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SYSTEMS OF DIFFERENTIAL EQUATIONS MODELING NON-MARKOV PROCESSES

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Abstract. The work deals with non-Markov processes and the construction of systems of differential equations with delay that describe the probability vectors of such processes. The generating stochastic operator and properties of stochastic operators are used to construct systems that define non-Markov processes.

1. Introduction concepts

In recent years, studies of the non-Markovian dynamics of open systems have become increasingly popular, with a diverse range of researchers involved. The theory of non-Markovian random processes is constantly developing and meets modern requirements. This interest arose from the fundamental problem of defining and quantifying memory effects in the quantum realm, how to use and develop applications based on them, and also because of the question of what are the ultimate limits for controlling the dynamics of open systems.

In addition, there are many important control problems that are not naturally formulated as Markov decision processes. For example, if the agent cannot directly observe the state of the environment, then it is more appropriate to use a partially observable Markov model of the decision process. Even with complete observability, the probability distribution over the next states may not depend only on the current state.

Some postulated problems and also models with non-Markov parameters using fractional dynamics, predictive control or stabilization are considered in [2].

In the presented work, constructions of certain non-Markovian random processes are proposed using stochastic operators, which are called generating operators. Naturally, other methods of constructing non-Markov random processes can be

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proposed. In any case, these processes will be determined by equations with a delayed argument.

In the constructions proposed by us, the stochastic operator plays an important role, so we will present its definition and some basic properties. For a deeper understanding of this term see works [1, 4].

Definition 1.1. Let on the probability space $(\Omega,\ \mathcal{F},\ P)$ be defined two random variables $x \equiv x(\omega)\colon \Omega \to \mathbb{R}$ and $y \equiv y(\omega)\colon \Omega \to \mathbb{R}$ with probability density functions $f_1(x)$ and $f_2(y)$ respectively. Then the operator $L\colon f_1(x) \to f_2(y)$,

$$f_2(y) = Lf_1(x)\,,$$

is said to be the *stochastic or generating operator.*

Theorem 1.2 ([1])**.** *Let on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be defined two random variables x, y with probability density functions $f_1(x)$ and $f_2(y)$ respectively. Let $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable monotonically increasing function such that $\lim\limits_{x\to-\infty} g(x) = -\infty\,,\quad \lim\limits_{x\to+\infty} g(x) = +\infty$. Then*

$$f_1(x) = f_2(g(x))\frac{dg(x)}{dx}\,.$$

It should be noted that, in general, there is considered a set S of functions $f(x), x = (x_1,\, x_2, \dots, x_m) \in \mathbb{R}^m$ such that

$$f(x) \geq 0, \quad \int\limits_{\mathbb{R}^m} f(x)dx = 1\,,$$

and the operator L is mapping a set S to the itself.
If $f_1(x) \in S$ implies $f_2(y) = Lf_1(x) \in S$, then the operator L is stochastic.

A similar statement as Theorem 1.2 can also be formulated if f_1, f_2 are vector functions. For this we will use the following notation

$$\det\frac{Dg(x)}{Dx} := \begin{vmatrix} \dfrac{\partial g_1(x)}{\partial x_1} & \dfrac{\partial g_1(x)}{\partial x_2} & \dots & \dfrac{\partial g_1(x)}{\partial x_m} \\ \dfrac{\partial g_2(x)}{\partial x_1} & \dfrac{\partial g_2(x)}{\partial x_2} & \dots & \dfrac{\partial g_2(x)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_m(x)}{\partial x_1} & \dfrac{\partial g_m(x)}{\partial x_2} & \dots & \dfrac{\partial g_m(x)}{\partial x_m} \end{vmatrix} \neq 0\,,$$

where $g\colon \mathbb{R}^m \to \mathbb{R}^m$ is a continuously differentiable function.

Theorem 1.3 ([1])**.** *Let $g\colon \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable function for which there exists the inverse function $h\colon \mathbb{R}^m \to \mathbb{R}^m$ to g, i. e., $y = g(x),\ \ x = h(y)$, $\det\dfrac{Dh(y)}{Dy} \neq 0$, and $\lim\limits_{\|x\|\to\infty} \|g(x)\| = \infty,\ \lim\limits_{\|x\|\to\infty} \|h(x)\| = \infty$. If on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are defined two random variables $x\colon \Omega \to \mathbb{R}^m$, $y\colon \Omega \to \mathbb{R}^m$ with* *probability density functions* $f_1(x)$, $f_2(y)$, *respectively,* $x,\, y \in \mathbb{R}^m$, $f_1,\, f_2 \in S$, *such that* $y = g(x)$, $x = h(y)$, *then*

$$f_1(x) = f_2(g(x)) \left| \det \frac{Dg(x)}{Dx} \right| ,$$

$$f_2(y) = f_1(h(y)) \left| \det \frac{Dh(y)}{Dy} \right| .$$

First, we show a possible construction of a differential equation determining some random process using a stochastic operator (for details see [5, 6]).
Let $L_\tau(t,\varepsilon)$ be a stochastic operator that depends on the parameter ε and is defined for an m-dimensional probability vector $P(t+\tau)$, $\tau < 0$, such that there exists the limit

$$\lim_{\varepsilon\to 0} L_\tau(t,\varepsilon)P(t+\tau) = P(t)\,.$$

In addition, for any continuous vector function P there exists an operator

$$A_\tau(t)P(t+\tau) = \lim_{\varepsilon\to\infty} \frac{L_\tau(t,\varepsilon)P(t+\tau) - P(t)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\partial L_\tau(t,\varepsilon)}{\partial\varepsilon}\, P(t+\tau)\,.$$

Then the difference equation

$$P(t+\varepsilon) = L_\tau(t,\varepsilon)P(t+\tau), \qquad \varepsilon > 0,\ \tau \le 0\,,$$

determines the vector of probabilities of some random process $\xi(t) = \xi(t,\omega)$, $\omega \in \Omega$. This equation can be written in the form

$$\frac{P(t+\varepsilon) - P(t)}{\varepsilon} = \frac{L_\tau(t,\varepsilon)P(t+\tau) - P(t)}{\varepsilon}\,.$$

If $\varepsilon \to 0$ in this relation, assuming that the vector $P(t)$ is differentiable, we obtain a system of differential equations

$$\frac{dP(t)}{dt} = A_\tau(t)P(t+\tau)\,, \tag{1.1}$$

which describes some random process $\xi(t) = \xi(t,\omega), \omega \in \Omega$.
The operator $A_\tau(t)$ in system of differential equations (1.1) is so called the **generating (stochastic) operator**.

2. Main results

Using the properties of stochastic operators given in Theorems 1.2 and 1.3 (see [1] for other properties), we show some constructions of generating operators that can be used to construct systems of differential equations whose solutions are non-Markov stochastic processes.

Theorem 2.1. *Let* $A_\tau(t)$ *be a generating operator and let* $0 < \alpha(t) \le c, c \in \mathbb{R}^+$. *Then* $\alpha(t)A_\tau(t)$ *is also a generating operator.*

Proof. The difference equation

$$P\left(t + \frac{\varepsilon}{\alpha(t)}\right) = L_\tau(t,\varepsilon)P(t+\tau), \qquad 0 < \alpha(t) \le c$$

determines the probability vector of some random process. This equation we rewritten into the form

$$\frac{P(t+\varepsilon\alpha^{-1}(t)) - P(t)}{\varepsilon} = \frac{L_\tau(t,\varepsilon)P(t+\tau) - P(t)}{\varepsilon}$$

and if $\varepsilon \to 0$, we obtain the system of differential equations

$$\frac{dP(t)}{dt} = \alpha(t)A_\tau(t)P(t+\tau)\,,$$

which corresponds to equation (1.1). This proves the theorem. □

Theorem 2.2. *Let $A_\tau^{(1)}(t), A_\tau^{(2)}(t)$ be generating operators. Then $A_\tau^{(1)}(t) + A_\tau^{(2)}(t)$ is also a generating operator.*

Proof. Let $A_\tau^{(k)}(t), k = 1, 2$ be generating operators such that

$$\lim_{\varepsilon\to\infty} \varepsilon^{-1}\left(L_\tau^{(k)}(t,\varepsilon)P(t+\tau) - P(t)\right) = A_\tau^{(k)}(t)P(t+\tau)\,.$$

The difference equation

$$P\left(t+\frac{\varepsilon}{2}\right) = \frac{1}{2}\sum_{k=1}^{2} L_\tau^{(k)}(t,\varepsilon)P(t+\tau)\,,$$

determines the probability vector of some random process. This equation we rewritten into the form

$$\varepsilon^{-1}\left(P\left(t+\frac{\varepsilon}{2}\right) - P(t)\right) = \frac{1}{2}\sum_{k=1}^{2}\varepsilon^{-2}\left(L_\tau^{(k)}(t,\varepsilon)P(t+\tau) - P(t)\right)\,.$$

If $\varepsilon \to 0$ than we obtain the system of differential equations

$$\frac{dP(t)}{dt} = \sum_{k=1}^{2} A_\tau^{(k)}(t)P(t+\tau)\,.$$

This proves the theorem. □

Theorem 2.3. *Let $A_\tau^{(k)}(t),\ k = 1, 2, \dots, N$ be generating operators and let functions $\alpha_k(t),\ k = 1, 2, \dots, N$ satisfy the conditions $0 < \alpha_k(t) \le c, c \in \mathbb{R}^+$. Then*

$$\sum_{k=1}^{N} \alpha_k(t)A_\tau^{(k)}(t)$$

is also a generating operator.

Proof. The proof follows from Theorems 2.1, 2.2.

□

Now, as a consequence of Theorem 2.2, we consider possible options for constructing systems of differential equations that describe the probability vector of various random processes.

Corollary 2.4. *Let $A_0(t)$ be an $m\times m$ matrix with elements $a_{js}(t), j, s = 1, 2, \dots, m$ such that*

$$\sum_{j=1}^{m} a_{js}(t) = 0, \qquad a_{js}(t) \geq 0,\ j \neq s, \qquad a_{jj}(t) \leq 0\,, \quad j, s = 1, 2, \dots, m\,. \tag{2.1}$$

If the elements $a_{js}(t), j, s = 1, 2, \dots, m$ are bounded, then the operator

$$L_\tau(t,\varepsilon)P(t+\tau) \equiv P(t) + A_0(t)P(t)$$

will be stochastic for sufficiently small values of $\varepsilon > 0$, where the matrix $A_0(t)$ defines the generating operator, and the system of differential equations

$$\frac{dP(t)}{dt} = A_0(t)P(t)$$

determines the vector of probabilities of the finite-valued Markov process.

Corollary 2.5. *Let $\Pi(t)$ be an arbitrary stochastic matrix. The operator given by the equality*

$$L_\tau(t,\varepsilon) = (1-\varepsilon)P(t) + \varepsilon\Pi(t)P(t-\tau(t))\,, \qquad \tau(t) \geq 0\,,$$

is stochastic when $0 \leq \varepsilon \leq 1$. Then the system of linear differential equations

$$\frac{dP(t)}{dt} = \Pi(t)\big(P(t-\tau) - P(t)\big)\,, \qquad \tau(t) \geq 0$$

determines the vector of probabilities of some non-Markov random process.

Corollary 2.6. *Let elements $a_{js}(t), j, s = 1, 2, \dots, m$ of matrix $A_0(t)$ satisfy* (2.1) *and $0 \leq \alpha_k(t) \leq c$, $c = \mathrm{const}$, $\tau_k(t) \geq 0$, $\Pi_k(t) \in L_\tau$, $k = 0, 1, 2, \dots, N$. Then the system of differential equations*

$$\frac{dP(t)}{dt} = A_0(t)P(t) + \sum_{k=0}^{N} \alpha_k(t)\big(\Pi_k(t)P(t-\tau_k(t)) - P(t)\big) \tag{2.2}$$

determines the vector of probabilities of some non-Markov random process.

Corollary 2.7. *Let $\alpha(t,\tau) \geq 0$, $t \geq 0$, $\tau \geq 0$, $\int\limits_0^\infty \alpha(t,\tau)d\tau \leq c$, $c = \mathrm{const}$, $\Pi(t,\tau) \in L_\tau$, and elements $a_{js}(t), j, s = 1, 2, \dots, m$ of matrix $A_0(t)$ satisfy* (2.1)*. Then, if $N \to \infty$, system* (2.2) *yields the system of integro-differential equations*

$$\frac{dP(t)}{dt} = A_0(t)P(t) + \int_0^\infty \alpha(t,\tau)\big(\Pi(t,\tau)P(t-\tau) - P(t)\big)d\tau\,,$$

which determines the vector of probabilities of some non-Markov random process.

Corollary 2.8. *Let $F(t,x)$ be a vector of partial probability densities*

$$F(t,x) = \big(f_1(t,x), \dots, f_n(t,x)\big), \quad f_k(t,x) \geq 0,\ k = 1, 2, \dots, n,$$

$$\int\limits_{\mathbb{R}^m} \sum_{k=1}^{n} f_k(t,x)dx = 1\,,$$

and let $L_\tau(t,\varepsilon)$ be a stochastic operator that depends on $\varepsilon \geq 0$ and is defined for the vector $F(t+\tau,x)$ at $\tau \leq 0$. We assume that $\lim\limits_{\varepsilon\to 0^+} L_\tau(t,\varepsilon)F(t+\tau,x) = F(t,x)$ and there exists an operator $A_\tau(t)$ such that

$$\begin{aligned} A_\tau(t)F(t+\tau,x) &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\big(L_\tau(t,\varepsilon)F(t+\tau,x) - F(t,x)\big) \\ &= \lim_{\varepsilon\to 0^+}\frac{\partial L_\tau(t,\varepsilon)}{\partial\varepsilon}\,F(t+\tau,x)\,. \end{aligned}$$

Then the operator equation $F(t+\varepsilon,x) = L_\tau(t,\varepsilon)F(t+\tau,x)$ determines the partial densities of the distribution of some non-Markov random process when $\varepsilon > 0$.

Remark 2.9. Assuming differentiation $F(t,x)$ with respect to t, if $\varepsilon \to 0^+$ in the equation

$$\varepsilon^{-1}\big(F(t+\varepsilon,x) - F(t,x)\big) = \varepsilon^{-1}\big(L_\tau(t,\varepsilon)F(t+\tau,x) - F(t,x)\big)\,,$$

we obtain a system of differential equations

$$\frac{\partial F(t,x)}{\partial t} = A_\tau(t)F(t+\tau,x) \tag{2.3}$$

that describes the partial distribution densities of some random process.

Theorem 2.10. *Let $A_\tau^{(k)}(t)$, $k = 1,2,\dots,N$ be generating operators and let functions $\alpha_k(t)$, $k = 1,2,\dots,N$ satisfy the conditions $0 < \alpha_k(t) \leq c, c \in \mathbb{R}^+$. Then the system of differential equations*

$$\frac{\partial F(t,x)}{\partial t} = \sum_{k=0}^{N}\alpha_k(t)A_\tau^{(k)}(t)F(t+\tau,x)$$

determines the partial distribution densities of some random process.

Proof. It follows from Theorem 2.3, the operator

$$\alpha_\tau(t) = \sum_{k=0}^{N}\alpha_k(t)A_\tau^{(k)}(t)$$

is also a generating operator. The statement then follows from (2.3). □

Corollary 2.11. *Let Π be an arbitrary stochastic matrix. The generating operator*

$$A_\tau(t)F(t+\tau,x) = \Pi F(t+\tau,x) - F(t,x)\,,\qquad \tau(t)\geq 0$$

corresponds to the stochastic operator

$$L_\tau(t,\varepsilon)F(t+\tau,x) = \varepsilon\Pi F(t+\tau,x) + (1-\varepsilon)F(t,x)\,,\qquad \tau(t)\geq 0\,,$$

and the generating operator

$$A_\tau(t)F(t+\tau,x) = F\big(t+\tau,\Psi(t,x)\big)\left|\frac{D\Psi(t,x)}{Dx}\right| - F(t,x)\,,\qquad \tau(t)\geq 0$$

corresponds to the stochastic operator

$$A_\tau(t,x)F(t+\tau,x) = \varepsilon F\big(t+\tau,\Psi(t,x)\big)\left|\frac{D\Psi(t,x)}{Dx}\right| + (1-\varepsilon)F(t,x),\qquad \tau(t)\geq 0,$$

where $y = \Psi(t,x)$ is differentiable vector function defined for $x \in \mathbb{R}^m, t \geq 0$.

Corollary 2.12. *Let $\Pi_k, k = 0,1,2,\dots,N$ be $n \times n$ stochastic matrices, and let $y = \Psi_k(t,x), k = 0,1,2,\dots,N$ be differentiable vector functions which mutually uniquely map $\mathbb{R}^m$ to $\mathbb{R}^m$. Then the system of differential equations*

$$\frac{\partial F(t,x)}{\partial t} = \sum_{k=0}^{N} \alpha_k(t) \left(\Pi_k F\big(t - \tau_k(t), \Psi_k(t,x)\big) \left| \det \frac{D\Psi_k(t,x)}{Dx} \right| - F(t,x) \right),$$
$$\tau_k(t) \geq 0,\ k = 0,1,2,\dots,N,$$

determines the partial densities of the distribution of some non-Markov process. When $\tau_k(t) \equiv 0, k = 0,1,2,\dots,N$, the random process will be Markov.

Corollary 2.13. *Let $f(t,x)$ be differentiable vector function with respect to t, x and let $y = \Psi(t,x)$ be differentiable vector function with respect to x with projections $\phi_k(t,x), k = 1,2,\dots,m$. If $\varepsilon \to 0$, then the stochastic operator*

$$L^{(1)} f(t,x) = f\big(t, x + \varepsilon\Psi(t,x)\big) \det\left(E + \varepsilon \frac{D\Psi(t,x)}{Dx}\right)$$

reduces to the generating operator

$$A^{(1)} f(t,x) = \operatorname{div}\big(f(t,x), \Psi(t,x)\big) = \sum_{k=1}^{m} \frac{\partial\big(f(t,x)\phi_k(t,x)\big)}{\partial x_k}.$$

Corollary 2.14. *Let the stochastic operator*

$$\begin{aligned} L^{(2)} f(t,x) = &\frac{1}{4} f\left(t, x + \sqrt{2\varepsilon}\Phi(t,x)\right) \det\left(E + \sqrt{2\varepsilon}\frac{D\Phi(t,x)}{Dx}\right) \\ &+ \frac{1}{4} f\left(t, x + \sqrt{2\varepsilon}\Phi(t,x)\right) \det\left(E - \sqrt{2\varepsilon}\frac{D\Phi(t,x)}{Dx}\right) \\ &+ \frac{1}{2} f\left(t, x + \frac{\varepsilon}{2}\Phi(t,x)\right) \det\left(E - \sqrt{2\varepsilon}\frac{D\Phi(t,x)}{Dx}\right) \end{aligned}$$

be given, where $\Phi(t,x)$ is a vector-function twice differentiable with respect to x, with projections

$$\varphi_k(t,x) = \operatorname{grad}\big(\varphi_k(t,x), \Phi(t,x)\big) = \sum_{s=1}^{m} \frac{\partial \varphi_k(t,x)}{\partial x_s} \varphi_k(t,s),\ k = 1,2,\dots,m.$$

Then the corresponding generating operator takes the form

$$
\begin{aligned}
A^{(2)}(t)f(t,x) &= \operatorname{div}\Big(\Phi(t,x)\operatorname{div}\big(f(t,x)\Phi(t,x)\big)\Big) = \frac{1}{2}f(t,x) \\
&= \big(\operatorname{div}\Phi(t,x)\big)^2 + \frac{1}{2}\Big(\operatorname{grad}\operatorname{div}\Phi(t,x)\Phi(t,x)\Big) \\
&\quad + \operatorname{div}\Phi(t,x)\Bigg(\operatorname{grad} f(t,x) + \frac{1}{2}\sum_{k,s=1}^{m}\frac{\partial^2 f(t,x)}{\partial x_k \partial x_s}\varphi_k(t,x)\varphi_s(t,x)\Bigg) \\
&\quad + \frac{1}{2}\sum_{k,s=1}^{m}\frac{\partial f(t,x)}{\partial x_k}\big(\operatorname{grad}\varphi_k(t,x)\Phi(t,x)\big)\,.
\end{aligned}
$$

3. Conclusion and further research direction

The paper shows possible procedures for constructing stochastic operators that can be used to construct differential equations with delay for analytically given non-Markov processes. In our opinion, in the situation developing in the modern world, when modeling with the study of decision making under uncertainty, non-Markovian processes will dominate.

There are still many unsolved questions in the field of the construction of non-Markovian models. Although non-Markov models describe events more realistically in many situations, there is a need to focus on building such models that will be even more personalized by incorporating domain knowledge.

The results presented here, in particular Theorem 2.3, make it possible to construct systems of differential equations with delay of the Kolmogorov-Feller type (see [3]), which can be used to construct moment equations for systems of differential equations, as well as differential equations with non-Markov coefficients.

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