurrence relations, a generalized Binet formula and different arithmetic properties for $P^{(k)}$. They also generalized some well-known properties of $P^{(2)}$ to the sequence $P^{(k)}$ and showed the exponential growth of $P^{(k)}$. For instance, the following nice formula involving Fibonacci and generalized Pell numbers was proved in [3].

$$P_n^{(k)} = F_{2n-1} - \sum_{j=1}^{n-k-1} F_{2j} P_{n-k-j}^{(k)} \quad \text{holds for all} \quad k \geq 2 \quad \text{and} \quad n \geq k+2\,.$$

Note that the above identity immediately shows that the *nth* k-Pell number does not exceed the Fibonacci number with index $2n-1$. In fact

$$P_n^{(k)} < F_{2n-1} \quad \text{holds for all} \quad k \geq 2 \quad \text{and} \quad n \geq k+2\,.$$

Recall that a positive integer is called a *repdigit* if it has only one distinct digit in its decimal expansion. In particular, these numbers have the form $a(10^m - 1)/9$ for some $m \geq 1$ and $1 \leq a \leq 9$. There is a lot of literature dealing with Pell numbers and repdigits. For instance, in 2018 Normenyo, Luca and Togbé [15] found all repdigits expressible as sums of three Pell numbers, and shortly afterwards they extended their work to four Pell numbers [16]. In 2015, Faye and Luca [7] looked for repdigits in the usual Pell sequence and using some elementary methods concluded that there are no Pell numbers larger than 10 which are repdigits.

In this paper, we extend the previous work [7] and search for k-Pell numbers which are repdigits, i.e., we determine all the solutions of the Diophantine equation

$$P_n^{(k)} = a\Big(\frac{10^\ell - 1}{9}\Big)\,, \tag{2}$$

in positive integers n, k, ℓ, a with $k \geq 2$, $\ell \geq 2$ and $a \in \{1, 2, \ldots, 9\}$. We clarify that the condition $\ell \geq 2$ in the above equation is only meant to ensure that $P_n^{(k)}$ has at least two digits and so to avoid trivial solutions.

Similar problems as the one discussed in this paper have been investigated for the Fibonacci sequence and its generalizations. For example, in 2000 Luca [12] showed that 55 and 11 are the largest repdigits in the Fibonacci and Lucas sequences, respectively. A conjecture (proposed by Marques [13]) about repdigits in k-generalized Fibonacci sequences was proved by Bravo–Luca [4]. The k-generalized Fibonacci sequence starts with $k-1$ consecutive 0's followed by a 1 and each term afterwards is the sum of the k preceding terms. A similar work for k-generalized Lucas sequences was performed in [5], where the k-generalized Lucas sequence follows the same recursive pattern as the k-generalized Fibonacci numbers but starting with $0, \ldots, 0, 2, 1$ (k terms).

Before presenting our main theorem, we mention that in the Pell case, namely when $k = 2$, several well known divisibility properties of the Pell numbers were used by Faye and Luca in [7] to solve equation (2). Unfortunately, divisibility properties similar to those used in [7] are not known for $P^{(k)}$ when $k \geq 3$ and therefore it is necessary to attack the problem in a different way. Our result is the following.

Theorem 1. *The only solutions of the Diophantine equation* (2) *are*

$$(n, k, \ell, a) \in \{(5, 3, 2, 3), (6, 4, 2, 8)\}\,,$$

namely, $P_5^{(3)} = 33$ *and* $P_6^{(4)} = 88$.

Our method is roughly as follows. We use lower bounds for linear forms in logarithms of algebraic numbers to bound n and ℓ polynomially in terms of k. When k is small, the theory of continued fractions suffices to lower such bounds and complete the calculations. When k is large, we deduce an important estimate by using the fact that the dominant root of the k-Pell sequence is exponentially close to φ^2 where φ denotes the golden section, so we use this estimation in our calculations and finish the job.

2. Preliminary results

In this section we present some basic properties of the k-Pell sequences and give some important estimations needed for the sequel. One of them is given by Lemma 2 and will play a crucial role in addressing the large values of k. Additionally, we present a lower bound for a nonzero linear form in logarithms of algebraic numbers and state a reduction lemma which will be the key tool used in this paper to reduce some upper bounds. All these facts will be used in the proof of Theorem 1.

2.1. **The k-Pell sequence.** It is known that the characteristic polynomial of $P^{(k)}$, namely

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1\,,$$

has just one real root outside the unit circle and all the roots are simple. Throughout this paper, $\gamma := \gamma(k)$ denotes that single root which is a Pisot number of degree k since the other roots of $\Phi_k(x)$ are strictly inside the unit circle. This important property of γ leads us to call it *the dominant root* of $P^{(k)}$. Since γ is a Pisot number with minimal polynomial $\Phi_k(x)$, it follows that this polynomial is irreducible over $\mathbb{Q}[x]$ (for more details on $\Phi_k(x)$ see Section 2 of [3]). Moreover, it is also known that $\gamma(k)$ is exponentially close to φ^2. In fact, it was proved in [3, Lemma 3.2] that $\gamma(k)$ is located between $\varphi^2(1-\varphi^{-k})$ and φ^2. To simplify notation, we shall omit the dependence on k of γ whenever no confusion may arise.

Let us now define, for an integer $k \geq 2$, the function

$$g_k(z) = \frac{z-1}{(k+1)z^2 - 3kz + k - 1} = \frac{z-1}{k(z^2-3z+1) + z^2 - 1}\,. \tag{3}$$

If we consider the function $g_k(x)$ defined in (3) as a function of a real variable, then it is not difficult to see that $g_k(x)$ has a vertical asymptote in

$$c_2(k) := \frac{3k + \sqrt{5k^2+4}}{2(k+1)}\,, \tag{4}$$

and is positive and continuous in $(c_2(k), +\infty)$. The following lemma, which is an elementary result, will be needed for some estimations in the next subsection.

Lemma 1. *Keep the above notation and let $k \geq 2$ be an integer. Then*

$$0.276 < g_k(\gamma) < 0.5 \quad \textit{and} \quad |g_k(\gamma_i)| < 1 \quad \textit{for} \quad 2 \leq i \leq k\,,$$

where $\gamma := \gamma_1, \gamma_2, \ldots, \gamma_k$ are the roots of the characteristic polynomial $\Phi_k(x)$.

Proof. The proof of the first part can be found in [3, Lemma 3.2]. For the second part, i.e., for $2 \leq i \leq k$, we consider the function $h_k(x)$ defined by

$$h_k(x) = (x-1)\Phi_k(x) = x^{k-1}(x^2 - 3x + 1) + 1\,. \tag{5}$$

Evaluating expression (5) at γ_i and rearranging some terms of the resulting expression, we get the relation $\gamma_i^2 - 3\gamma_i + 1 = -1/\gamma_i^{k-1}$, and so

$$k(\gamma_i^2 - 3\gamma_i + 1) + \gamma_i^2 - 1 = \gamma_i^2 - 1 - \frac{k}{\gamma_i^{k-1}}\,.$$

Hence,

$$|k(\gamma_i^2 - 3\gamma_i + 1) + \gamma_i^2 - 1| = \Big|\frac{k}{\gamma_i^{k-1}} - (\gamma_i^2 - 1)\Big| \geq \frac{k}{|\gamma_i|^{k-1}} - |\gamma_i^2 - 1| > k - 2\,,$$

where we used the fact that $|\gamma_i| < 1$ because $2 \leq i \leq k$. Consequently,

$$|g_k(\gamma_i)| = \frac{|\gamma_i - 1|}{|k(\gamma_i^2 - 3\gamma_i + 1) + \gamma_i^2 - 1|} < \frac{2}{k-2} \leq 1 \quad \text{for all} \quad k \geq 4\,.$$

Finally, the cases $k = 2$ and 3 can be checked computationally. $\square$

The following “Binet–like” formula for $P^{(k)}$ appears in [3]:

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\gamma_i)\gamma_i^n\,, \tag{6}$$

where, as before, $\gamma := \gamma_1, \gamma_2, \ldots, \gamma_k$ are the roots of the characteristic polynomial $\Phi_k(x)$. It was also proved in [3] that the contribution of the roots which are inside the unit circle to the formula (6) is very small, namely that the approximation

$$|P_n^{(k)} - g_k(\gamma)\gamma^n| < 1/2 \quad \text{holds for all} \quad n \geq 2 - k\,. \tag{7}$$

From (7) we can write

$$P_n^{(k)} = g_k(\gamma)\gamma^n + e_k(n) \quad \text{where} \quad |e_k(n)| < 1/2\,. \tag{8}$$

Furthermore, in [3], it is shown that the inequality

$$\gamma^{n-2} \leq P_n^{(k)} \leq \gamma^{n-1} \quad \text{holds for all} \quad n \geq 1\,, \tag{9}$$

extending a result known for the usual Pell numbers.

We finish this subsection by giving an important estimate of the dominant term in the Binet–like formula for $P^{(k)}$. This estimation, based on the fact that the dominant root of the k-Pell sequence is exponentially close to φ^2, will be the key point in addressing the large values of k, and we believe is of independent interest.

Lemma 2. *Let $\gamma = \gamma(k)$ be the dominant root of the characteristic polynomial $\Phi_k(x)$ of the k-Pell sequence and consider the function $g_k(x)$ defined in* (3)*. If $k \geq 30$ and $n > 1$ are integers satisfying $n < \varphi^{k/2}$, then*

$$g_k(\gamma)\gamma^n = \frac{\varphi^{2n}}{\varphi + 2}(1 + \zeta) \quad \textit{where} \quad |\zeta| < \frac{4}{\varphi^{k/2}}\,.$$

Proof. Let $\lambda > 0$ be such that $\gamma + \lambda = \varphi^2$. Since γ is located between $\varphi^2(1 - \varphi^{-k})$ and φ^2, we get that $\lambda < \varphi^2 - \varphi^2(1 - \varphi^{-k}) = 1/\varphi^{k-2}$, i.e., $\lambda \in (0, 1/\varphi^{k-2})$. Besides,

$$\gamma^n = (\varphi^2 - \lambda)^n = \varphi^{2n}\Big(1 - \frac{\lambda}{\varphi^2}\Big)^n = \varphi^{2n}e^{n\log(1-\lambda/\varphi^2)} \geq \varphi^{2n}e^{-2\lambda n/\varphi^2}\,,$$

where we used the fact that $\log(1-x) \geq -2x$ for all $x \in (0, 1/2)$. But we also have that $e^{-x} \geq 1 - x$ for all $x \in \mathbb{R}$, so, $\gamma^n \geq \varphi^{2n}(1 - 2\lambda n/\varphi^2)$. Moreover,

$$2\lambda n/\varphi^2 < 2n/\varphi^k < 2\varphi^{k/2}/\varphi^k = 2/\varphi^{k/2}\,.$$

Hence,

$$\gamma^n > \varphi^{2n}(1 - 2/\varphi^{k/2})\,.$$

It then follows that the following inequalities hold

$$\varphi^{2n} - \frac{2\varphi^{2n}}{\varphi^{k/2}} < \gamma^n < \varphi^{2n}\,,$$

or

$$\left|\gamma^n - \varphi^{2n}\right| < \frac{2\varphi^{2n}}{\varphi^{k/2}}. \tag{10}$$

Using now the Mean-Value Theorem, we get that there exists some $\theta \in (\gamma, \varphi^2)$ such that

$$g_k(\gamma) = g_k(\varphi^2) + (\gamma - \varphi^2)g_k'(\theta)\,. \tag{11}$$

It is a simple matter to show that

$$g_k'(x) = -\frac{s_k(x)}{t_k(x)}\,,$$

where

$$s_k(x) = (k+1)x^2 - 2(k+1)x + 2k + 1 \quad \text{and} \quad t_k(x) = ((k+1)x^2 - 3kx + k - 1)^2\,.$$

Since the discriminant of $s_k(x)$ is negative and $x = 1$ is the only critical point of $s_k(x)$, we deduce that $s_k(x) > 0$ for all $x \in \mathbb{R}$ and is increasing in $(1, \infty)$. Additionally, it is not difficult to see that $t_k(x)$ is increasing in $(c_2(k), \infty)$ where $c_2(k)$ is given by (4). From the above and using the fact that $c_2(k) < \varphi^2 - 1/k < \gamma < \theta < \varphi^2$ (see [3, Lemma 3.2(*c*)]), we obtain, after some elementary algebra, that

$$\begin{aligned} t_k(\theta) \geq t_k(\varphi^2 - 1/k) &= (\varphi - 2\varphi/k - 1/k + 1/k^2 + 2)^2 \\ &\geq (\varphi - 2\varphi/k - 1/k + 2)^2 \\ &> 12 \end{aligned}$$

for all $k \geq 30$. As a consequence

$$|g_k'(\theta)| = \frac{s_k(\theta)}{t_k(\theta)} \leq \frac{s_k(\varphi^2)}{12} = \frac{(\varphi+2)k + \varphi + 1}{12} < k$$

for all $k \geq 30$. Hence, by (11), we get

$$|g_k(\gamma) - g_k(\varphi^2)| = |\gamma - \varphi^2||g_k'(\theta)| = \lambda\,|g_k'(\theta)| < \frac{k}{\varphi^{k-2}}\,. \tag{12}$$

Writing

$$\gamma^n = \varphi^{2n} + \delta \quad \text{and} \quad g_k(\gamma) = g_k(\varphi^2) + \eta\,,$$

then inequalities (10) and (12) yield

$$|\delta| < \frac{2\varphi^{2n}}{\varphi^{k/2}} \quad \text{and} \quad |\eta| < \frac{k}{\varphi^{k-2}}\,.$$

Moreover, since $g_k(\varphi^2) = 1/(\varphi+2)$, we have

$$g_k(\gamma)\gamma^n = (g_k(\varphi^2) + \eta)(\varphi^{2n} + \delta) = \frac{\varphi^{2n}}{\varphi+2}(1+\zeta) \ ,$$

where

$$\zeta = \frac{\delta}{\varphi^{2n}} + (\varphi+2)\eta + \frac{(\varphi+2)\eta\delta}{\varphi^{2n}} \, .$$

Finally, we note that

$$|\zeta| < \frac{2}{\varphi^{k/2}} + \frac{(\varphi+2)k}{\varphi^{k-2}} + \frac{2(\varphi+2)k}{\varphi^{3k/2-2}} < \frac{4}{\varphi^{k/2}} \quad \text{for all} \quad k \geq 30 \, .$$

This finishes the proof of the lemma. $\square$

2.2. **Linear forms in logarithms.** In order to prove our main result, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers, and such a bound, which plays an important role in this paper, was given by Matveev [14]. We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(x - \eta^{(i)}) \, ,$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta) = \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log \Big(\max\{|\eta^{(i)}|, 1\} \Big) \Big)$$

is called the *logarithmic height* of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$.

The following are basic properties of the logarithmic height. For α, β algebraic numbers and $s \in \mathbb{Z}$, we have

- $h(\alpha \pm \beta) \leq h(\alpha) + h(\beta) + \log 2$.
- $h(\alpha\beta^{\pm 1}) \leq h(\alpha) + h(\beta)$.
- $h(\alpha^s) = |s| h(\alpha)$.

Matveev [14] proved the following deep theorem.

Theorem 2 (Matveev's theorem). *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1 \qquad \textit{and} \qquad B \geq \max\{|b_1|, \ldots, |b_t|\} \, .$$

Let $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \ldots A_t) \, .$$

We now give an estimation for the logarithmic height of the algebraic number $g_k(\gamma)$ that will be used later.

Lemma 3. *For $k \geq 2$, we have that $h(g_k(\gamma)) < 4k\log\varphi + k\log(k+1)$.*

Proof. Put

$$f_k(x) = \prod_{i=1}^{k}(x - g_k(\gamma_i)) \in \mathbb{Q}[x],$$

where $\gamma := \gamma_1, \gamma_2, \ldots, \gamma_k$ are the roots of characteristic polynomial $\Phi_k(x)$ as mentioned before. Then the leading coefficient a_0 of the minimal polynomial of $g_k(\gamma)$ of degree d over the integers divides $\prod_{i=1}^{k}((k+1)\gamma_i^2 - 3k\gamma_i + k - 1)$. But,

$$\Big|\prod_{i=1}^{k}((k+1)\gamma_i^2 - 3k\gamma_i + k - 1)\Big| = (k+1)^k\Big|\prod_{i=1}^{k}(c_1(k) - \gamma_i)(c_2(k) - \gamma_i)\Big|,$$

where

$$(c_1(k), c_2(k)) := \Big(\frac{3k - \sqrt{5k^2+4}}{2(k+1)}, \frac{3k + \sqrt{5k^2+4}}{2(k+1)}\Big).$$

Since

$$|\Phi_k(y)| < \max\{y^k, 1 + y + \cdots + y^{k-2} + 2y^{k-1}\} < \varphi^{2k} \quad \text{for all} \quad 0 < y < \varphi^2,$$

and $0 < c_1(k) < c_2(k) < \varphi^2$, which are easily seen, it follows that $a_0 < \varphi^{4k}(k+1)^k$. By using this and Lemma 1, we obtain

$$h(g_k(\gamma)) = \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d}\log\max\{|g_k(\gamma_i)|, 1\}\Big) < 4k\log\varphi + k\log(k+1).$$

□

2.3. **Reduction lemma.** In the course of our calculations, we get some upper bounds on the variables n and ℓ which are very large, so we need to reduce them to a size that can be easily handled. To do this, we use the following lemma which is a slight variation of a result due to Dujella and Pethö [6] and itself is a generalization of a result of Baker and Davenport [1]. We shall use the version given by Bravo, Gómez and Luca in [2].

Lemma 4. *Let M be a positive integer, let p/q be a convergent of the continued fraction expansion of the irrational $\widehat{\gamma}$ such that $q > 6M$, and let A, B, $\widehat{\mu}$ be some real numbers with $A > 0$ and $B > 1$. Put $\epsilon := \|\widehat{\mu}q\| - M\|\widehat{\gamma}q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no positive integer solution (u, v, w) to the inequality*

$$0 < |u\widehat{\gamma} - v + \widehat{\mu}| < AB^{-w},$$

subject to the restrictions that

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

3. Proof of Theorem 1

Assume throughout that (n, k, ℓ, a) is a solution of equation (2). Suppose further that $k \geq 3$ since the case $k = 2$ was already treated by Faye and Luca in [7]. In the case $1 \leq n \leq k+1$, we obtain from (1) and (2) that $F_{2n-1} = a(10^\ell - 1)/9$, which is not possible because $F_{10} = 55$ is the only nontrivial repdigit in the Fibonacci sequence (see [12]). Thus, we can assume that $n \geq k+2$. So, we get easily that $n \geq 5$.

3.1. **An initial relation.** Since $P_n^{(k)}$ is a repdigit with ℓ digits, it follows from (2) and (9) that $10^{\ell-1} < P_n^{(k)} \leq \gamma^{n-1}$ and $\gamma^{n-2} \leq P_n^{(k)} < 10^\ell$. So we get

$$(n-2)\frac{\log\gamma}{\log 10} < \ell < (n-1)\frac{\log\gamma}{\log 10} + 1\,.$$

From this and taking into account that

$$0.2 < \frac{\log\varphi}{\log 10} < \frac{\log\gamma}{\log 10} < \frac{2\log\varphi}{\log 10} < 0.5,$$

because $\varphi < \gamma < \varphi^2$ for all $k \geq 2$, we obtain

$$\frac{1}{5}(n-2) < \ell < \frac{1}{2}(n+1), \tag{13}$$

which is an estimate on ℓ in terms of n. We shall have some use for it later.

3.2. **An inequality for n and ℓ in terms of k.** We now use (2) and (8) to obtain

$$\left|g_k(\gamma)\gamma^n - \frac{a10^\ell}{9}\right| = \left|e_k(n) + \frac{a}{9}\right| < \frac{1}{2} + \frac{a}{9} \leq \frac{3}{2}\,. \tag{14}$$

Dividing the above inequality by $g_k(\gamma)\gamma^n$, we get that

$$\left|10^\ell \cdot \gamma^{-n} \cdot \frac{a}{9}\left(g_k(\gamma)\right)^{-1} - 1\right| < \frac{3}{2g_k(\gamma)\gamma^n} < \frac{6}{\gamma^n}\,, \tag{15}$$

where we used the fact $3/(2g_k(\gamma)) < 6$ (see Lemma 1). In order to use the result of Matveev Theorem 2, we take $t := 3$ and

$$\gamma_1 := 10\,, \qquad \gamma_2 := \gamma\,, \qquad \gamma_3 := \frac{a}{9}\left(g_k(\gamma)\right)^{-1}\,.$$

We also take $b_1 := \ell$, $b_2 := -n$ and $b_3 := 1$. We begin by noticing that the three numbers γ_1, γ_2, γ_3 are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\gamma)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. The left-hand side of (15) is not zero. Indeed, if this were zero, we would then get that

$$\frac{a}{9}10^\ell = g_k(\gamma)\gamma^n\,.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Phi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i \geq 2$, we have

$$\frac{a}{9}10^\ell = \left|g_k(\gamma_i)\gamma_i^n\right|\,. \tag{16}$$

But the last equality above is not possible for $i \geq 2$ because, in view of Lemma 1, the right-hand side of (16) is at most 1, whereas its left-hand side is $\geq 100/9$.

Since $h(\gamma_1) = \log 10$ and $h(\gamma_2) = (\log \gamma)/k < (2\log\varphi)/k$ we can take $A_1 := k\log 10$ and $A_2 := 2\log\varphi$. Further, by the previous properties of the logarithmic height and Lemma 3, we have $h(\gamma_1) < \log 9 + 4k\log\varphi + k\log(k+1) < 4k\log k$ for all $k \geq 3$. So, we can take $A_3 := 4k^2\log k$. In addition, since $\ell < n$ (see (13)), we take $B := n$. Then, Matveev's theorem together with a straightforward calculation gives

$$\left|10^\ell \cdot \gamma^{-n} \cdot \frac{a}{9}\left(g_k(\gamma)\right)^{-1} - 1\right| > \exp\left(-5.1 \times 10^{12} k^5 \log^2 k \log n\right), \tag{17}$$

where we used that $1 + \log k < 2\log k$ for all $k \geq 3$ and $1 + \log n \leq 2\log n$ for all $n \geq 3$. Taking logarithms in inequality (17) and comparing the resulting inequality with (15), we get

$$\frac{n}{\log n} < 5.7 \times 10^{12} k^5 \log^2 k\,. \tag{18}$$

In the above we have used the fact that the dominant root $\gamma > 2.5$ for all $k \geq 3$. In order to get an upper bound for n depending on k we next use the fact that $x/\log x < A$ implies $x < 2A\log A$ whenever $A \geq 3$. Indeed, taking $x := n$ and $A := 5.7 \times 10^{12} k^5 \log^2 k$, and using the fact that $29.4 + 5\log k + 2\log\log k < 32\log k$ for all $k \geq 3$, inequality (18) yields $n < 3.7 \times 10^{14} k^5 \log^3 k$. We record what we have proved so far as a lemma.

Lemma 5. *If* (n, k, ℓ, a) *is a solution of equation* (2) *with* $k \geq 3$, *then* $n \geq k + 2$ *and*

$$\ell < n < 3.7 \times 10^{14} k^5 \log^3 k\,.$$

3.3. **The case of large k.** Suppose that $k > 330$. In this case the following inequalities hold

$$n < 3.7 \times 10^{14} k^5 \log^3 k < \varphi^{k/2}.$$

At this point, we require the estimation from Lemma 2 in order to find absolute upper bounds for n and ℓ. Indeed, since $n < \varphi^{k/2}$, Lemma 2 and (14) imply

$$\left|\frac{\varphi^{2n}}{\varphi+2} - \frac{a10^\ell}{9}\right| \leq \left|g_k(\gamma)\gamma^n - \frac{a10^\ell}{9}\right| + \frac{\varphi^{2n}|\zeta|}{\varphi+2} \leq \frac{\varphi^{2n}}{\varphi+2}\left(\frac{3(\varphi+2)}{2\varphi^{2n}} + \frac{4}{\varphi^{k/2}}\right). \tag{19}$$

Since $n \geq k+2$ we get that $2n > k/2$ and so $2\varphi^{2n} > \varphi^{k/2}$. Hence, from inequality (19) we obtain

$$\left|\frac{\varphi^{2n}}{\varphi+2} - \frac{a10^\ell}{9}\right| < \frac{\varphi^{2n}}{\varphi+2}\left(\frac{3\varphi+10}{\varphi^{k/2}}\right) < \frac{15\varphi^{2n}}{(\varphi+2)\varphi^{k/2}}, \tag{20}$$

where we use the fact that $3\varphi + 10 < 14.8541\ldots < 15$. Dividing both sides of the above inequality (20) by $\varphi^{2n}/(\varphi+2)$, we get that

$$\left|\frac{a}{9}(\varphi+2) \cdot 10^\ell \cdot \varphi^{-2n} - 1\right| < \frac{15}{\varphi^{k/2}}\,. \tag{21}$$

Again, in order to use the result of Matveev Theorem 2, we take $t := 3$ and

$$\gamma_1 := \frac{a}{9}(\varphi+2)\,, \qquad \gamma_2 := 10\,, \qquad \gamma_3 := \varphi^2\,.$$

We also take $b_1 := 1$, $b_2 := \ell$ and $b_3 := -n$. We begin by noticing that the three numbers γ_1, γ_2, γ_3 are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\varphi) = \mathbb{Q}(\sqrt{5})$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = 2$. To see why the left-hand side of (21) is not zero, note that otherwise, we would get that

$$\frac{a}{9}(\varphi+2)10^{\ell} = \varphi^{2n}\,. \tag{22}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\frac{a}{9}(\overline{\varphi}+2)10^{\ell} = \overline{\varphi}^{2n}\,, \tag{23}$$

where $\overline{\varphi} = (1-\sqrt{5})/2$. Combining (22) and (23) we obtain that $F_{2n} = a10^{\ell}/9$, which itself implies $a = 9$ and therefore $F_{2n} = 10^{\ell}$. However, this is impossible because there are no powers of 10 in the Fibonacci sequence. Hence, the left-hand side of inequality (21) is nonzero.

It is easy to see that $h(\gamma_1) = (\log 5)/2$, $h(\gamma_2) = \log 10$ and $h(\gamma_3) = \log\varphi$. Hence, we can take $A_1 := \log 5$, $A_2 := 2\log 10$ and $A_3 := 2\log\varphi$. Here, we can also take $B := n$. Then, Matveev's theorem together with a straightforward calculation gives

$$\left|\frac{a}{9}(\varphi+2)\cdot 10^{\ell}\cdot\varphi^{-2n} - 1\right| > \exp\left(-1.4\times 10^{13}\log n\right)\,, \tag{24}$$

where we used that $1+\log n < 2\log n$ for all $n \geq 3$. Comparing (21) and (24), taking logarithms and then performing the respective calculations, we get that

$$k < 6\times 10^{13}\log n\,.$$

But, recall that by Lemma 5, we have $n < 3.7\times 10^{14}k^5\log^3 k$. Thus

$$k < (6\times 10^{13})(34 + 5\log k + 3\log\log k) < 7.2\times 10^{14}\log k\,,$$

where we used the fact that the inequality $34 + 5\log k + 3\log\log k < 12\log k$ holds for all $k > 330$. *Mathematica* gives $k < 3\times 10^{16}$. By Lemma 5 once again, we obtain $n < 5\times 10^{101}$. We record our conclusion as follows.

Lemma 6. *If* (n,k,ℓ,a) *is a solution on positive integers of equation* (2) *with* $k > 330$*, then all inequalities*

$$k < 3\times 10^{16} \quad \textit{and} \quad \ell < n < 5\times 10^{101}$$

hold.

3.4. **Reducing the bound on k.** We now want to reduce our bound on k by using Lemma 4. Let

$$z_1 := \ell\log 10 - 2n\log\varphi + \log\mu_a\,,$$

where $\mu_a = a(\varphi+2)/9$. Then, from the linear form (21) we get

$$|e^{z_1} - 1| < \frac{15}{\varphi^{k/2}}\,. \tag{25}$$

Note that $z_1 \neq 0$, thus we distinguish the following cases. If $z_1 > 0$, then $e^{z_1} - 1 > 0$, so from (25) we obtain

$$0 < z_1 \leq e^{z_1} - 1 < \frac{15}{\varphi^{k/2}}\,.$$

Suppose now that $z_1 < 0$. Since $|e^{z_1} - 1| < 15/\varphi^{165} < 1/2$, we conclude that $e^{|z_1|} < 2$. Thus,

$$0 < |z_1| \leq e^{|z_1|} - 1 = e^{z_1}\,|e^{z_1} - 1| < \frac{30}{\varphi^{k/2}}\,.$$

In any case, we have that the inequality

$$0 < |z_1| < \frac{30}{\varphi^{k/2}}$$

always holds. Replacing z_1 by its expression in the above inequality and dividing through by $2\log\varphi$, we get

$$0 < |\ell\widehat{\gamma} - n + \widehat{\mu}_a| < AB^{-k}\,, \tag{26}$$

where

$$\widehat{\gamma} := \frac{\log 10}{2\log\varphi}, \quad \widehat{\mu}_a := \frac{\log\mu_a}{2\log\varphi}, \quad A := 32 \quad \text{and} \quad B := \varphi^{1/2}\,.$$

Note that $\widehat{\gamma}$ is an irrational number. We now put $M := 5 \times 10^{101}$ which is upper bound on ℓ by Lemma 6. Applying Lemma 4 to the inequality (26) for all choices $a \in \{1, 2, \ldots, 9\}$, we obtain that $k < 1010$. Then Lemma 5 tells us that $\ell < 1.3 \times 10^{32}$. With this new upper bound for ℓ we repeated the process, i.e., we now take $M := 1.3 \times 10^{32}$ to obtain, in view of Lemma 4, $k < 370$. Thus, $\ell < 5.4 \times 10^{29}$. A third application of Lemma 4 gives $k < 330$, which contradicts our assumption that $k > 330$. Hence, we deduce that the possible solutions (n, k, ℓ, a) of the equation (2) all have $k \in [3, 330]$.

3.5. **The case of small k.** Suppose now that $k \in [3, 330]$. Note that for each of these values of k, Lemma 5 gives us absolute upper bounds for n and ℓ. However, these upper bounds are so large and will be reduced by using Lemma 4 once again. In order to apply Lemma 4 we put

$$z_2 := \ell\log 10 - n\log\gamma + \log\mu_a\,,$$

where now $\mu_a = a(g_k(\gamma))^{-1}/9$. Thus, (15) can be rewritten as

$$|e^{z_2} - 1| < \frac{6}{\gamma^n}\,. \tag{27}$$

Note that $z_2 \neq 0$. Here, in a similar way as in the case of z_1, we obtain from (27) that the inequality

$$0 < |z_2| < \frac{12}{\gamma^n}$$

holds for all $n \geq 5$ no matter whether z_2 is positive or negative. Replacing z_2 in the above inequality by its formula and dividing it across by $\log\gamma$, we conclude that

$$0 < |\ell\widehat{\gamma}_k - n + \widehat{\mu}_{a,k}| < A \cdot B^{-n}\,, \tag{28}$$

where now

$$\widehat{\gamma}_k := \frac{\log 10}{\log \gamma}, \quad \widehat{\mu}_{a,k} := \frac{\log \mu_a}{\log \gamma}, \quad A := 14 \quad \text{and} \quad B := \gamma.$$

Let us show that $\widehat{\gamma}_k$ is an irrational number. If it were not, then with $\widehat{\gamma}_k = a/b$ with coprime positive integers a and b, we would get that $10^b = \gamma^a$. As before, conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Phi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that $10^b = |\gamma_i|^a$ for any $i \geq 2$. But this is not possible because $|\gamma_i| < 1$.

We put $M_k := \lfloor 3.7 \times 10^{14} k^5 \log^3 k \rfloor$ which is an upper bound on ℓ from Lemma 5. A computer search with *Mathematica* revealed that if $k \in [3, 330]$, then the maximum value of $\log(Aq_k/\epsilon_k)/\log B$ is ≤ 100, where $q_k > 6M_k$ is a denominator of a convergent of the continued fraction of $\widehat{\gamma}_k$ such that $\epsilon_k = ||\widehat{\mu}_{a,k} q_k|| - M_k ||\widehat{\gamma}_k q_k|| > 0$. It then follows from Lemma 4, applied to inequality (28) for each $k \in [3, 330]$, that $n \leq 100$. Since $k + 2 \leq n$, we also deduce that $k \leq 98$.

Finally, a brute force search with *Mathematica* in the range

$$3 \leq k \leq 98 \quad \text{and} \quad k + 2 \leq n \leq 100$$

gives the solutions shown in the statement of Theorem 1. This completes the analysis in the case $k \in [3, 330]$ and therefore the proof of Theorem 1.

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Departamento de Matemáticas,
Universidad del Cauca, Calle 5 No 4–70,
Popayán, Colombia
E-mail: jbravo@unicauca.edu.co joseherrera@unicauca.edu.co