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FOREWORD TO PROCEEDINGS OF EQUADIFF 15

The Conference on Differential Equations and Their Applications – abbreviated as Equadiff – is one of the oldest active series of mathematical conferences in the world. The tradition of the Czechoslovak Equadiff dates back to 1962 when Equadiff 1 took place in Prague. The subsequent Czechoslovak Equadiff conferences are held since then periodically in Prague, Bratislava, and Brno every four years (with few exceptions). The Western Equadiff conferences are organized in various cities in Western Europe, starting in Marseille in 1970 and with the last meeting in Leiden in 2019.

The last Equadiff was held in Brno in summer 2022 as the 15th conference within the Czechoslovak Equadiff series, and hence it bears the name Equadiff 15. The conference was rescheduled to the year 2022 from the original date in July 2021 due to an unstable pandemic situation in the world. The proceedings from all previous Czechoslovak Equadiff conferences are available via the Czech Digital Mathematics Library at

`https://dml.cz/handle/10338.dmlcz/700001`.

The conference Equadiff 15 was organized by joint efforts of the Faculty of Science of Masaryk University (and its Department of Mathematics and Statistics) with the Faculty of Civil Engineering of Brno University of Technology, the Institute of Mathematics of the Czech Academy of Sciences, and the Brno branch of The Union of Czech Mathematicians and Physicists. The conference took place at the campus of the Faculty of Economics and Administration of Masaryk University from July 11 till July 15, 2022. More than 250 participants from 37 countries from all over the world attended the 241 talks of the conference, including 6 plenary talks, 17 invited talks, 124 talks in 33 organized minisymposia, 75 contributed talks, and 19 posters.

The proceedings of Equadiff 15 cover the theory of differential equations in a broad sense, including their theoretical aspects, numerical methods, and applications. The proceedings contain 29 scientific articles written by participants of Equadiff 15. The papers are divided into three sections according to the program of the conference:

- ordinary differential equations (15 papers),
- partial differential equations (9 papers),
- numerical analysis and applications (5 papers).

Each manuscript underwent a rigorous refereeing process to ensure its scientific quality. This issue contains the contributions from section Partial differential equations.

We would like to take this opportunity to express our special thanks to all the participants for their active contributions to the success of the Equadiff 15 conference. Our gratitude and appreciation belong to the members of the Scientific Committee who ensured the high standards of the scientific activities of the conference, to the organizers and supporting PhD students for their efforts towards the realization of the conference, to the administration of the Faculty of Economics and Administration of Masaryk University for providing the venue for the conference and for their organizational support, to the management and employees of the Accommodation and Catering Services of Masaryk University for their help with the organization and realization of the catering during the conference, to the workers of the Botanical Garden of the Faculty of Science of Masaryk University for providing the flower decoration, and to the director of the Department of Mathematics and Statistics of the Faculty of Science of Masaryk University for financial support. We also thank to Ilona Lukešová from the Editorial Office of Archivum Mathematicum for her extensive editorial work on these proceedings.

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ARCHIVUM MATHEMATICUM (BRNO)
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A PRIORI BOUNDS FOR POSITIVE RADIAL SOLUTIONS OF QUASILINEAR EQUATIONS OF LANE–EMDEN TYPE

Soohyun Bae

Abstract. We consider the quasilinear equation $\Delta_p u + K(|x|)u^q = 0$, and present the proof of the local existence of positive radial solutions near 0 under suitable conditions on K. Moreover, we provide a priori estimates of positive radial solutions near ∞ when $r^{-\ell}K(r)$ for $\ell \geq -p$ is bounded near ∞.

1. Introduction

We consider the equation

$$\Delta_p u + K(|x|)u^q = 0\,, \tag{1.1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $n > p > 1$ and $q > p-1$. Let $r = |x|$ and $\frac{d}{dr}u(r) = u_r(r)$. Then, the radial version of (1.1) is

$$r^{1-n}(r^{n-1}|u_r|^{p-2}u_r)_r + K(r)u^q = 0\,. \tag{1.2}$$

For $p = 2$, the basic assumption of K for local solutions is (K):

(i) $K(r) \geq 0, \not\equiv 0$; $K(r)$ is continuous on $(0,\infty)$;

(ii) $\int_0 rK(r)\,dr < \infty$, i.e., $rK(r)$ is integrable near 0.

Under condition (K), (1.2) with $p = 2$ and $u(0) = \alpha > 0$, has a unique positive solution $u_\alpha \in C^2(0,\varepsilon) \cap C[0,\varepsilon)$ for small $\varepsilon > 0$. In order to obtain local solutions (1.2) near 0, we assume (KP): (i) of (K), and for $r > 0$ small,

$$\int_0^r t^{\frac{1-n}{p-1}}\Big(\int_0^t s^{n-1}K(s)ds\Big)^{\frac{1}{p-1}}dt < \infty\,.$$

For $p = 2$, this integrability is (ii) of (K). If $K(r) = r^l$, then it is easy to see that (KP) holds for $l > -p$. As a typical example, the equation

$$\Delta_p u + |x|^l u^q = 0 \tag{1.3}$$

possesses a local radial solution $\overline{u}_\alpha$ with $\overline{u}_\alpha(0) = \alpha$ for each $\alpha > 0$, and has the scaling invariance:

$$\overline{u}_\alpha(r) = \alpha\overline{u}_1(\alpha^{\frac{1}{m}}r) \tag{1.4}$$

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with $m = \frac{p+l}{q-(p-1)}$. Moreover, (1.3) has a singular solution which is invariant under the scaling in (1.4), the so-called self-similar solution. That is,

$$U(x) = L|x|^{-m},$$

where L is defined by

$$L = L(n,p,q,l) = [m^{p-1}(n-1-(m+1)(p-1))]^{\frac{1}{q-(p-1)}} \,. \tag{1.5}$$

This singular solution can be defined for $l > -p$ and $q > \frac{(p-1)(n+l)}{n-p}$ because $n-1-(m+1)(p-1) > 0$. Then, we observe the asymptotic self-similar behavior.

Theorem 1.1. *Let $n > p > 1$ and $q > \frac{(p-1)(n+l)}{n-p}$ with $l > -p$. If $r^{-l}K(r) \to 1$ as $r \to \infty$, then any positive solution u of (1.2) near ∞ satisfies one of the two asymptotic behavior: either*

$$\liminf_{r\to\infty} r^m u(r) \le L \le \limsup_{r\to\infty} r^m u(r) < \infty \tag{1.6}$$

with $L = L(n,p,q,l)$ given by (1.5) or $r^{(n-p)/(p-1)}u(r) \to C > 0$ as $r \to \infty$.

Moreover, (1.6) can be the asymptotic self-similarity

$$\lim_{r\to\infty} r^m u(r) = L\,.$$

In a forthcoming paper, we study entire solutions of (1.2) with this asymptotic behavior in a supercritical range.

1.1. **Lower bound.** The p-Laplace equation has the radial form

$$(|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r = 0\,, \tag{1.7}$$

where $n > p > 1$. Then, (1.7) possesses a solution $|x|^{-\theta}$ with $\theta = \frac{n-p}{p-1}$. Let u be a positive radial solution satisfying the quasilinear inequality

$$r^{1-n}(r^{n-1}|u_r|^{p-2}u_r)_r = (|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r \le 0\,. \tag{1.8}$$

If $u_r(r_0) \le 0$ for some $r_0 > 0$, then $u_r(r) \le 0$ for $r > r_0$. Hence, u is monotone near ∞. Assume $u_r \le 0$ for $r \ge r_0$ with some $r_0 > 1$. Setting $V(t) = r^\theta u(r)$ for $t = \log r \ge t_0 = \log r_0$, we see that $g(t) = \theta V(t) - V'(t) = r^{\theta+1}(-u_r(r)) = r^{\frac{n-1}{p-1}}(-u_r(r))$ satisfies

$$\frac{d}{dt}(g^{p-1}(t)) = (n-1)g^{p-1}(t) + r^n[(-u_r)^{p-1}]_r \ge 0$$

for $t \ge t_0$. Hence, g is increasing for $t \ge t_0$. Then, V satisfies that for $t > T \ge t_0$,

$$V'(t) - \theta V(t) \le V'(T) - \theta V(T)\,.$$

Suppose $V'(T) < 0$. Setting $c = \theta V(T) - V'(T)$, we have $(e^{-\theta t}V(t))' \le -ce^{-\theta t}$ and

$$V(t) \le e^{\theta(t-T)}(V(T) - \frac{c}{\theta}) + \frac{c}{\theta} = e^{\theta(t-T)}\frac{V'(T)}{\theta} + \frac{c}{\theta}\,.$$

Hence, V has a finite zero. Therefore, in order for u to be positive near ∞, V must be increasing and $(r^\theta u(r))_r \ge 0$ near ∞. This is true obviously in the other case that $u_r > 0$ near ∞.

Lemma 1.2. *Let $n > p > 1$. If u is a positive radial solution satisfying* (1.8) *near ∞, then $r^{\frac{n-p}{p-1}} u(r)$ is increasing.*

Now, we classify positive solutions of (1.8) near ∞ into two groups according to their behaviors. If $r^{\frac{n-p}{p-1}} u$ converges to a positive constant at ∞, then we call u a fast decaying solution. Otherwise, u is a slowly decaying solution if $r^{\frac{n-p}{p-1}} u(r) \to \infty$ as $r \to \infty$.

1.2. **Known results.** One of Liouville's theorems related to p-Laplace equation is the nonexistence of nontrivial nonnegative solutions in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ to the following quasilinear inequality

$$-\Delta_p u \geq c|x|^l u^q$$

with $c > 0$ and $l > -p$, when $n > p > 1$ and

$$q \leq \frac{(p-1)(n+l)}{n-p}.$$

See [1, Theorem 3.3 (iii)]. For the existence of nontrivial solutions to

$$\Delta_p u + u^q = 0\,,$$

on $\mathbf{R}^n$ with $n > p > 1$ and $q > p-1$, it is necessary and sufficient that $q \geq \frac{n(p-1)+p}{n-p}$ [6]. On the other hand, (1.3) with $q = q_s := \frac{n(p-1)+p+pl}{n-p}$ admits the one-parameter family of positive solutions given by

$$\overline{u}_\alpha(x) = \frac{\alpha}{(1 + \xi(\alpha^{\frac{p}{n-p}}|x|)^{\frac{p+l}{p-1}})^{\frac{n-p}{p+l}}}$$

with $\xi = \xi_{p,n} = \frac{p-1}{(n-p)(n+l)^{1/(p-1)}}$ and $\overline{u}_\alpha(0) = \alpha > 0$. A radial solution $u(x) = u(|x|)$ of (1.3) satisfies the equation

$$(|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r + r^l u^q = 0\,. \tag{1.9}$$

For $l > -p$, (1.9) with $u(0) = \alpha > 0$, has a unique positive solution $u \in C^1(0,\epsilon) \cap C[0,\epsilon)$ for small $\epsilon > 0$ such that $|u_r|^{p-2}u_r \in C^1[0,\epsilon)$. If $q < q_s$, then every local solution of (1.9) has a finite zero [2, 5]. In the opposite case $q > q_s$, every local solution of (1.9) is to be a slowly decaying solution [2, 3, 5].

2. Local existence

Let $n \geq p > 1$, $l > -p$ and $q \geq p-1$. First, in order to prove the local existence of positive radial solutions of (1.3), we consider the integral equation

$$u(r) = \alpha - \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} u^q(s)ds)^{\frac{1}{p-1}}\, dt$$

with $\alpha > 0$.

2.1. **Integral representation.** On a space

$$S = \{u \in C[0,\varepsilon] \,|\, 0 \le u \le \alpha\}$$

we study a nonlinear operator T from S to $C[0,\varepsilon]$ by

$$T(u)(r) = \alpha - T_1(u)(r),$$

where

$$T_1(u)(r) = \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} u^q(s) ds)^{\frac{1}{p-1}}\, dt\,.$$

For $\varepsilon > 0$ small enough, T_1 satisfies that

$$0 \le T_1 \le \alpha^{\frac{q}{p-1}} \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} ds)^{\frac{1}{p-1}} dt \le (\frac{\alpha^q}{n+l})^{\frac{1}{p-1}} \frac{p-1}{p+l} \varepsilon^{\frac{p+l}{p-1}} \le \alpha\,.$$

Hence, $T(S) \subset S$. Minkowski's inequality for $p \ge 2$ shows that for $u_1, u_2 \in S$,

$$\begin{aligned}
\|T(u_2) - T(u_1)\| &\le \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} \big|u_2^{\frac{q}{p-1}} - u_1^{\frac{q}{p-1}}\big|^{p-1} ds)^{\frac{1}{p-1}}\, dt \\
&\le \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l}\, ds)^{\frac{1}{p-1}}\, dt\, \|u_2 - u_1\| \\
&= \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} (\frac{1}{n+l})^{\frac{1}{p-1}} \frac{p-1}{p+l} \varepsilon^{\frac{p+l}{p-1}} \|u_2 - u_1\|\,.
\end{aligned}$$

For $1 < p < 2$, we observe that for $u_1, u_2 \in S$,

$$\begin{aligned}
\|T(u_2) - T(u_1)\| &\le \int_0^r t^{\frac{1-n}{p-1}} \frac{\alpha^{\frac{q(2-p)}{p-1}}}{p-1} (\int_0^t s^{n-1+l}\, ds)^{\frac{2-p}{p-1}} (\int_0^t s^{n-1+l} |u_2^q - u_1^q|\, ds)\, dt \\
&\le \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l}\, ds)^{\frac{1}{p-1}}\, dt\, \|u_2 - u_1\| \\
&= \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} (\frac{1}{n+l})^{\frac{1}{p-1}} \frac{p-1}{p+l} \varepsilon^{\frac{p+l}{p-1}} \|u_2 - u_1\|\,.
\end{aligned}$$

Now, we assume that

$$\frac{p-1}{p+l} \max\{(\frac{\alpha^q}{n+l})^{\frac{1}{p-1}}, \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} (\frac{1}{n+l})^{\frac{1}{p-1}}\} \varepsilon^{\frac{p+l}{p-1}} < \min\{\alpha, 1\}.$$

Then, T is a contraction mapping in S and thus T has a unique fixed point $\bar{u}_\alpha$.

Generally, we consider the integral equation under condition (KP),

$$u(r) = \alpha - \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1} K(s) u^q(s) ds)^{\frac{1}{p-1}}\, dt\,.$$

Then, the integrability of (KP) shows in the same way the local existence of a positive solution u_α with $u_\alpha(0) = \alpha > 0$ to (1.2). Then, it is easy to see that there exists a sequence $\{r_j\}$ going to 0 such that

$$\lim_{j\to\infty} r_j^{n-1} |u_r(r_j)|^{p-2} u_r(r_j) = 0\,, \tag{2.1}$$

and $u_\alpha(r)$ is decreasing as long as u remains positive. Moreover, u_α is strictly decreasing after K becomes positive.

2.2. **Fowler transform.** Let $n > p > 1$ and $q > \frac{(n+l)(p-1)}{n-p}$ with $l > -p$. Set $m = \frac{p+l}{q-(p-1)}$. Fowler transform $V(t) = r^m u(r)$, $t = \log r$, of a positive solution to (1.2) satisfies

$$(p-1)(mV - V')^{p-2}(V'' - mV') - \xi(mV - V')^{p-1} + k(t)V^q = 0\,, \tag{2.2}$$

where $\xi = n-1-(m+1)(p-1) = \frac{L^{q-(p-1)}}{m^{p-1}}$ with L given by (1.5), and $k(t) = r^{-l}K(r)$. Furthermore, if $-r^{m+1}u_r(r) = mV - V' > 0$, then (2.2) can be rewritten as

$$(p-1)(V'' - mV') - \xi(mV - V') = -\frac{k(t)V^q}{(mV - V')^{p-2}}$$

and

$$(p-1)V'' + aV' - \xi mV = -\frac{k(t)V^q}{(mV - V')^{p-2}},$$

where $a = n-1-(2m+1)(p-1)$. Setting $b = \xi m = \frac{L^{q-(p-1)}}{m^{p-2}}$, we have

$$(p-1)V'' + aV' - (b - \frac{k(t)V^{q-1}}{(mV - V')^{p-2}})V = 0\,.$$

That is,

$$(p-1)V'' + aV' - \frac{1}{m^{p-2}}L^{q-(p-1)}V + \frac{k(t)}{(mV - V')^{p-2}}V^q = 0\,, \tag{2.3}$$

which holds as long as the local solution remains positive.

3. A priori estimates

In order to obtain upper bounds, we argue similarly as in Lemma 2.16, Lemma 2.20, Theorem 2.25 in [4].

3.1. **Upper bound.** Let $n > p \geq -\ell$. If u is a positive solution satisfying the inequality

$$(r^{n-1}|u_r|^{p-2}u_r)_r \leq -cr^{n-1+\ell}u^q \tag{3.1}$$

near ∞ for some $c > 0$, then

$$r^{n-1}|u_r|^{p-2}u_r \leq r_0^{n-1}|u_r(r_0)|^{p-2}u_r(r_0) - c\int_{r_0}^{r} s^{n-1+\ell}u^q(s)\,ds \tag{3.2}$$

for $r > r_0$, if r_0 is sufficiently large. Then, we may assume that $u_r(r_0) \leq 0$. Indeed, if $u_r(r_0) > 0$, then

$$r^{n-1}|u_r|^{p-2}u_r \leq r_0^{n-1}|u_r(r_0)|^{p-2}u_r(r_0) - cu^q(r_0)\frac{1}{n+\ell}(r^{n+\ell} - r_0^{n+\ell})$$

as long as u_r is positive. Hence, u_r is eventually negative. Therefore, (3.2) gives

$$r^{n-1}|u_r|^{p-2}u_r \leq -cu^q(r)\frac{1}{n+\ell}(r^{n+\ell} - r_0^{n+\ell})$$

and thus,

$$\frac{u_r}{u^{q/(p-1)}} \leq -c_1 r^{\frac{1+\ell}{p-1}}$$

for some $c_1 > 0$. Hence, we obtain

$$u(r) \leq \begin{cases} Cr^{-\frac{p+\ell}{q-(p-1)}} & \text{if} \quad \ell > -p\,, \\ C(\log r)^{-\frac{p-1}{q-(p-1)}} & \text{if} \quad \ell = -p \end{cases}$$

for some $C > 0$. Combining the a priori estimates and Lemma 1.2, we have the following assertion.

Theorem 3.1. *Let $n > p \geq -\ell$ and $q > \frac{(p-1)(n+\ell)}{n-p}$. Then, every positive solution to* (3.1) *near ∞ satisfies that*

$$C_1 r^{-\frac{p+\ell}{q-(p-1)}} \geq u(r) \geq C_2 r^{-\frac{n-p}{p-1}}$$

for $\ell > -p$ and

$$C_1 (\log r)^{-\frac{p-1}{q-(p-1)}} \geq u(r) \geq C_2 r^{-\frac{n-p}{p-1}}$$

for $\ell = -p$.

In Theorem 3.1, we use the notation ℓ instead of l to consider the case of $\ell = -p$. It is interesting to study the existence of positive entire solutions of (1.1) with the logarithmic asymptotic behavior at ∞.

Lemma 3.2. *Let $q > \frac{(p-1)(n+l)}{n-p}$. Assume $K(r) = O(r^l)$ at ∞ for some $l > -p$. If u is a positive solution to* (3.1) *near ∞ and $u(r) = O(r^{-m-\varepsilon})$ with some $\varepsilon > 0$ at ∞, then $u(r) = O(r^{\frac{p-n}{p-1}})$ at ∞.*

Proof. Integrating (1.2) over $[r, \infty)$, we obtain

$$u(r) = \int_r^\infty t^{\frac{1-n}{p-1}} \Big(\int_0^t K(s) u^q(s) s^{n-1}\, ds \Big)^{\frac{1}{p-1}}\, dt\,.$$

On the other hand, we have

$$\begin{aligned} \int_0^t K(s) u^q(s) s^{n-1}\, ds &\leq C + C \int_1^t s^{n-1+l-q(m+\varepsilon)}\, ds \\ &= \begin{cases} C + Ct^{n+l-q(m+\varepsilon)} & \text{if} \quad n+l \neq q(m+\varepsilon)\,, \\ C + C\log t & \text{if} \quad n+l = q(m+\varepsilon)\,. \end{cases} \end{aligned}$$

If $n + l < q(m+\varepsilon)$, we are done. If $n + l \geq q(m+\varepsilon)$, then

$$u(r) \leq \begin{cases} Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p-n}{p-1}} (\log r)^{\frac{1}{p-1}} & \text{if} \quad n+l = q(m+\varepsilon)\,, \\ Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p+l}{p-1} - \frac{q(m+\varepsilon)}{p-1}} & \text{if} \quad p+l < q(m+\varepsilon) < n+l\,. \end{cases}$$

In case $n + l = q(m+\varepsilon)$, we replace ε by $\frac{n-p}{p-1} - m - \delta$ in the above arguments, where $\delta > 0$ is so small that $\delta < \frac{n-p}{p-1} - m$. Note that $m < \frac{n-p}{p-1}$ iff $q > \frac{(p-1)(n+l)}{n-p}$.

$$u(r) \leq \begin{cases} Cr^{\frac{p-n}{p-1}} & \text{if} \quad n+l = q(m+\varepsilon)\,, \\ Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p+l}{p-1} + \frac{q(p+l)}{(p-1)^2} - \frac{q^2(m+\varepsilon)}{(p-1)^2}} & \text{if} \quad p+l < q(m+\varepsilon) < n+l\,. \end{cases}$$

In case $q(m+\varepsilon) < n+l$, we iterate this process to obtain

$$\begin{aligned} u(r) &\le Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p+l}{p-1}\sum_{i=0}^{j-1}(\frac{q}{p-1})i - \frac{q^j(m+\varepsilon)}{(p-1)^j}} \\ &= Cr^{\frac{p-n}{p-1}} + Cr^{-m-\varepsilon(\frac{q}{p-1})^j} \end{aligned}$$

for any positive integer j. Since $q > p-1$, we reach the conclusion after a finite number of iterations. $\square$

Lemma 3.3. *Let $q > \frac{(p-1)(n+l)}{n-p}$. Assume $K(r) = O(r^l)$ at ∞ for some $l > -p$. If $u(r) = o(r^{-m})$ at ∞, then $(r^m u(r))_r < 0$ near ∞.*

Proof. Let $V(t) = r^m u(r)$, $t = \log r$. Then, V satisfies (2.3). Suppose $V'(T) = 0$ for some T near ∞ and $k(t)V^{q-(p-1)}(t) < m^{p-2}b$ for $t \in [T, \infty)$. Then, $V''(T) > 0$ and $V(t)$ is strictly increasing near T but for $t > T$. Since $V \to 0$ at ∞, there exists $T_1 > T$ such that $V'(T_1) = 0$ and

$$V''(T_1) = \frac{1}{p-1}(b - \frac{1}{m^{p-2}}k(T_1)V^{q-(p-1)}(T_1))V(T_1) \le 0\,,$$

a contradiction. $\square$

Theorem 3.4. *Let $q > \frac{(p-1)(n+l)}{n-p}$. Assume $K(r) = O(r^l)$ at ∞ for some $l > -p$. If $u(r) = o(r^{-m})$ at ∞, then $u(r) = O(r^{\frac{p-n}{p-1}})$ at ∞.*

Proof. Let $\varphi(r) = r^m u(r)$. Then, φ satisfies

$$\varphi_{rr} + (1 + \frac{a}{p-1})\frac{1}{r}\varphi_r - \frac{b}{(p-1)r^2}\varphi + \frac{k}{(p-1)(m\varphi - r\varphi_r)^{p-2}r^2}\varphi^q = 0\,.$$

For $\varepsilon > 0$, define the elliptic operator

$$\mathcal{L}_\varepsilon\varphi = \Delta\varphi - [2m + (n-1)\frac{p-2}{p-1}]\frac{x\cdot\nabla\varphi}{|x|^2} - m(\frac{L^{q-(p-1)}}{m^{p-1}} - \varepsilon)\frac{\varphi}{|x|^2}\,,$$

where $\frac{L^{q-(p-1)}}{m^{p-1}} = n-1-(m+1)(p-1)$. It follows from Lemma 3.3 that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\mathcal{L}_\varepsilon\varphi = m\varepsilon\frac{\varphi}{r^2} - \frac{k\varphi^q}{(p-1)r^2(m\varphi - r\varphi_r)^{p-2}} \ge (m\varepsilon - \frac{k\varphi^{q-(p-1)}}{(p-1)m^{p-2}})\frac{\varphi}{r^2} \ge 0$$

in $\mathbf{R}^n \backslash B_{R_\varepsilon}(0)$. For $0 < \varepsilon < n-1-(m+1)(p-1)$, let $\eta_\varepsilon(x) = |x|^{\sigma_\varepsilon}$ with σ_ε being the negative root of $\sigma(\sigma-1) + (n-1-2m-(n-1)\frac{p-2}{p-1})\sigma - m(\frac{L^{q-(p-1)}}{m^{p-1}} - \varepsilon) = 0$, i.e.,

$$\sigma_\varepsilon = \frac{1}{2}\left[-(n-2-2m-(n-1)\frac{p-2}{p-1}) - \sqrt{D}\right]\,,$$

where $D = (n-1-2m-(n-1)\frac{p-2}{p-1})^2 + 4m(\frac{L^{q-(p-1)}}{m^{p-1}} - \varepsilon)$. Setting $C_\varepsilon = \varphi(R_\varepsilon)R_\varepsilon^{-\sigma_\varepsilon}$, we see that $\mathcal{L}_\varepsilon(\varphi - C_\varepsilon\eta_\varepsilon) \ge 0$ in $\mathbf{R}^n\backslash B_{R_\varepsilon}(0)$ and $\varphi(R_\varepsilon) = C_\varepsilon\eta_\varepsilon(R_\varepsilon)$, $\varphi - C_\varepsilon\eta_\varepsilon \to 0$ as $r \to \infty$. Then, the maximum principle implies that $\varphi - C_\varepsilon\eta_\varepsilon \le 0$ in $\mathbf{R}^n\backslash B_{R_\varepsilon}(0)$. Hence, $\varphi(r) \le C_\varepsilon\eta_\varepsilon(r)$ at ∞. Then, Lemma 3.2 implies the conclusion. $\square$

Proof of Theorem 1.1. When $k(t) = r^{-l}K(r) \to 1$ as $t = \log r \to +\infty$, it follows from Theorem 3.1 and (2.3) that slowly decaying solutions satisfy

$$\liminf_{r\to\infty} r^m u(r) \le L \le \limsup_{r\to\infty} r^m u(r) < \infty\,.$$

Indeed, at every local minimum (maximum) point of $V(t) = r^m u(r)$, V satisfies

$$\frac{1}{m^{p-2}} L^{q-(p-1)} V \ge (\le) \frac{k(t)}{(mV)^{p-2}} V^q\,.$$

If V is monotonically increasing near $+\infty$, then it is easy to see that $V \to L$ as $t \to +\infty$ by (2.3). If V is monotonically decreasing and $V \to 0$, then it follows from Lemma 1.2 and Theorem 3.4 that $r^{\frac{n-p}{p-1}} u(r) \to C$ for some $C > 0$. $\square$

References

[1] Bidaut-Véron, M.-F., Pohozaev, S., *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84** (2001), 1–49.

[2] Kawano, N., Ni, W.-M., Yotsutani, S., *A generalized Pohozaev identity and its applications*, J. Math. Soc. Japan **42** (1990), 541–564.

[3] Kawano, N., Yanagida, E., Yotsutani, S., *Structure theorems for positive radial solutions to* $\operatorname{div}(|Du|^{m-2}Du) + K(|x|)u^q = 0$ *in* $\mathbf{R}^n$, J. Math. Soc. Japan **45** (1993), 719–742.

[4] Li, Y., Ni, W.-M., *On conformal scalar curvature equation in* $\mathbf{R}^n$, Duke Math. J. **57** (1988), 895–924.

[5] Ni, W.-M., Serrin, J., *Existence and non-existence theorems for ground states of quasilinear partial differential equations: The anomalous case*, Atti Convegni Lincei **77** (1986), 231–257.

[6] Serrin, J., Zou, H., *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002), 79–142.

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LARGE TIME BEHAVIOR IN A QUASILINEAR PARABOLIC-PARABOLIC-ELLIPTIC ATTRACTION-REPULSION CHEMOTAXIS SYSTEM

Yutaro Chiyo

Abstract. This paper deals with a quasilinear parabolic-parabolic-elliptic attraction-repulsion chemotaxis system. Boundedness, stabilization and blow-up in this system of the fully parabolic and parabolic-elliptic-elliptic versions have already been proved. The purpose of this paper is to derive boundedness and stabilization in the parabolic-parabolic-elliptic version.

1. Introduction and main result

In this paper we consider the quasilinear attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w \big)\,, \\ v_t = \Delta v + \alpha u - \beta v\,, \\ 0 = \Delta w + \gamma u - \delta w\,, \\ (\nabla u \cdot \nu)|_{\partial\Omega} = (\nabla v \cdot \nu)|_{\partial\Omega} = (\nabla w \cdot \nu)|_{\partial\Omega} = 0\,, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$. Here m, p, $q \geq 1$, χ, ξ, α, β, γ, $\delta > 0$ are constants, ν is the outward normal vector to $\partial\Omega$,

$$u_0 \in C^0(\overline{\Omega})\,, \quad u_0 \geq 0 \ \text{ in } \overline{\Omega} \quad \text{and} \quad u_0 \neq 0, \tag{1.2}$$

$$v_0 \in W^{1,\theta}(\Omega) \text{ for some } \theta > n\,, \quad v_0 \geq 0 \ \text{ in } \overline{\Omega} \quad \text{and} \quad v_0 \neq 0. \tag{1.3}$$

The model (1.1) was proposed by [12] to describe the aggregation of microglial cells in Alzheimer's disease. Also, u, v and w represent the cell density, concentrations of attractive and repulsive chemical substances; α and γ idealize the rates at which the cell produces substances; β and δ represent the rates at which substances are transformed into another ones which do not involve in the movement of the cell.

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Let us overview previous results on the attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla \cdot \left(\nabla u - \chi u \nabla v + \xi u \nabla w\right), \\ \tau v_t = \Delta v + \alpha u - \beta v, \\ \tau w_t = \Delta w + \gamma u - \delta w, \end{cases} \tag{1.4}$$

where χ, ξ, α, β, γ, $\delta > 0$ are constants and $\tau \in \{0, 1\}$. This system has been investigated in several studies. For instance, in the case that $\tau = 1$ boundedness (including global existence) was studied in [5], finite-time blow-up (blow-up for short) was analyzed in [9] and stabilization was studied in [11]. Also, in the simplified case that $\tau = 0$ there are more precise studies. Indeed, blow-up with logistic source was discussed in [2] and stabilization was investigated in [10, 13]. On the other hand, as to the quasilinear version, such as (1.1), of the above system (1.4) with $\tau = 0$, there are several studies. Indeed, boundedness and blow-up were classified by the size of p, q in [4] and stabilization was obtained in [1, 3].

In summary, boundedness, stabilization and blow-up in the attraction-repulsion system (1.4) have been well studied in the fully parabolic case ($\tau = 1$) and in the parabolic-elliptic-elliptic case ($\tau = 0$). However, the *quasilinear* parabolic-*parabolic*-elliptic attraction-repulsion chemotaxis system has not been analyzed. The purpose of this paper is to derive boundedness and stabilization in (1.1).

The main result of this paper reads as follows.

Theorem 1.1. *Let $n \in \mathbb{N}$. Let m, $p \geq 1$ fulfill $p - m \in [0, 1]$ when $n = 1$, $p - m \in [0, \frac{2}{n}]$ when $n \geq 2$ and let $q \geq 1$. Assume that u_0, v_0 satisfy* (1.2), (1.3)*. Then there exists a unique triplet (u, v, w) which solves* (1.1) *in the classical sense and is bounded, that is,*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C$$

for all $t > 0$ with some $C > 0$ in the cases that $p - m \in [0, 1)$ for $n = 1$ and that $p - m \in [0, \frac{2}{n})$ for $n \geq 2$. Also, there exists $\lambda_0 > 0$ such that if

$$\|u_0\|_{L^1(\Omega)} < \lambda_0 \tag{1.5}$$

only in the cases that $p - m = 1$ for $n = 1$ and that $p - m = \frac{2}{n}$ for $n \geq 2$, then the same conclusion on boundedness holds. Moreover, assume further that u_0 satisfies

$$\chi \|u_0\|_{L^1(\Omega)}^{p-m} < \frac{1}{C_{\langle p-m \rangle}}, \tag{1.6}$$

where $C_{\langle p-m \rangle} > 0$ is a constant appearing in the Poincaré-Sobolev inequality (*see* (2.14))*. Then the bounded solution (u, v, w) has the property that*

$$(u(\cdot, t), v(\cdot, t), w(\cdot, t)) \to \left(\overline{u_0}, \frac{\alpha}{\beta}\overline{u_0}, \frac{\gamma}{\delta}\overline{u_0}\right) \text{ in } [L^\infty(\Omega)]^3 \quad \text{as } t \to \infty, \tag{1.7}$$

where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$.

Remark 1.2. We need the condition (1.5) only to assert boundedness.

2. Proof of Theorem 1.1

We first give a result on local existence in (1.1).

Lemma 2.1. *Let $m, p, q \geq 1$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Then for all u_0, v_0 satisfying the conditions (1.2), (1.3) there exists $T_{\max} \in (0, \infty]$ such that (1.1) admits a unique classical solution (u, v, w) such that $u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$, $v, w \in C^0([0, T_{\max}); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$. Moreover, if $T_{\max} < \infty$, then $\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$.*

Proof. Let $T \in (0, 1]$, $M := \|u_0\|_{L^\infty(\Omega)} + 1$ and $N := \|v_0\|_{W^{1,\theta}(\Omega)}$. We introduce the set $\mathcal{S} := \{\varphi \in X \mid 0 \leq \varphi \leq M \text{ in } \overline{\Omega} \times [0, T]\}$, where $X := C^0(\overline{\Omega} \times [0, T])$. Also, we define $\Phi(\widehat{u}) := u$ for $\widehat{u} \in \mathcal{S}$, where u is the solution of

$$u_t = \nabla \cdot \big((\widehat{u} + 1)^{m-1} \nabla u - \chi \widehat{u} (\widehat{u} + 1)^{p-2} \nabla v + \xi \widehat{u} (\widehat{u} + 1)^{q-2} \nabla w\big) \text{ in } \Omega \times (0, T)$$

with $(\nabla u \cdot \nu)|_{\partial\Omega} = 0$, $u(x, 0) = u_0(x)$, where v and w are the solutions of

$$v_t = \Delta v + \alpha \widehat{u} - \beta v \text{ in } \Omega \times (0, T)$$

with $(\nabla v \cdot \nu)|_{\partial\Omega} = 0$, $v(x, 0) = v_0(x)$ and

$$0 = \Delta w + \gamma \widehat{u} - \delta w \text{ in } \Omega \times (0, T)$$

with $(\nabla w \cdot \nu)|_{\partial\Omega} = 0$, respectively. Then, by an argument similar to that in [8,15], we can verify that Φ is a continuous and compact map of $\mathcal{S}$ into $\mathcal{S}$. Therefore, from the Schauder fixed point theorem and standard regularity theory for parabolic and elliptic equations, we obtain local existence in (1.1). $\square$

The first purpose of this section is to derive global existence and boundedness. To achieve this, we obtain an L^r-estimate for u with sufficiently large r.

Lemma 2.2. *Let $s \in (0, T_{\max})$. Let $m, p \geq 1$ fulfill $p - m \in [0, 1]$ when $n = 1$, $p - m \in [0, \frac{2}{n}]$ when $n \geq 2$ and let $q \geq 1$. Let u_0, v_0 satisfy (1.2), (1.3). Then there exist $r_0 > 1$ and $\lambda_0 > 0$ such that if u_0 satisfies $\|u_0\|_{L^1(\Omega)} < \lambda_0$ only in the cases that $p - m = 1$ for $n = 1$ and that $p - m = \frac{2}{n}$ for $n \geq 2$, then for all $r > r_0$,*

$$\sup_{t \in (s, T_{\max})} \|u(\cdot, t)\|_{L^r(\Omega)} \leq K_r \tag{2.1}$$

with some $K_r > 0$.

Proof. Let $s \in (0, T_{\max})$ and $r > 1$. By the first equation of (1.1) and integration by parts, we have

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|u(\cdot, t)\|^r_{L^r(\Omega)} &= -\int_\Omega (u + 1)^{m-1} \nabla u \cdot \nabla u^{r-1} \\ &\quad + \chi \int_\Omega u (u + 1)^{p-2} \nabla v \cdot \nabla u^{r-1} \\ &\quad - \xi \int_\Omega u (u + 1)^{q-2} \nabla w \cdot \nabla u^{r-1} \\ &=: I_1(\cdot, t) + I_2(\cdot, t) + I_3(\cdot, t) \end{aligned} \tag{2.2}$$

for all $t \in (s, T_{\max})$. This corresponds to [6, (28) with $D(s) = s^{m-1}$, $S(s) = s^{p-1}$, $\varepsilon = 1$] with additional term I_3, but we use [4, (3.13) and (3.16)] to derive

$$\begin{aligned} I_3(\cdot, t) &\le \frac{\xi(r-1)}{r+q-2}\Big(2\delta \int_\Omega u^{r+q-2} w + \delta c_1 \int_\Omega w - \gamma \int_\Omega u^{r+q-1}\Big) \\ &\le \frac{\xi(r-1)}{r+q-2}\Big[2\delta\Big(\frac{\gamma}{2\delta}\int_\Omega u^{r+q-1} + c_2\Big) + c_3 - \gamma \int_\Omega u^{r+q-1}\Big] \\ &= \frac{\xi(r-1)}{r+q-2}(2\delta c_2 + c_3) =: c_4 \end{aligned} \tag{2.3}$$

for all $t \in (s, T_{\max})$ with some $c_1, c_2, c_3 > 0$. Thus, combining (2.3) with (2.2), we can observe from [6, p. 223, lines 12 and 13] that there exist $r_1, r_2 > 1$ such that

$$\begin{aligned} &\frac{d}{dt}\|u(\cdot,t)\|^r_{L^r(\Omega)} \\ &\quad \le -\|u(\cdot,t)\|^r_{L^r(\Omega)} + (c_5 r)^{c_6 r} - \frac{1}{2}A(r,m,p,u_0)\|\nabla u^{\frac{r+m-1}{2}}(\cdot,t)\|^2_{L^2(\Omega)} \\ &\qquad + c_7\|\Delta v(\cdot,t)\|^{r+p-1}_{L^{r+p-1}(\Omega)} + c_8\|\Delta v(\cdot,t)\|^{r+1}_{L^{r+1}(\Omega)} + c_9 \end{aligned} \tag{2.4}$$

for all $t \in (s, T_{\max})$ and all $r > \max\{\frac{n}{2}(p-m) - p + 1, \frac{n}{2}(2-m) - 1, r_1, r_2\}$ with some $c_5, c_6, c_7, c_8, c_9 > 0$, where $A(r,m,p,u_0) > 0$ is a constant defined as

$$A(r,m,p,u_0) := \begin{cases} \frac{2r(r-1)}{(r+m-1)^2} & \text{if } p-m \in [0,1)\ (n=1), \\ & \quad p-m \in [0, \frac{2}{n})\ (n \ge 2), \\ \frac{4r(r-1)}{(r+m-1)^2} - c_{10} r \|u_0\|^{c_{11}(r+p-1)}_{L^1(\Omega)} & \text{if } p-m = 1\ (n=1), \\ & \quad p-m = \frac{2}{n}\ (n \ge 2) \end{cases}$$

with $c_{10}, c_{11} > 0$, where the value $-c_{10} r \|u_0\|^{c_{11}(r+p-1)}_{L^1(\Omega)}$ in the critical case is derived from [6, p. 222, line 4]. Then, from an argument parallel to that in the derivation of [6, (38)], the differential inequality (2.4) implies that

$$\begin{aligned} \|u(\cdot,t)\|^r_{L^r(\Omega)} &\le \|u(\cdot,s)\|^r_{L^r(\Omega)} \\ &\quad + \Big[(c_5 r)^{c_6 r} + c_9 + (c_{12} r C_{\mathrm{MR}}^{r+p-1})^{c_{13} r} + (c_{14} r C_{\mathrm{MR}}^{r+1})^{c_{15} r}\Big] \\ &\quad + c_7 r C_{\mathrm{MR}}^{r+p-1}\|\Delta v(\cdot,s)\|^{r+p-1}_{L^{r+p-1}(\Omega)} + c_8 r C_{\mathrm{MR}}^{r+1}\|\Delta v(\cdot,s)\|^{r+1}_{L^{r+1}(\Omega)} \end{aligned}$$

for all $t \in (s, T_{\max})$ and all $r > \max\{\frac{n}{2}(p-m) - p + 1, \frac{n}{2}(2-m) - 1, r_1, r_2\}$ with some $C_{\mathrm{MR}}, c_{12}, c_{13}, c_{14}, c_{15} > 0$ via estimates for $\int_s^t \|\Delta e^{\frac{\sigma-t}{r+p-1}} v(\cdot,\sigma)\|^{r+p-1}_{L^{r+p-1}(\Omega)}\, d\sigma$ and $\int_s^t \|\Delta e^{\frac{\sigma-t}{r+1}} v(\cdot,\sigma)\|^{r+1}_{L^{r+1}(\Omega)}\, d\sigma$ by the maximal Sobolev regularity ([6, Lemma 2.1]) and the Young inequality. More precisely, we estimate these two terms as

$$\begin{aligned} \int_s^t \|\Delta e^{\frac{\sigma-t}{\theta_1}} v(\cdot,\sigma)\|^{\theta_1}_{L^{\theta_1}(\Omega)}\, d\sigma &\le c_{16} r C_{\mathrm{MR}}^{\theta_1}\|\Delta v(\cdot,s)\|^{\theta_1}_{L^{\theta_1}(\Omega)} \\ &\quad + \int_s^t e^{\sigma-t}\Big[\frac{1}{4}A(r,m,p,u_0)\|\nabla u^{\frac{r+m-1}{2}}(\cdot,t)\|^2_{L^2(\Omega)} + (c_{17} r C_{\mathrm{MR}}^{\theta_1})^{c_{18} r}\Big]\, d\sigma \end{aligned}$$

with $\theta_1 \in \{r+p-1, r+1\}$ and $c_{16}, c_{17}, c_{18} > 0$. Therefore, by following an argument similar to that in [6] and taking $\|u_0\|_{L^1(\Omega)}$ sufficiently small such that $A(r, m, p, u_0) > 0$ only in the cases that $p - m = 1$ for $n = 1$ and that $p - m = \frac{2}{n}$ for $n \geq 2$, we arrive at (2.1). $\square$

Proof of Theorem 1.1 (*Boundedness*)**.** Taking $r = r^* > 1$ in Lemma 2.2 sufficiently large such that r^* fulfills the assumption of [14, Lemma A.1], we have $\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$, which means that $T_{\max} = \infty$ by the extensibility criterion, and boundedness holds. $\square$

The second purpose of this section is to prove stabilization. To this end, we introduce the function

$$\Phi(s) := \int_1^s \int_1^\sigma \frac{1}{\eta(\eta+1)^{p-2}}\, d\eta d\sigma\,, \quad s \geq 0\,,$$

where $p \geq 1$ is a constant appearing in the attraction term in (1.1). In order to obtain an energy inequality we first calculate and estimate $\frac{d}{dt}\int_\Omega \Phi(u)$.

Lemma 2.3. *The first component u satisfies that*

$$\frac{d}{dt}\int_\Omega \Phi(u) + \int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 \leq \chi \int_\Omega \nabla u\cdot\nabla v \tag{2.5}$$

for all $t > 0$.

Proof. We see from the first equation in (1.1) and the identity $\Phi''(u) = \frac{1}{u(u+1)^{p-2}}$ as well as straightforward calculations that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Phi(u) = &-\int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 + \chi\int_\Omega \nabla u\cdot\nabla v\\ &-\xi\int_\Omega (u+1)^{q-p}\nabla u\cdot\nabla w\end{aligned} \tag{2.6}$$

for all $t > 0$. Here we can estimate the third term on the right-hand side by zero. Indeed, we rewrite the third equation in (1.1) as

$$0 = \Delta\Big(w + \frac{\gamma}{\delta}\Big) + \gamma(u+1) - \delta\Big(w+\frac{\gamma}{\delta}\Big)\,, \tag{2.7}$$

and thereby we invoke integration by parts to obtain

$$\begin{aligned} I &:= -\xi\int_\Omega (u+1)^{q-p}\nabla u\cdot\nabla w\\ &= \frac{\xi}{q-p+1}\int_\Omega (u+1)^{q-p+1}\Delta\Big(w+\frac{\gamma}{\delta}\Big)\\ &= \frac{\xi\delta}{q-p+1}\int_\Omega (u+1)^{q-p+1}\Big(w+\frac{\gamma}{\delta}\Big) - \frac{\xi\gamma}{q-p+1}\int_\Omega (u+1)^{q-p+2}.\end{aligned} \tag{2.8}$$

Moreover, applying the Hölder inequality to (2.8) and noticing that (2.7) yields

$$\Big\|w(\cdot,t)+\frac{\gamma}{\delta}\Big\|_{L^{q-p+2}(\Omega)} \leq \frac{\gamma}{\delta}\|u(\cdot,t)+1\|_{L^{q-p+2}(\Omega)}$$

for all $t > 0$, we obtain

$$I \le \frac{\xi\delta}{q-p+1} \|u(\cdot,t)+1\|_{L^{q-p+2}(\Omega)}^{q-p+1} \cdot \left(\left\| w(\cdot,t) + \frac{\gamma}{\delta} \right\|_{L^{q-p+2}(\Omega)} - \frac{\gamma}{\delta} \|u(\cdot,t)+1\|_{L^{q-p+2}(\Omega)} \right) \le 0\,,$$

which along with (2.6) implies that (2.5) holds. □

In order to state the next lemma we define the function

$$V(x,t) := v(x,t) - \frac{\alpha}{\beta}\overline{u_0} \quad \text{for } x \in \Omega\,,\ t > 0\,.$$

Lemma 2.4. *The first component u satisfies that for all $t > 0$,*

$$\begin{aligned} &\frac{d}{dt}\Big[\int_\Omega \Phi(u) + \frac{\chi}{2\alpha}\int_\Omega |\nabla V|^2 + \frac{\chi\beta}{\alpha}\int_\Omega V^2 \Big] \\ &\quad + \int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 + \frac{\chi\beta}{\alpha}\int_\Omega |\nabla V|^2 + \frac{\chi\beta^2}{\alpha}\int_\Omega V^2 + \frac{\chi}{\alpha}\int_\Omega V_t^2 \\ &\le \chi \int_\Omega (u-\overline{u_0})^2\,. \end{aligned} \tag{2.9}$$

Proof. Noting from the second equation in (1.1) that $V_t = \Delta V + \alpha(u - \overline{u_0}) - \beta V$ and testing this equation by V_t and V, we can see that

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega |\nabla V|^2 + \frac{\beta}{2}\int_\Omega V^2 \Big] + \int_\Omega V_t^2 \\ &\quad = -\alpha\int_\Omega \nabla u\cdot\nabla v + \alpha\int_\Omega (u-\overline{u_0})^2 - \alpha\beta\int_\Omega (u-\overline{u_0})V, \end{aligned} \tag{2.10}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega V^2 + \int_\Omega |\nabla V|^2 + \beta\int_\Omega V^2 = \alpha\int_\Omega (u-\overline{u_0})V \tag{2.11}$$

for all $t > 0$, respectively. Thus, multiplying (2.10) and (2.11) by $\frac{\chi}{\alpha}$ and $\frac{\chi\beta}{\alpha}$, respectively, and adding them to (2.5), we obtain (2.9). □

We finally derive an energy inequality.

Lemma 2.5. *Let m, p fulfill $p - m \in [0,1]$ when $n = 1$, $p - m \in [0, \frac{2}{n}]$ when $n \ge 2$ and let $C_{\langle p-m\rangle} > 0$ be a constant appearing in the Poincaré–Sobolev inequality (see (2.14)). Then the first component u satisfies that*

$$\frac{d}{dt}\int_\Omega F(u,v) + \Big[\frac{1}{C_{\langle p-m\rangle}\|u_0\|_{L^1(\Omega)}^{p-m}} - \chi \Big] \int_\Omega (u-\overline{u_0})^2 \le 0 \tag{2.12}$$

for all $t > 0$, where

$$F(u,v) := \int_\Omega \Phi(u) + \frac{\chi}{2\alpha}\int_\Omega |\nabla v|^2 + \frac{\chi\beta}{\alpha}\int_\Omega \Big(v - \frac{\alpha}{\beta} \Big)^2.$$

In particular, if u_0 meets (1.6), then

$$\int_0^\infty \int_\Omega (u-\overline{u_0})^2 < \infty\,. \tag{2.13}$$

Proof. We first see from the fact $(u+1)^{m-p+1} \ge u^{m-p+1}$, the mass conservation $\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ $(t>0)$ and the Poincaré–Sobolev inequality that

$$\begin{aligned}\int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 &\ge \frac{1}{\|u_0\|^{p-m}_{L^1(\Omega)}}\Big(\int_\Omega |\nabla u|^{\frac{2}{p-m+1}}\Big)^{p-m+1} \\ &\ge \frac{1}{C_{\langle p-m\rangle}\|u_0\|^{p-m}_{L^1(\Omega)}}\int_\Omega (u-\overline{u_0})^2 \end{aligned} \tag{2.14}$$

for all $t>0$, which along with (2.9) implies that

$$\frac{d}{dt}\int_\Omega F(u,v) + \frac{1}{C_{\langle p-m\rangle}\|u_0\|^{p-m}_{L^1(\Omega)}}\int_\Omega (u-\overline{u_0})^2 \le \chi\int_\Omega (u-\overline{u_0})^2$$

for all $t>0$, which entails (2.12). Also, integrating (2.12) over $(0,t)$, using the positivity of F and (1.6), and taking the limit $t\to\infty$, we derive (2.13). □

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 (*Stabilization*)**.** We first derive L^∞-convergence of u. Since the first component u is bounded in time, we see from parabolic regularity theory ([7]) that there exist $\sigma\in(0,1)$ and $c_1>0$ such that

$$\|u\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{\Omega}\times[1,\infty))} \le c_1\,, \tag{2.15}$$

which implies that the function $t\mapsto \|u(\cdot,t)-\overline{u_0}\|^2_{L^2(\Omega)}$ is uniformly continuous in $[0,\infty)$. Hence, in light of time integrability of $\|u(\cdot,t)-\overline{u_0}\|^2_{L^2(\Omega)}$ (see (2.13)), we infer that $\|u(\cdot,t)-\overline{u_0}\|_{L^2(\Omega)}\to 0$ as $t\to\infty$. Also, employing the Gagliardo–Nirenberg inequality, we can find $c_2>0$ such that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \le c_2\|u(\cdot,t)-\overline{u_0}\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|u(\cdot,t)-\overline{u_0}\|^{\frac{2}{n+2}}_{L^2(\Omega)}\,. \tag{2.16}$$

Noting from the estimate (2.15) that $\|u(\cdot,t)-\overline{u_0}\|_{W^{1,\infty}(\Omega)}\le c_3 := c_1+\overline{u_0}$, we derive from L^2-convergence of u and the estimate (2.16) that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)}\to 0 \quad \text{as } t\to\infty\,.$$

We next show L^∞-convergences of v and w. We put $U(x,t):=u(x,t)-\overline{u_0}$, $V(x,t):=v(x,t)-\frac{\alpha}{\beta}\overline{u_0}$ and $W(x,t):=w(x,t)-\frac{\gamma}{\delta}\overline{u_0}$ for $x\in\Omega$, $t>0$. Then the second equation and boundary condition in (1.1) yield

$$V_t = \Delta V + \alpha U - \beta V\,,\quad (\nabla V\cdot\nu)|_{\partial\Omega} = 0\,.$$

Recalling that $(e^{t\Delta})_{t>0}$ acts as a contraction on $L^\infty(\Omega)$, we have that for all $t>0$,

$$\begin{aligned}\|V(\cdot,t)\|_{L^\infty(\Omega)} &\le e^{-t\beta}\|e^{t\Delta}V(\cdot,0)\|_{L^\infty(\Omega)} + \alpha\int_0^t e^{-(t-s)\beta}\|e^{(t-s)\Delta}U(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\ &\le e^{-t\beta}\|V(\cdot,0)\|_{L^\infty(\Omega)} + \alpha\Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\Big)e^{-(t-s)\beta}\|U(\cdot,s)\|_{L^\infty(\Omega)}\,ds\,.\end{aligned}$$

Also, using boundedness of U i.e. $\|U(\cdot,s)\|_{L^\infty(\Omega)} \le c_3\,(= c_1 + \overline{u_0})$ and the estimate $e^{-(t-s)\beta} \le e^{-\frac{t}{2}\beta}$ for $s \in [0, \frac{t}{2}]$, and for all $\varepsilon > 0$, $\|U(\cdot,s)\|_{L^\infty(\Omega)} < \varepsilon$ for $s \in [\frac{t}{2}, t]$ with sufficiently large t by L^∞-convergence of U, we see that

$$\|V(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty\,.$$

On the other hand, since $0 = \Delta W + \gamma U - \delta W$ and $(\nabla W \cdot \nu)|_{\partial\Omega} = 0$, in view of the maximum principle we see from L^∞-convergence of U that

$$\|W(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\gamma}{\delta}\|U(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty\,.$$

Therefore we arrive at (1.7). □

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References

[1] Chiyo, Y., *Stabilization for small mass in a quasilinear parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with density-dependent sensitivity: repulsion-dominant case*, Adv. Math. Sci. Appl. **31** (2) (2022), 327–341.

[2] Chiyo, Y., Marras, M., Tanaka, Y., Yokota, T., *Blow-up phenomena in a parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with superlinear logistic degradation*, Nonlinear Anal. **212** (2021), 14 pp., Paper No. 112550.

[3] Chiyo, Y., Yokota, T., *Stabilization for small mass in a quasilinear parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with density-dependent sensitivity: balanced case*, Matematiche (Catania), to appear.

[4] Chiyo, Y., Yokota, T., *Boundedness and finite-time blow-up in a quasilinear parabolic-elliptic elliptic attraction-repulsion chemotaxis system*, Z. Angew. Math. Phys. **73** (2) (2022), 27 pp., Paper No. 61.

[5] Fujie, K., Suzuki, T., *Global existence and boundedness in a fully parabolic 2D attraction-repulsion system: chemotaxis-dominant case*, Adv. Math. Sci. Appl. **28** (2019), 1–9.

[6] Ishida, S., Yokota, T., *Boundedness in a quasilinear fully parabolic Keller-Segel system via maximal Sobolev regularity*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 2112–232.

[7] Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, 1968.

[8] Lankeit, J., *Infinite time blow-up of many solutions to a general quasilinear parabolic-elliptic Keller-Segel system*, Discrete Contin. Dyn. Syst. Ser. S **13** (2) (2020), 233–255.

[9] Lankeit, J., *Finite-time blow-up in the three-dimensional fully parabolic attraction-dominated attraction-repulsion chemotaxis system*, J. Math. Anal. Appl. **504** (2) (2021), 16 pp., Paper No. 125409.

[10] Li, Y., Lin, K., Mu, C., *Asymptotic behavior for small mass in an attraction-repulsion chemotaxis system*, Electron. J. Differential Equations **2015** (146) (2015), 13 pp.

[11] Lin, K., Mu, C., Wang, L., *Large-time behavior of an attraction-repulsion chemotaxis system*, J. Math. Anal. Appl. **426** (1) (2015), 105–124.

[12] Luca, M., Chavez-Ross, A., Edelstein-Keshet, L., Mogliner, A., *Chemotactic signalling, microglia, and Alzheimer's disease senile plague: Is there a connection?*, Bull. Math. Biol. **65** (2003), 673–730.

[13] Tao, Y., Wang, Z.-A., *Competing effects of attraction vs. repulsion in chemotaxis*, Math. Models Methods Appl. Sci. **23** (2013), 1–36.

[14] Tao, Y., Winkler, M., *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations **252** (1) (2012), 692–715.

[15] Winkler, M., *Global classical solvability and generic infinite-time blow-up in quasilinear Keller-Segel systems with bounded sensitivities*, J. Differential Equations **266** (12) (2019), 8034–8066.

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CRITICAL POINTS FOR REACTION-DIFFUSION SYSTEM WITH ONE AND TWO UNILATERAL CONDITIONS

Jan Eisner and Jan Žilavý

Abstract. We show the location of so called *critical points*, i.e., couples of diffusion coefficients for which a non-trivial solution of a linear reaction-diffusion system of activator-inhibitor type on an interval with Neumann boundary conditions and with additional non-linear unilateral condition at one or two points on the boundary and/or in the interior exists. Simultaneously, we show the profile of such solutions.

1. Introduction

Let us consider a reaction-diffusion system

$$u_t = d_1 u_{xx} + f(u,v)\,, \quad v_t = d_2 v_{xx} + g(u,v) \qquad \text{in} \quad (0,\ell) \tag{1.1}$$

with Neumann boundary conditions for u

$$u_x(0) = u_x(\ell) = 0 \tag{1.2}$$

and at first with Neumann boundary conditions also for v

$$v_x(0) = v_x(\ell) = 0\,. \tag{1.3}$$

Let us assume there is (U_c, V_c) a stationary and spatially constant solution to (1.1) with (1.2), (1.3), in particular $f(U_c,V_c) = g(U_c,V_c) = 0$. We can assume without loss of generality that the *trivial solution* $(U_c,V_c) = (0,0)$ but keep in mind that in application where u and v represent e.g. concentrations of two chemicals or of two population species they are assumed to be positive.

We will allways assume the Jacobi matrix $B = (b_{ij})$ of (f,g) at (U_c,V_c) satisfies

$$\operatorname{Tr} B < 0 \quad \text{and} \quad \det B > 0\,. \tag{1.4}$$

Then it follows from Hurwitz criteria that the trivial solution (U_c,V_c) is stable as a solution to the corresponding ODE system without diffusion, i.e., for $d_1 = d_2 = 0$.

Finally, we will assume that

$$b_{11} > 0, \quad b_{12}b_{21} < 0, \quad b_{22} < 0\,. \tag{1.5}$$

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Then the RD-system (1.1) is of an activator-inhibitor or a depletion-substrate type if $b_{12} > 0$ or $b_{21} > 0$, respectively. It is well known [7] that under (1.5) an effect of *Turing instability* appears: Only for some diffusion coefficients $(d_1, d_2) \in \mathbb{R}^2_+$ the trivial solution (U_c, V_c) remains stable but becomes unstable for the rest of positive diffusion coefficients. More precisely, there is a system of hyperbolas

$$H_n = \{(d_1, d_2) \in \mathbb{R}_2^+ : d_2 = (\det B - \kappa_n d_1 b_{22})/(b_{21}\kappa_n - d_1\kappa_n^2)\}\,, \tag{1.6}$$

where $\kappa_n > 0$ is a sequence of *positive* eigenvalues to Neumann BVP

$$u_{xx} + \kappa u = 0 \quad \text{on } (0, \ell)$$

with (1.2). Let us remark that there is no H_0 for $\kappa_0 = 0$. Now, the *domain of stability* D_S of the trivial solution is the set of couples (d_1, d_2) lying to the right from all hyperbolas H_n and the *domain of instability* D_U is the set of (d_1, d_2) lying to the left from at least one hyperbola.

In the rest of this paper we will study only stationary solutions of (1.1) and consider only a linear ODE system

$$d_1 u'' + b_{11}u + b_{12}v = 0\,, \quad d_2 v'' + b_{21}u + b_{22}v = 0 \qquad \text{in } (0, \ell) \tag{1.7}$$

where the prime denotes the derivative w.r.t. to the variable $x \in (0, \ell)$. We will still refer to (1.2) and/or (1.3) where $u_x = u'$ and $v_x = v'$.

It is easy to see that for any $(d_1, d_2) \in \mathbb{R}^2_+$ the pair $(0, 0)$ is a solution to (1.7) with (1.2), (1.3). *Critical points* of a given boundary value problem will be the set of diffusion coefficients $(d_1, d_2) \in \mathbb{R}^2_+$ for which a *nontrivial (spatially nonconstant) solution* exists. It follows from [5] (cf. also Lemma 2.2 below) that the set of critical points of the Neumann BVP (1.7), (1.2), (1.3) is just the system of hyperbolas (1.6).

We will describe and locate the set of critical points if we prescribe, in addition to Neumann BCs, some unilateral condition(s) for the inhibitor v. More precisely, we will describe in the following sections the sets of critical points for the BVPs (1.7) with Neumann boudary conditions (1.2) for activator u and with several types of unilateral conditions for v. Let us remark that we choose the simplest examples in order to be at least partially analytically and numerically tractable. This is the reason to consider only one dimensional space domain and only point-wise unilateral obstacles. This method could be applied for the higher dimensional domain only of a very special form (e.g. a rectangle with unilateral conditions on (a part of) one edge) but we could obtain only a subset of possible critial points only because we can not analytically express all non-trivial solutions of a given unilateral BVP.

Let us finally remark that even the system (1.7) is linear, the unilateral conditions break the linearity, the BVP remains only *positively homogeneous*: only a *non-negative* multiple of a solution is also a solution.

2. A unilateral obstacle for inhibitor

We will start with one point-wise unilateral (one-sided) obstacle.

2.1. **A unilateral obstacle on the boundary.** The simplest unilateral obstacle is given by a Signorini condition prescribed at one boundary point, without loss of generality at $x = \ell$

$$v'(0) = 0, \qquad v(\ell) \geq 0, \quad v'(\ell) \geq 0\,, \quad v(\ell)v'(\ell) = 0\,. \tag{2.1}$$

The last three conditions allow $v(\ell)$ to be non-negative with a non-negative derivative $v'(\ell)$, but only one of them can be positive. If the value is positive, zero Neumann condition must be fulfilled. This BC can be considered as a certain regulation allowing the concentration to be above a prescribed value (here $V_c = 0$) and then the boundary is closed, there is no flux through this part of the boundary. But if $v(\ell)$ decreases below this value, the boudary opens and the inhibitor income from outside is large enough to stop the decrease of $v(\ell)$ below V_c (v satisfies Dirichlet BC in that case). In other words, the simple point of view is that v satifies Signorini BC at $x = \ell$ if and only if it satisfies either Neumann BC with a proper (non-negative) sign of $v(\ell)$ or Dirichlet BC with a proper (non-negative) sign of $v'(\ell)$. Of course, it can exceptionally happen that both $v(\ell) = v'(\ell) = 0$.

Looking for critical points of the BVP (1.7) with (1.2), (2.1), these considerations allow us to decompose this unilateral and hence non-linear Neumann-Signorini BVP onto two problems: on purely Neumann BVP (1.7), (1.2), (1.3) with a proper sign of $v(\ell)$ and on Neumann-Dirichlet BVP (1.7), (1.2),

$$v'(0) = 0, \qquad v(\ell) = 0 \tag{2.2}$$

with a proper sign of $v'(\ell)$.

Lemma 2.1. *Let (u, v) be a solution to one of linear BVPs* (1.7)*,* (1.2)*,* (1.3) *or* (1.7)*,* (1.2)*,* (2.2)*. Then (u, v) or $(-u, -v)$ is a solution of the unilateral BVP* (1.7)*,* (1.2)*,* (2.1)*.*

Proof. If (u, v) is a solution to a *linear* BVP then also $(-u, -v)$ is a solution. Now it is necessary to realize that in both BVPs we need to control a sign only of one object. □

Lemma 2.2. *The set of critical points K_N to the BVP* (1.7)*,* (1.2)*,* (1.3) *are just the hyperbolas H_n from* (1.6)*,*

$$K_N = \bigcup_{n=1}^{\infty} H_n\,.$$

The profiles of the corresponding non-trivial solutions for $(d_1, d_2) \in H_n$ are

$$\begin{aligned} u_n(x) &= A(d_2\kappa_n - b_{22})\cos(nx)/b_{21}\,, \\ v_n(x) &= A\cos(nx) \end{aligned} \tag{2.3}$$

with arbitrary $A \in \mathbb{R}$.

Proof. The assertion follows e.g. from [5]. □

Characteristic equation corresponding to the system (1.7) is biquadratic

$$d_1 d_2 r^4 + (d_2 b_{11} + d_1 b_{22})r^2 + \det B = 0$$

and has the (possibly complex) roots $\pm r_1$ and $\pm r_2$. We obtain for any $(d_1, d_2) \in \mathbb{R}^2_+$ with the exception of two half-lines

$$(b_{11}d_2 + b_{22}d_1)^2 - 4d_1d_2 \det B = 0$$

(where the upper one is a joint tangent to all hyperbolas H_n) a general solution

$$\begin{aligned} u(x) &= Ae^{r_1x} + Be^{-r_1x} + Ce^{r_2x} + De^{-r_2x}\,, \\ v(x) &= -(d_1u''(x) + b_{11}u(x))/b_{12} \end{aligned} \tag{2.4}$$

with arbitrary A, B, C, $D \in \mathbb{R}$.

Let us define on $(0, \ell)$ some auxiliary functions

$$\begin{aligned} C_1(x) &:= e^{r_1x} + e^{-r_1x}\,, \qquad & S_1(x) &:= e^{r_1x} - e^{-r_1x}\,, \\ C_2(x) &:= e^{r_2x} + e^{-r_2x}\,, \qquad & S_2(x) &:= e^{r_2x} - e^{-r_2x} \end{aligned}$$

and denote

$$R_1 := r_1^2 + \frac{b_{11}}{d_1}\,, \quad R_2 := r_2^2 + \frac{b_{11}}{d_1}\,.$$

Lemma 2.3. *The set of critical points K_D to the BVP* (1.7), (1.2), (2.2) *are the positive roots of the complex-valued function*

$$F_D(d_1, d_2) = d_1r_1R_2S_1(\ell)C_2(\ell) - d_1r_2R_1S_2(\ell)C_1(\ell)\,.$$

The profiles of the corresponding non-trivial solutions for $d = (d_1, d_2) \in K_D$ are

$$\begin{aligned} u(x) &= A(C_1(x) - C_2(x)\beta(d))\,, \\ v(x) &= -A(d_1(r_1^2C_1(x) - r_2^2C_2(x)\beta(d)) + b_{11}(C_1(x) - C_2(x)\beta(d)))/b_{12} \end{aligned} \tag{2.5}$$

with arbitrary $A \in \mathbb{R}$ and $\beta(d) = r_2S_2(\ell)/(r_1S_1(\ell))$.

Proof. The function F_D corresponds to the determinant of the linear system of 4 equations for coefficients A, B, C, D from (2.4) derived by using BCs (1.2), (2.2). Since these conditions are linear, a nontrivial quadruplet exists if and only if this determinant is zero. The form (2.5) then follows from (2.4). See e.g. [3] or [6] for details. □

Remark 2.4. Let us emphasize that the coefficients r_1, r_2 and therefore also the functions $C_i(x)$ and $S_i(x)$, $i = 1, 2$, and the numbers R_1, R_2 and β are in general complex and depend on diffusion parameters $(d_1, d_2) \in \mathbb{R}^2_+$. The form (2.4) and hence also (2.5) are written in a complex form, nevertheless one can rewrite them to obtain a couple (u, v) of non-trivial real solutions to the corresponding BVP.

Theorem 2.5. *The set of critical points K_S to the unilateral BVP* (1.7), (1.2), (2.1) *is given by*

$$K_S = K_N \cup K_D = \bigcup_{n=1}^{\infty} H_n \cup \{(d_1, d_2) \in \mathbb{R}^2_+ \,:\, F_D(d_1, d_2) = 0\}\,.$$

The profiles of the corresponding non-trivial solutions for (d_1, d_2) lying on some H_n or in K_D are given by (2.3) *or* (2.5) *with any $A \in \mathbb{R}$ having the proper sign, i.e. such that $v(\ell) \geq 0$ or $v'(\ell) \geq 0$, respectively.*

Proof. The assertion follows from Lemmas 2.1, 2.2 and 2.3 and considerations above. □

2.2. **A unilateral obstacle in the interior of the domain.** Let us consider our system (1.7) with (1.2), (1.3) and let us add for v a one-sided obstacle given by a unilateral condition at $x = x_1 \in (0,\ell)$ of the form

$$v(x_1) \geq 0, \quad v'(x_1-) \geq v'(x_1+)\,, \quad v(x_1)\,(v'(x_1-) - v'(x_1+)) = 0\,. \tag{2.6}$$

It is clear that if (u,v) is a non-trivial solution to (1.7), (1.2), (1.3) then (u,v) or $(-u,-v)$ (or exceptionally both) satisfies also (2.6). Such pairs are the C^2-smooth solutions to the unilateral problem (1.7), (1.2), (1.3), (2.6) and hence K_N is one part of the set of corresponding critical points.

The other type of solutions are those, for which the obstacle is 'active' and they are broken in the derivative of v (we write one-sided derivatives in (2.6)). The smoothness of activator u remains 'full', i.e. $u \in C[0,\ell] \cap C^2(0,\ell)$ but

$$v \in C[0,\ell] \cap C^2(0,x_1) \cap C^2(x_1,\ell)$$

and (1.7) separates to two systems, on $(0,x_1)$ and on (x_1,ℓ), and four conditions connecting the left (u_L,v_L) and right (u_R,v_R) solutions appear from (2.6)

$$u_L(x_1) = u_R(x_1)\,, \quad u_L'(x_1-) = u_R'(x_1+)\,, \quad v_L(x_1) = 0\,, \quad v_R(x_1) = 0\,, \tag{2.7}$$

together with the proper sign of the jump of derivatives

$$v'(x_1-) \geq v'(x_1+)\,. \tag{2.8}$$

Expressing general solution on $(0,x_1)$ and on (x_1,ℓ) and using BCs (1.2), (1.3) and conditions (2.7) we obtain a linear system for 8 coefficients A_L, B_L, C_L, D_L and A_R, B_R, C_R, D_R. Determinant of the matrix corresponding to this linear system is the desired function $F_{x_1}(d_1,d_2)$, positive roots of which are critical points corresponding to solutions satisfying $v(x_1) = 0$ (they touch the obstacle) and which can be (obstacle is not active) or are not (obstacle is active and breaks v) C^1-smooth on the whole domain $(0,\ell)$.

Lemma 2.6 ([3])**.** *The set of critical points K_{x_1} to the BVP* (1.7) *on* $(0,x_1)$ *and on* (x_1,ℓ) *with* (1.2), (1.3), (2.7) *are the roots of the complex-valued function*

$$F_{x_1}(d_1,d_2) = \tfrac{r_1}{r_2}\left(S_1(x_1) + S_1(\ell - x_1)\tfrac{C_1(x_1)}{C_1(\ell-x_1)}\right) - \tfrac{R_1}{R_2}C_1(x_1)\left(\tfrac{S_2(\ell-x_1)}{C_2(\ell-x_1)} + \tfrac{S_2(x_1)}{C_2(x_1)}\right)\,.$$

The profiles of the corresponding non-trivial solutions for $(d_1,d_2) \in K_{x_1}$ are

$$\begin{aligned} u_L(x) &= A_L(C_1(x) - \beta_1(d)C_2(x))\,, \\ v_L(x) &= -A_L\tfrac{d_1\left(r_1^2C_1(x) - \beta_1(d)r_2^2C_2(x)\right) + b_{11}(C_1(x) - \beta_1(d)C_2(x))}{b_{12}} \end{aligned} \tag{2.9}$$

on $(0,x_1)$ *and*

$$\begin{aligned} u_R(x) &= A_L\beta_3(d)(C_1(\ell-x) - \beta_2(d)C_2(\ell-x))\,, \\ v_R(x) &= -A_L\beta_3(d)\tfrac{d_1\left(r_1^2C_1(\ell-x) - \beta_2(d)r_2^2C_2(x)\right) + b_{11}(C_1(\ell-x) - \beta_2(d)C_2(\ell-x))}{b_{12}} \end{aligned} \tag{2.10}$$

on (x_1,ℓ) *with arbitrary* $A_L \in \mathbb{R}$ *and*

$$\beta_1(d) = \frac{R_1C_1(x_1)}{R_2C_2(x_1)}\,, \quad \beta_2(d) = \frac{R_1C_1(\ell-x_1)}{R_2C_2(\ell-x_1)}\,, \quad \beta_3(d) = \frac{C_1(x_1)}{C_1(\ell-x_1)}\,.$$

Proof. The expressions follow from [3, Section 5.5]. ☐

Theorem 2.7. *The set of critical points U_{x_1} to the unilateral BVP* (1.7), (1.2), (1.3) *and* (2.6) *is given by*

$$U_{x_1} = K_N \cup K_{x_1} = \bigcup_{n=1}^{\infty} H_n \cup \{(d_1, d_2) \in \mathbb{R}_+^2 \,:\, F_{x_1}(d_1, d_2) = 0\} .$$

The profiles of the corresponding non-trivial solutions for (d_1, d_2) lying on H_n or in K_{x_1} are given by (2.3) *or by* (2.9), (2.10), *with any A or A_L having the proper sign, i.e. such that $v(x_1) \geq 0$ or* (2.8) *holds, respectively.*

Proof. The assertion follows from the analogy of Lemma 2.1 together with Lemmas 2.2 and 2.6. □

3. Two unilateral obstacles for inhibitor

3.1. **Two obstacles from below.** Let us focus now on the example of two one-sided obstacles at $x = x_1$ and at $x = \ell$ (both *acting from below*) for v, i.e., we will consider the BVP (1.7), (1.2), (2.1) and (2.6). Two obstacles mean that there is no analogy of Lemma 2.1. We can still decompose the task: the critical points are such pairs (d_1, d_2) for which the corresponding solutions have no active contact with obstacles or for which only one or even both obstacles are active. In the last cas we have

Lemma 3.1. *The set of critical points $K_{x_1\ell}$ to the BVP* (1.7) *on $(0, x_1) \cup (x_1, \ell)$ with* (1.2), (2.2), (2.7) *are positive pairs (d_1, d_2) for which the algebraic linear system*

$$(3.1)\qquad \begin{aligned} & r_1(A_R - B_R) + r_2(C_R - D_R) = 0\,, \\ & R_1(A_R + B_R) + R_2(C_R + D_R) = 0\,, \\ & R_1(A_R e^{r_1 x_1} + B_R e^{-r_1 x_1}) + R_2(C_R e^{r_2 x_1} + D_R e^{-r_2 x_1}) = 0\,, \\ & A_R e^{r_1 x_1} + B_R e^{-r_1 x_1} + C_R e^{r_2 x_1} + D_R e^{-r_2 x_1} = A_L C_1(x_1) \left(1 - \tfrac{R_1}{R_2}\right) , \\ & \tfrac{r_1}{r_2}(A_R e^{r_1 x_1} - B_R e^{-r_1 x_1}) + C_R e^{r_2 x_1} - D_R e^{-r_2 x_1} = A_L C_1(x_1) \left(\tfrac{r_1}{r_2} - \tfrac{R_1}{R_2}\right) , \end{aligned}$$

has a non-trivial solution $(A_L, A_R, B_R, C_R, D_R)$. Then the nontrivial left and right solutions (u_L, v_L) and (u_R, v_R) of our BVP are given by (2.9) *and* (2.4) *with this A_L and (A_R, B_R, C_R, D_R), respectively.*

Proof. We obtain (3.1) by using boundary and inner conditions (1.2), (2.2), (2.7) for general solution (2.4) considered on $(0, x_1)$ and on (x_1, ℓ). □

Theorem 3.2. *The set of critical points $U_{x_1\ell}$ to the unilateral BVP* (1.7), (1.2), (2.1) *and* (2.6) *is given by*

$$(3.2)\qquad U_{x_1\ell} \subset (K_N \cup K_{x_1} \cup K_D \cup K_{x_1\ell})$$

such that the profiles of the corresponding non-trivial solutions satisfy both (2.1) *and* (2.6).

Remark 3.3. Nodal properties of the v-part of corresponding non-trivial solutions are preserved along the individual branches of critical points only to purely Neumann BVP (i.e. only along hyperbolas H_n). This is not true in general for the unilateral

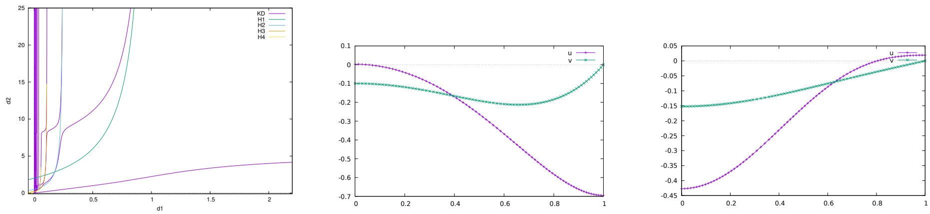

Fig. 1: Critical points K_S (H_n for $n = 1, 2, 3, 4$, and violet K_D) for BVP (1.7), (1.2), (2.1). Profile of solution (u, v) for $(d_1, d_2) = (0.913, 2) \in K_D$ on the horizontal branch and for $(d_1, d_2) = (0.407, 10) \in K_D$ on the second violet branch.

BVPs. Therefore, it strongly depends on the location of $x_1 \in (0, \ell)$ which parts of these branches are simultaneously critical points also for the 2-obstacles BVP. Hence we can not characterize explicitely $U_{x_1\ell}$ by the equality in (3.2).

Remark 3.4. Numerically it seems that the nodal properties of v are preserved along a large part of the right-most branches (going to the right which seems to be bounded in d_2) of K_D, K_{x_1} as well as of $K_{x_1\ell}$. This boundedness perfectly fits with the theoretical results for BVPs with unilateral conditions prescribed on the boundary, see [1, 2, 4]. As far as (more precisely, as close as to the origin) we can go with $d = (d_1, d_2)$ along the right-most branches while the profile of v satisfies simultaneously sharp inequality in (2.8) and $v(\ell) \geq 0$, such d belongs also to $U_{x_1\ell}$.

3.2. **Two obstacles from opposite sides.** Let us consider the similar BVP but with obstacles acting from the opposite sides and without loss of generality take

$$v(x_1) \leq 0, \quad v'(x_1-) \leq v'(x_1+), \quad v(x_1)(v'(x_1-) - v'(x_1+)) = 0 \tag{3.3}$$

instead of (2.6). We obtain an analogue of Theorem 3.2 with different subset $U^-_{x_1\ell}$ of $K_N \cup K_{x_1} \cup K_D \cup K_{x_1\ell}$. Irrespectively to Remark 3.4, if $d \in K_D$ or $d \in K_{x_1}$ lies on the righ-most branch and close enough to the origin, v with proper sign of A or A_L, resp., satisfies both (3.3) and $v(\ell) \geq 0$, hence such $d \in U^-_{x_1\ell}$.

Remark 3.5. The second right-most branch of $K_{x_1\ell}$ lies completely to the right from all H_n, i.e., in the domain of stability D_S of the trivial solution.

Let $d \in K_{x_1\ell}$ be from the second right-most branch. Let the corresponding v satisfy (2.8). Then numerically we observe that this inequality is sharp. Moreover, $v'(\ell) \geq 0$ (hence $d \in U_{x_1\ell}$) or $v'(\ell) \leq 0$ (hence $(-u, -v)$ satisfies (2.1) and (3.3), so $d \in U^-_{x_1\ell}$) for d being sufficiently close to or far from, respectively, the origin.

4. Examples and numerical results for given obstacles

Let us consider unilateral BVP (1.7), (1.2), (2.1) with a matrix $B = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}$. The set of critical points K_S from Lemma 2.3 and Theorem 2.5 is visible on Fig. 1. One can observe just one branch going to the right and being bounded in d_2. This branch are the only critical points from K_D and hence from K_S lying in D_S, i.e. to the right from all hyperbolas H_n. The other branches belong to D_U.

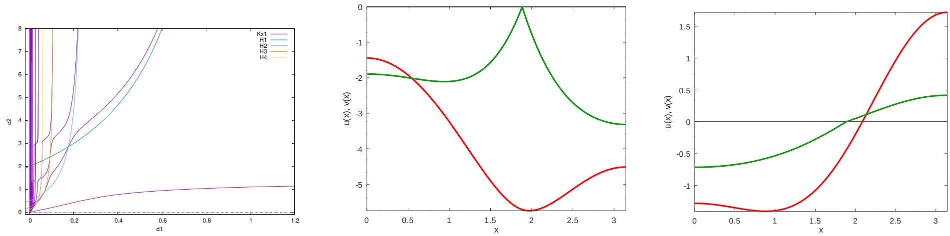

Fig. 2: Critical points K_{x_1} (in violet) for BVP (1.7), (1.2), (1.3), (2.6) with $x_1 = 0.6\pi$. Profile of solution u in red and v in green for $(d_1, d_2) = (0.6, 0.944) \in K_{x_1}$ on the horizontal branch and for $(d_1, d_2) = (0.218, 3.52) \in K_{x_1}$ on the second violet branch.

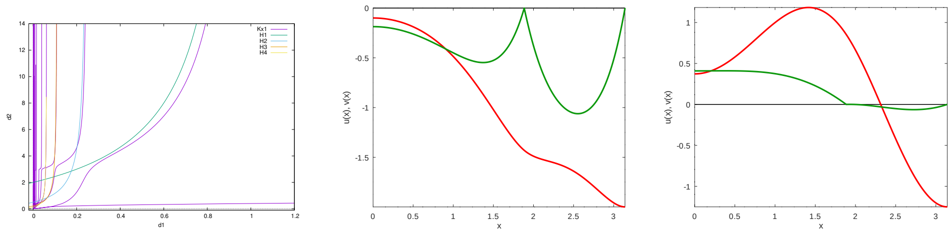

Fig. 3: Critical points $K_{x_1\ell}$ for BVP (1.7), (1.2), (2.1), (2.6) with $x_1 = 0.6\pi$. Profile of solution u in red and v in green for $(d_1, d_2) = (0.4, 0.286) \in K_{x_1\ell}$ on the horizontal branch and for $(d_1, d_2) = (0.251, 2.7) \in K_{x_1\ell}$ on the second violet branch.

References

[1] Eisner, J., Kučera, M., Väth, M., *Global bifurcation of a reaction-diffusion system with inclusions*, J. Anal. Appl. **28** (4) (2009), 373–409.

[2] Eisner, J., Väth, M., *Degree, instability and bifurcation of reaction-diffusion systems with obstacles near certain hyperbolas*, Nonlinear Anal. **135** (2016), 158–193.

[3] Kouba, P., *Existence of nontrivial solutions for reaction-diffusion systems of activator-inhibitor type with dependence on parameter*, Master's thesis, Č. Budějovice, Faculty of Science, University of South Bohemia, 2015, (in Czech).

[4] Kučera, M., Väth, M., *Bifurcation for reaction-diffusion systems with unilateral and Neumann boundary conditions*, J. Differential Equations **252** (2012), 2951–2982.

[5] Mimura, M., Nishiura, Y., Yamaguti, M., *Some diffusive prey and predator systems and their bifurcation problems*, Ann. N.Y. Acad. Sci. **316** (1979), 490–510.

[6] Pšenicová, M., *Newton boundary value problem for reaction-diffusion system of activator-inhibitor type with parameter*, Bachelor thesis, Č. Budějovice (2018), Faculty of Science, University of South Bohemia, 2018, (in Czech).

[7] Turing, A.M., *The chemical basis of morphogenesis*, Philos. Trans. Roy. Soc. London Ser. B **237** (641) (1952), 37–72.

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STABILIZATION IN DEGENERATE PARABOLIC EQUATIONS IN DIVERGENCE FORM AND APPLICATION TO CHEMOTAXIS SYSTEMS

Sachiko Ishida and Tomomi Yokota

Abstract. This paper presents a stabilization result for weak solutions of degenerate parabolic equations in divergence form. More precisely, the result asserts that the global-in-time weak solution converges to the average of the initial data in some topology as time goes to infinity. It is also shown that the result can be applied to a degenerate parabolic-elliptic Keller-Segel system.

1. Introduction: stabilization result

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary $\partial\Omega$. Then we consider the initial-boundary value problem for the degenerate parabolic equation,

$$(1.1)\qquad \begin{cases} u_t = \nabla \cdot \big(f(u)\nabla u + g(u,x,t)\big)\,, & x \in \Omega,\ t > 0\,,\\ \big(f(u)\nabla u + g(u,x,t)\big)\cdot \nu = 0\,, & x \in \partial\Omega,\ t > 0\,,\\ u(x,0) = u_0(x)\,, & x \in \Omega\,, \end{cases}$$

where f is supposed to be a non-negative function satisfying

$$(1.2)\quad f \in C([0,\infty)) \cap C^2((0,\infty))\,,$$

$$(1.3)\quad f(\sigma) \ge k_0\sigma^{m-1} \text{ with some } k_0 > 0,\ m \ge 1\ (\forall \sigma \ge 0)\,,\ \limsup_{\sigma \searrow 0} \sigma f'(\sigma) < \infty\,,$$

and moreover, g is assumed to be a vector-valued function approximated by $g_\varepsilon \in C([0,\infty) \times \overline{\Omega} \times [0,\infty);\mathbb{R}^N) \cap C^{1,1,0}([0,\infty) \times \Omega \times (0,\infty);\mathbb{R}^N)$ with some

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$\{\varepsilon\} \subset (0,1)$ fulfilling $\varepsilon \to 0$ such that for all $T > 0$,

$$\begin{aligned}
&0 \le w_\varepsilon,\ w \in L^\infty(0,T;L^\infty(\Omega))\,, \\
&w_\varepsilon \to w \text{ a.e. on } \Omega \times (0,T) \text{ and weakly* in } L^\infty(0,T;L^\infty(\Omega)) \\
&\quad \Rightarrow\ g_\varepsilon(w_\varepsilon,\cdot,\cdot) \to g(w,\cdot,\cdot) \text{ weakly in } L^2(0,T;(L^2(\Omega))^N)\,,
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
&0 \le w \in L^\infty(0,\infty;L^\infty(\Omega)) \text{ with } \|w\|_{L^\infty(0,\infty;L^\infty(\Omega))} \le c \\
&\quad \Rightarrow\ \|g_\varepsilon(w,\cdot,\cdot)\|_{L^2(0,\infty;(L^2(\Omega))^N)} \le M(c)\,,
\end{aligned} \tag{1.5}$$

where $M(c) \ge 0$ is a constant depending on c.

We first state the definition of weak solutions to (1.1) as follows:

Definition 1.1. A non-negative function $u(x,t)$ defined in $\Omega \times (0,\infty)$ is called *a global weak solution* of (1.1) if the following conditions are satisfied for all $T > 0$:

- $u \in L^\infty(0,T;L^\infty(\Omega))$,
- $\int_0^u f(\sigma)\,d\sigma \in L^2(0,T;H^1(\Omega))$, $g(u,x,t) \in L^2(0,T;(L^2(\Omega))^N)$,
- u fulfills (1.1) in the distributional sense: for every $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ (a.a. $x \in \Omega$),

$$\begin{aligned}
&\int_0^T \int_\Omega \left(\nabla\left(\int_0^u f(\sigma)\,d\sigma\right) \cdot \nabla\varphi - g(u,x,t)\cdot\nabla\varphi - u\varphi_t \right) dx\,dt \\
&\quad = \int_\Omega u_0(x)\varphi(x,0)\,dx\,.
\end{aligned}$$

We next give the following approximate problem:

$$\begin{cases}
(u_\varepsilon)_t = \nabla\cdot\left(f(u_\varepsilon+\varepsilon)\nabla u_\varepsilon + g_\varepsilon(u_\varepsilon,x,t)\right), & x \in \Omega,\ t > 0\,, \\
\big(f(u_\varepsilon+\varepsilon)\nabla u_\varepsilon + g_\varepsilon(u_\varepsilon,x,t)\big)\cdot\nu = 0\,, & x \in \partial\Omega,\ t > 0\,, \\
u_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, & x \in \Omega\,,
\end{cases} \tag{1.6}$$

where $g_\varepsilon \in C([0,\infty)\times\overline{\Omega}\times[0,\infty);\mathbb{R}^N) \cap C^{1,1,0}([0,\infty)\times\Omega\times(0,\infty);\mathbb{R}^N)$ with some $\{\varepsilon\} \subset (0,1)$ fulfilling $\varepsilon \to 0$ is an approximation of g, which also appears in (1.4), (1.5). The initial data $u_{0\varepsilon}$ is the regularization of u_0 such that $u_{0\varepsilon} \in C_0^\infty(\Omega)$ and $u_{0\varepsilon} \to u_0$ in $L^p(\Omega)$ as $\varepsilon \to 0$ for any $p \in [1,\infty)$. For example, we define it as $u_{0\varepsilon} := [\zeta_\varepsilon(\rho_\varepsilon * \widetilde{u_0})]|_\Omega$, where $\widetilde{u_0}$ denotes the zero extension of u_0 on $\mathbb{R}^N$. The function ρ_ε is the mollifier such that $0 \le \rho_\varepsilon \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\rho_\varepsilon \subset \overline{B(0,\varepsilon)}$, $\int_{\mathbb{R}^N}\rho_\varepsilon(x)\,dx = 1$, and ζ_ε is the cut-off function defined as $\zeta_\varepsilon(x) := \zeta(\varepsilon x)$, where ζ is a fixed function in $C_0^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$, $\zeta(x) = 1$ ($|x| \le 1$), $\zeta(x) = 0$ ($|x| \ge 2$). We assume that (1.6) possesses global classical solutions $u_\varepsilon \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty))$.

We now present a stabilization result established in [7].

Theorem 1.2. *Let f, g satisfy* (1.2), (1.3), (1.4), (1.5) *and* $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$. *Let u_ε be a global classical solution of* (1.6). *Suppose that there exists a constant $u_{\max} > 0$, which is independent of ε and t, such that*

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \textit{for all } t > 0\,.$$

Then there exists a global weak solution to (1.1)*, which is given by*

$$u_\varepsilon \to u \text{ weakly* in } L^\infty(0,\infty;L^\infty(\Omega)) \text{ as } \varepsilon \to 0$$

for some subsequence of $\{\varepsilon\}$*, satisfying*

$$\begin{aligned}
&u \in C_{w-*}([0,\infty);L^\infty(\Omega))\,,\\
&\|u(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \text{for all } t \ge 0\,,\\
&u(t) \to \overline{u_0} \quad \text{weakly* in } L^\infty(\Omega) \quad \text{as } t \to \infty\,,
\end{aligned}$$

where $\overline{u_0} := \fint_\Omega u_0(x)\,dx$*.*

The above theorem is applicable to some degenerate parabolic equations with drift terms in divergence form, whereas a similar result on stabilization in the case of non-divergence form with reaction terms has already been developed by [9]. In [7] we applied Theorem 1.2 to a parabolic–*parabolic* Keller-Segel system with degenerate diffusion. In this paper we give another application.

2. Application to chemotaxis systems

Consider the following degenerate parabolic–*elliptic* Keller-Segel system:

$$\tag{2.1}
\begin{cases}
u_t = \nabla\cdot(D(u)\nabla u - u\nabla v)\,, & x\in\Omega,\ t>0\,,\\
0 = \Delta v - v + u\,, & x\in\Omega,\ t>0\,,\\
\big(D(u)\nabla u + S(u)\nabla v\big)\cdot\nu = \nabla v\cdot\nu = 0\,, & x\in\partial\Omega,\ t>0\,,\\
u(x,0) = u_0(x)\,, & x\in\Omega\,,
\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\in\mathbb{N})$ is a bounded domain with smooth boundary $\partial\Omega$. Assume that the diffusivity function D fulfills the following conditions:

$$\tag{2.2} D\in C([0,\infty))\cap C^2((0,\infty)),$$

$$\tag{2.3} D(\sigma)\ge k_0\sigma^{m-1}\ (\sigma\ge 0) \text{ with some } k_0>0,\ m\ge 1,\ \limsup_{\sigma\searrow 0}\sigma D'(\sigma)<\infty$$

and that the initial data (u_0,v_0) satisfies

$$\tag{2.4} u_0\ge 0\,,\quad u_0\in L^\infty(\Omega)\,.$$

We define weak solutions of (2.1).

Definition 2.1. A couple (u,v) of non-negative functions satisfying the following is called *a global weak solution* of (2.1):

- $u\in L^\infty(0,T;L^\infty(\Omega))$, $\int_0^u D(\sigma)\,d\sigma\in L^2(0,T;H^1(\Omega))$ for all $T>0$,
- $v\in L^\infty(0,T;W^{1,\infty}(\Omega))$ for all $T>0$,

- (u,v) fulfills (2.1) in the distributional sense: for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ (a.a. $x \in \Omega$),

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(\nabla\Big(\int_0^u D(\sigma)\,d\sigma \Big) \cdot \nabla\varphi - u\nabla v \cdot \nabla\varphi - u\varphi_t \Big)\,dx\,dt \\
&\qquad = \int_\Omega u_0(x)\varphi(x,0)\,dx\,, \\
&\int_0^T \int_\Omega (\nabla v \cdot \nabla\varphi + v\varphi - u\varphi)\,dx\,dt = 0\,.
\end{aligned}$$

In this section we deal with the sub-critical case that $2 - \frac{2}{N} < m$, where $m = 2 - \frac{2}{N}$ is the critical exponent whether (2.1) possesses a global bounded solution or not. In view of the results in [10] which dealt with a general quasilinear chemotaxis term with $N \geq 3$, solutions are global and bounded if $2 - \frac{2}{N} < m$, whereas there are many initial data producing unbounded solutions if $m < 2 - \frac{2}{N}$. A similar situation is found in the parabolic-parabolic system: for boundedness in the case $2 - \frac{2}{N} < m$, see [5, 12, 14] on bounded domains, [6, 13] on the whole space; for blow-up in the case $m \leq 2 - \frac{2}{N}$, see [2, 4, 11] and [16].

We would like to turn to the asymptotic behavior of global solutions. To the best of our knowledge, there are few papers on this topic, e.g., the sub-critical parabolic–*parabolic* case is studied in [1, 3, 8] and [15]. For instance, the solution (u,v) of non-degenerate systems converges to $(\overline{u_0}, \overline{u_0})$ in $(L^\infty(\Omega))^2$, where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx$, under some smallness condition for initial data ([1, 3, 15]), whereas, when $m \geq 2$, an energy solution (u,v) tends to a non-negative stationary solution (U,V) which is potentially non-constant or constant equilibria ([8]). From these results, solvability has already been achieved for $2 - \frac{2}{N} < m$ and stabilization has not been achieved in the case that $2 - \frac{2}{N} < m < 2$. In [7] we could establish stabilization in the fully parabolic version of (2.1) by applying Theorem 1.2. However, there seems to be still room for consideration in the parabolic–*elliptic* Keller–Segel system (2.1). In this section, we will extend the range of the application of Theorem 1.2.

In stating the main theorem, we use the constant in the Poincaré inequality through the embedding $W^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ for any $\alpha \geq \frac{2N}{N+2}$:

$$\|\psi - \overline{\psi}\|_{L^2(\Omega)}^2 \leq k_{\mathcal{P}} \|\nabla\psi\|_{L^\alpha(\Omega)}^2 \ (\forall \psi \in W^{1,\alpha}(\Omega)), \tag{2.5}$$

where $\overline{\psi} := \frac{1}{|\Omega|}\int_\Omega \psi$ and $k_{\mathcal{P}} = k_{\mathcal{P}}(\alpha, N, \Omega)$ is a positive constant.

Theorem 2.2. *Let D satisfy the conditions* (2.2), (2.3) *with*

$$2 - \frac{2}{N} < m \leq 2\,.$$

Let (u_0, v_0) satisfy (2.4) *and assume that*

$$\|u_0\|_{L^1(\Omega)}^{2-m} < \frac{k_0}{k_{\mathcal{P}}}\,, \tag{2.6}$$

where $k_{\mathcal{P}}$ is the same one as in (2.5) *with* $\alpha = \frac{2}{3-m}$. *Then, there exists a global weak solution* (u, v) *of* (2.1) *which satisfies*

$$
\begin{aligned}
&u \in C_{w-*}([0,\infty); L^\infty(\Omega)),\\
&\|u(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \text{for all } t \ge 0,\\
&\|v(t)\|_{W^{1,\infty}(\Omega)} \le v_{\max} \quad \text{for all } t \ge 0,\\
&u(t) \to \overline{u_0} \quad \text{weakly* in } L^\infty(\Omega) \text{ as } t \to \infty,\\
&v(t) \to \overline{u_0} \quad \text{strongly in } L^\infty(\Omega) \text{ as } t \to \infty\,,
\end{aligned}
\tag{2.7}
$$

where $u_{\max}, v_{\max} \ge 0$ *are constants that appear in Lemma 2.3 and* $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$.

As in Theorem 1.2, we consider the approximate problem

$$
\begin{cases}
(u_\varepsilon)_t = \nabla\cdot(D(u_\varepsilon+\varepsilon)\nabla u_\varepsilon) - \nabla\cdot(u_\varepsilon\nabla v_\varepsilon)\,, & x\in\Omega,\ t>0\,,\\
0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon\,, & x\in\Omega,\ t>0\,,\\
\frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0 & x\in\partial\Omega,\ t>0\,,\\
u_\varepsilon(x,0) = [\zeta_\varepsilon(\rho_\varepsilon * \widetilde{u_0})]|_\Omega\,, & x\in\Omega\,,
\end{cases}
\tag{2.8}
$$

where $\widetilde{u_0}$ denotes the zero extension of u_0 on $\mathbb{R}^N$, ρ_ε is the mollifier and ζ_ε is the cut-off function.

We first give existence of global bounded solutions to the approximate problem (2.8), which can be proved by the same way as in [6] for the fully parabolic case; note that in the parabolic-elliptic case it suffices to replace Δv with $v-u$ instead of the use of the maximal Sobolev regularity in [6, (28)].

Lemma 2.3. *Assume that D satisfy the conditions* (2.2), (2.3) *with* $2 - \frac{2}{N} < m$. *Then for any initial data satisfying* (2.4), *there exists a pair* $(u_\varepsilon, v_\varepsilon)$ *of non-negative functions*

$$
u_\varepsilon, v_\varepsilon \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty))\,,
\tag{2.9}
$$

which solves (2.8) *classically, and* $(u_\varepsilon, v_\varepsilon)$ *fulfills*

$$
\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le u_{\max},\ \ \|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \le v_{\max} \quad \text{for all } t\in(0,T)\,,
$$

where $u_{\max}, v_{\max}$ *are positive constants which are independent of* t, ε. *Moreover, there exist a subsequence* $\{\varepsilon_n\}_n \subset \{\varepsilon\}$ *and non-negative functions*

$$
u \in L^\infty(0,\infty; L^\infty(\Omega)),\ \ v \in L^\infty(0,\infty; W^{1,\infty}(\Omega))
$$

such that

$$
\begin{aligned}
&u_{\varepsilon_n} \to u \quad \text{weakly* in } L^\infty(0,\infty; L^\infty(\Omega)),\\
&u_{\varepsilon_n} \to u \quad \text{a.e. on } \Omega\times(0,\infty),\\
&v_{\varepsilon_n} \to v \quad \text{weakly* in } L^\infty(0,\infty; W^{1,\infty}(\Omega))
\end{aligned}
\tag{2.10}
$$

as $n \to \infty$.

In order to apply Theorem 1.2, we will verify the conditions (1.2)–(1.5) with

$$f(\sigma) = D(\sigma)\,, \quad g(w,x,t) = w\nabla v\,, \quad g_\varepsilon(w,x,t) = w\nabla v_\varepsilon\,,$$

where $\varepsilon := \varepsilon_n$ for large n. In the following proof, c_i $(i = 1, 2, \cdots)$ denote positive constants independent of t and ε.

Proof of Theorem 2.2. We first observe that (1.2) and (1.3) are satisfied by (2.2) and (2.3). In view of (2.9) we can define $g_\varepsilon \in C([0,\infty)\times\overline{\Omega}\times[0,\infty);\mathbb{R}^N)\cap C^{1,1,0}([0,\infty)\times\Omega\times(0,\infty);\mathbb{R}^N)$ as

$$g_\varepsilon(w,x,t) := w\nabla v_\varepsilon\,,$$

where $\{\varepsilon\}\subset(0,1)$ fulfilling $\varepsilon\to 0$ is defined as $\varepsilon := \varepsilon_n$ appearing in Lemma 2.3 for large n. From now on we omit the proviso that $\varepsilon \to 0$.

Next, we will confirm (1.4). Let w_ε, w be non-negative functions which belong to $L^\infty(0,T;L^\infty(\Omega))$ for all $T>0$ and satisfy

$$w_\varepsilon \to w \text{ a.e. on } \Omega\times(0,T) \text{ and weakly* in } L^\infty(0,T;L^\infty(\Omega))\,.$$

Since there exists c_1 such that $\|w_\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega))}\le c_1$, we see from the Lebesgue dominated convergence theorem that $w_\varepsilon\to w$ strongly in $L^2(0,T;L^2(\Omega))$. Combining this convergence with (2.10) ensures that

$$g_\varepsilon(w_\varepsilon,\cdot,\cdot)\to w\nabla v = g(w,x,t) \quad \text{weakly in } L^2(0,T;(L^2(\Omega))^N)\,.$$

We next consider (1.5). Let $w\in L^\infty(0,\infty;L^\infty(\Omega))$ with $\|w\|_{L^\infty(0,\infty;L^\infty(\Omega))}\le c_2$. Then we have

$$\int_0^\infty\int_\Omega |g_\varepsilon(w)|^2\,dx\,dt \le c_2^2\int_0^\infty\int_\Omega |\nabla v_\varepsilon|^2\,dx\,dt\,. \tag{2.11}$$

Set

$$z_\varepsilon(t) := v_\varepsilon(t) - \overline{v_\varepsilon(t)} = v_\varepsilon(t) - \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t).$$

Then, due to $\overline{v_\varepsilon(t)} = \overline{u_{0\varepsilon}}$, which is obtained by integrating the second equation in (2.8) over Ω, z_ε satisfies

$$\begin{cases} 0 = \Delta z_\varepsilon - z_\varepsilon + (u_\varepsilon - \overline{u_{0\varepsilon}})\,, & x\in\Omega,\ t>0\,,\\ \nabla z_\varepsilon\cdot\nu = 0\,, & x\in\partial\Omega\,. \end{cases} \tag{2.12}$$

Testing the equation in (2.12) by $u_\varepsilon - \overline{u_{0\varepsilon}}$ and z_ε, we obtain

$$\begin{aligned} 0 &= -\int_\Omega \nabla u_\varepsilon\cdot\nabla v_\varepsilon\,dx - \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})z_\varepsilon\,dx + \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})^2\,dx,\\ 0 &= -\int_\Omega (|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2)\,dx + \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})z_\varepsilon\,dx\,. \end{aligned}$$

From the first equation in (2.8) we see that

$$\frac{d}{dt}\int_\Omega (u_\varepsilon\log u_\varepsilon - u_\varepsilon)\,dx = -\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx + \int_\Omega \nabla v_\varepsilon\cdot\nabla u_\varepsilon\,dx\,.$$

Adding the above three identities, we have

$$
\begin{aligned}
(2.13)\quad & \frac{d}{dt}\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx \\
&= -\int_\Omega (|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2)\,dx - \int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx + \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})^2\,dx\,.
\end{aligned}
$$

The condition (2.6) and $m \le 2$ help us to control the last term on the right-hand side of the above identity by the second term on the same side. The fact that $W^{1,\frac{2}{3-m}}(\Omega) \hookrightarrow L^2(\Omega)$ as $2-\frac{2}{N} < m$ and (2.5) provide the constant $k_{\mathcal{P}}$ such that

$$
\|u_\varepsilon(t) - \overline{u_{0\varepsilon}}\|^2_{L^2(\Omega)} \le k_{\mathcal{P}} \|\nabla u_\varepsilon(t)\|^2_{L^{\frac{2}{3-m}}(\Omega)}\,.
$$

From Hölder's inequality along with $\|u_\varepsilon(t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}$ $(\forall t \ge 0)$ we infer

$$
\|\nabla u_\varepsilon(t)\|^2_{L^{\frac{2}{3-m}}(\Omega)} \le \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{2-m}}\,dx\right)(\|u_0\|_{L^1(\Omega)} + \varepsilon|\Omega|)^{2-m}.
$$

Thanks to (2.3), it clearly holds that

$$
\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{2-m}}\,dx \le \frac{1}{k_0}\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx\,.
$$

Connecting the above three estimates, we obtain

$$
(2.14)\quad \|u_\varepsilon(t) - \overline{u_{0\varepsilon}}\|^2_{L^2(\Omega)} \le \frac{k_{\mathcal{P}}}{k_0}(\|u_0\|_{L^1(\Omega)} + \varepsilon|\Omega|)^{2-m}\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx\,.
$$

By virtue of (2.6), if we take ε_0 small enough to fit

$$
\frac{k_{\mathcal{P}}}{k_0}(\|u_0\|_{L^1(\Omega)} + \varepsilon|\Omega|)^{2-m} - 1 < 0 \ (\varepsilon \in (0,\varepsilon_0))\,,
$$

then (2.13) together with (2.14) warrants that for $\varepsilon \in (0,\varepsilon_0)$,

$$
\frac{d}{dt}\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx \le -\int_\Omega (|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2)\,dx - c_3\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx\,,
$$

where $c_3 = 1 - \frac{k_{\mathcal{P}}}{k_0}(\|u_0\|_{L^1(\Omega)} + \varepsilon_0|\Omega|)^{2-m} > 0$. Integrating this inequality with respect to the time variable provides c_4 such that for $\varepsilon \in (0,\varepsilon_0)$,

$$
\begin{aligned}
\int_0^\infty\int_\Omega |\nabla v_\varepsilon|^2\,dx\,dt &= \int_0^\infty\int_\Omega |\nabla z_\varepsilon|^2\,dx\,dt \\
&\le -\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx + \int_\Omega (u_{0\varepsilon}\log u_{0\varepsilon} - u_{0\varepsilon})\,dx \\
(2.15)\qquad\qquad &\le c_4
\end{aligned}
$$

in light of boundedness of $f(\xi) = |\xi\log\xi - \xi|$ for $\xi \in [0, u_{\max}]$. Plugging (2.15) into (2.11), we deduce that (1.5) holds. Thus, we can apply Theorem 1.2 to the parabolic-elliptic Keller-Segel system (2.1), so that there exists a global weak solution (u, v) fulfilling

$$\begin{aligned} &u \in C_{w-*}([0,\infty); L^\infty(\Omega))\,, \\ &\|u(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \text{for all } t \ge 0\,, \\ &u(t) \to \overline{u_0} \quad \text{weakly* in } L^\infty(\Omega) \text{ as } t \to \infty\,. \end{aligned}$$

Moreover, from the Sobolev embedding $W^{2,N+1}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and elliptic regularity as well as $\|u(t)\|_{L^\infty(\Omega)} \le u_{\max}$ $(t \ge 0)$ we have

$$\|v(t)\|_{W^{1,\infty}(\Omega)} \le c_5 \|v(t)\|_{W^{2,N+1}(\Omega)} \le c_6 \|u(t)\|_{L^{N+1}(\Omega)} \le c_6 |\Omega|^{\frac{1}{N+1}} u_{\max} = v_{\max}$$

with some $c_5, c_6 > 0$. We finally verify (2.7). Since $u(t) \to \overline{u_0}$ weakly in $L^{N+1}(\Omega)$ as $t \to \infty$, the compactness of $(I - \Delta)^{-1}$ from $L^{N+1}(\Omega)$ in $W^{1,N+1}(\Omega)$ implies that

$$v(t) - \overline{u_0} = (I - \Delta)^{-1}(u(t) - \overline{u_0}) \to 0 \quad \text{strongly in } W^{1,N+1}(\Omega) \quad \text{as } t \to \infty\,,$$

and also strongly in $L^\infty(\Omega)$ by the Sobolev embedding theorem, which implies (2.7). This completes the proof. □

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References

[1] Cao, X., *Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces*, Discrete Contin. Dyn. Syst. **35** (2015), 1891–1904.

[2] Cieślak, T., Stinner, C., *Finite-time blowup in a supercritical quasilinear parabolic-parabolic Keller-Segel system in dimension 2*, Acta Appl. Math. **129** (2014), 135–146.

[3] Cieślaka, T., Winkler, M., *Stabilization in a higher-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity*, Nonlinear Anal. **159** (2017), 129–144.

[4] Hashira, T., Ishida, S., Yokota, T., *Finite-time blow-up for quasilinear degenerate Keller-Segel systems of parabolic-parabolic type*, J. Differential Equations **264** (2018), 6459–6485.

[5] Ishida, S., Seki, K., Yokota, T., *Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains*, J. Differential Equations **256** (2014), 2993–3010.

[6] Ishida, S., Yokota, T., *Boundedness in a quasilinear fully parabolic Keller-Segel system via maximal Sobolev regularity*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 212–232.

[7] Ishida, S., Yokota, T., *Weak stabilization in degenerate parabolic equations in divergence form: application to degenerate Keller-Segel systems*, Calc. Var. Partial Differential Equations **61** (2022), Paper No. 105.

[8] Jiang, J., *Convergence to equilibria of global solutions to a degenerate quasilinear Keller-Segel system*, Z. Angew. Math. Phys. **69** (2018), Paper No. 130.

[9] Langlais, M., Phillips, D., *Stabilization of solutions of nonlinear and degenerate evolution equations*, Nonlinear Anal. **9** (1985), 321–333.

[10] Lankeit, J., *Infinite time blow-up of many solutions to a general quasilinear parabolic-elliptic Keller-Segel system*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 233–255.

[11] Laurençot, P., Mizoguchi, N., *Finite time blowup for the parabolic-parabolic Keller-Segel system with critical diffusion*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), 197–220.

[12] Senba, T., Suzuki, T., *A quasi-linear parabolic system of chemotaxis*, Abstr. Appl. Anal. **2006** (2006), 1–21.

[13] Sugiyama, Y., Kunii, H., *Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term*, J. Differential Equations **227** (2006), 333–364.

[14] Tao, Y., Winkler, M., *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations **252** (2012), 692–715.

[15] Winkler, M., *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Differential Equations **248** (2010), 2889–2905.

[16] Winkler, M., *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system*, J. Math. Pures Appl. **100** (2013), 748–767.

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UNIFORM ATTRACTORS IN SUP-NORM FOR SEMI LINEAR PARABOLIC PROBLEM AND APPLICATION TO THE ROBUST STABILITY THEORY

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Abstract. In this paper we establish the existence of the uniform attractor for a semi linear parabolic problem with bounded non autonomous disturbances in the phase space of continuous functions. We applied obtained results to prove the asymptotic gain property with respect to the global attractor of the undisturbed system.

1. Introduction

Stability property of stationary points plays an important role in robust control theory. The notion of input-to-state stability, firstly appeared in [23] now is widely used to nonlinear systems of different nature [24]. Other approaches in the control theory for nonlinear systems can be found in [2]–[11]. In recent years there have appeared many papers devoted to adaptation of input-to-state stability theory to infinite dimensional case [7]–[13]. One of the central object in the qualitative theory of dissipative infinite-dimensional systems is a global attractor [19], [22]. Stability properties of global attractors, including impulsive perturbations, can be found in [1]–[5], [9]. Recently in [6], [21] there have been obtained results about input-to-state stability and asymptotic gain properties with respect to global attractors of semi linear heat and wave equations in L^2 space. This results requires that the corresponding non autonomous problem generated semi process family with uniform attractor [3] which tends to the global attractor of undisturbed system. In the present paper we apply this scheme to the case of the phase space $\mathbb{C}_0$ of continuous functions supplied with sup-norm. Similar results for other type of perturbations were studied in [25], [26]. The work consists of two parts. In the first part we set the problem, provide necessary definitions and auxiliary results,

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and prove that under suitable assumptions mild solutions of the perturbed system generate a semi process family on $\mathbb{C}_0$ which has a uniform attractor. In the second part we use this result to establish the asymptotic gain properties with respect to the global attractor of the unperturbed system.

2. Setting of the problem and uniform attractors

We consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = Au + f(u) + h(t,x)\,, & (t,x) \in (0,\infty)\times\Omega\,, \\ u|_{\partial\Omega} = 0\,, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{2.1}$$

where $u(t,x)$ is an unknown function, $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary,

$$Au = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u\,.$$

Assume that

$$-A \text{ is a strongly elliptic self adjoint operator with bounded sufficiently smooth coefficients,} \tag{2.2}$$

$f\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $f(0) = 0$ and

$$\exists C > 0 \quad \text{such that } \forall\, |s| \ge C \quad s\cdot f(s) \le 0\,. \tag{2.3}$$

Assume that $h \in L^\infty(0,+\infty;X)$, where

$$X = C_0(\Omega) = \big\{v \in \mathbb{C}(\overline{\Omega}) | v = 0 \text{ on } \partial\Omega\big\}$$

supplied with the sup-norm $\|v\|_X = \sup\limits_{x\in\Omega}|v(x)|$. In the future we will use the spaces $H^1 = W^{1,2}(\Omega)$, $H_0^1 = \big\{v \in H^1, v|_{\partial\Omega} = 0\big\}$, $H^2 = W^{2,2}(\Omega)$, $L^2 = L^2(\Omega)$. We will study qualitative behaviour of mild solutions of (2.1) in the phase space X.

Definition 2.1. The function $u \in \mathbb{C}([0,T];X)$ is a mild solution of (2.1) with initial data $u_0 \in X$ if for all $t \in [0,T]$ we have

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))ds + \int_0^t T(t-s)h(s)\,ds\,, \tag{2.4}$$

where $F\colon X \to X$, $F(u)(x) = f(u(x))$, $T(t)$ is a C_0 semigroup of bounded operators, generated by A in X.

We prove that for all initial condition $u_0 \in X$ there exists a unique global mild solution of (2.1) with $u(0) = u_0$, which will be denoted by $u(t) = S_h(t,0,u_0)$.

Taking the set $\Sigma(h)$ of all time shifts of h we show that the semiprocess family $\{S_\sigma\}_{\sigma\in\Sigma(h)}$ (see definition below) has uniform attractor $\Theta_{\Sigma(h)}$ in the phase space

X and, moreover, for the global attractor Θ of the unperturbed system ($h \equiv 0$) we have:

$$\mathrm{dist}_X(\Theta_{\Sigma(h)}, \Theta) \to 0 \quad \text{as} \quad h \to 0\,, \tag{2.5}$$

where

$$\mathrm{dist}_X(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_X\,.$$

Limit equality (2.5) allow us to get the following result concerning robust stability: there exists a continuous strictly increasing function γ, vanishing at the origin, such that $\forall u_0 \in X$

$$\overline{\lim_{t\to\infty}} \|S_h(t, 0, u_0)\|_\Theta \le \gamma(\|h\|_\infty)\,, \tag{2.6}$$

where

$$\|u\|_\Theta := \inf_{\xi\in\Theta} \|\xi - u\|_X, \quad \|h\|_\infty = \sup_{t\ge 0} \|h(t)\|_X\,.$$

To prove (2.5), (2.6) we need some auxiliary results. First let us assume that $h \in L^2_{loc}(0, +\infty; X)$. Then, using Lipschitz continuity of f, we can use the classical result [17] (see Th. 1.4, Ch. 6) and claim that for every $u_0 \in X$ there exists $T = T(u_0, h) > 0$ such that there exists a unique mild solution of (2.1), $u \in \mathbb{C}([0, T]; X)$ with $u(0) = u_0$. Moreover, condition (2.3) allow us to use well-known comparison principle [12] and deduce the following estimate holds

$$\|u(t)\|_X \le Me^{-\lambda t}\|u_0\|_X + \frac{MC_1}{\lambda} + \int_0^t Me^{-\lambda(t-s)}\|h(s)\|_X\,ds\,, \tag{2.7}$$

where constant $C_1 > 0$ depends on f and positive constants M, λ are taken from the inequality

$$\|T(t)\| \le Me^{-\lambda t} \quad \forall\, t \ge 0\,. \tag{2.8}$$

This estimate shows that every mild solution is global, i.e., defined on $[0, +\infty)$.

In the sequel we will use the following facts. It is known that A is the infinitesimal generator of an analytic semigroup (still denoted by $T(t)$) in $L^p(\Omega)$, $p \ge 2$ [17]. Both in $L^p(\Omega)$, $p \ge 2$ and in X, we have the following estimates [3], [10]: there exist $c > 0$, $\alpha \in (0, 1)$, $\delta \in (\frac{1}{2}, 1)$ such that

$$\forall\, u_0 \in L^2(\Omega) \quad \|T(t)u_0\|_{H^2} \le \frac{c}{t}\|u_0\|_{L^2}\,, \tag{2.9}$$

$$]3pt]\forall\, u_0 \in X \qquad \|T(t)u_0\|_{C^{1+\alpha}} \le \frac{c}{t^\delta}\|u_0\|_X\,. \tag{2.10}$$

Let us consider linear nonhomogeneous problem

$$\begin{cases} \frac{du}{dt} = Au + g(t)\,, \\ u|_{t=0} = u_0 \in L^2(\Omega)\,, \end{cases} \tag{2.11}$$

where $g \in L^2(0, T; L^2(\Omega))$ is a given function.

We consider mild solution of (2.11), i.e. $u \in C([0,T];L^2(\Omega))$,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g(s)\,ds\,. \tag{2.12}$$

It is known [17] that mild solution of (2.11) is a weak solution of (2.11), i.e. $u \in L^2(0,T;H_0^1)$ such that $\forall\, v \in H_0^1$, $\forall\, \eta \in C_0^\infty(0,T)$

$$\int_0^T (u(t),v)\eta\,ds + \int_0^T (A^{\frac{1}{2}}u(t),A^{\frac{1}{2}}v)\eta\,ds = \int_0^T (g(t),v)\eta\,ds\,, \tag{2.13}$$

where $(\cdot,\cdot)$ is a scalar product in L^2, $\|u\| = \sqrt{(u,u)}$. Moreover, every weak solution u of (2.11) is a mild continuous of (2.11) in $[0,T]$. Additionally, if $u_0 \in H_0^1$ then $u \in C([0,T];H_0^1)\bigcap L^2(0,T;H^2)$, $u_t \in L^2(0,T;L^2)$. All this facts help us to prove the global existence result.

Now we are in position to construct the semi processes family, generated by the equation (2.1).

Let $h \in L^\infty(0,+\infty;X)$ and let $\Sigma(h) \subset L^2_{loc}(0,+\infty;X)$, $(\Sigma(0) = \{0\})$ be an arbitrary shift invariant (i.e. $\forall\, d \in \Sigma(h)$, $\forall\, s \geq 0$ $d(s+\cdot) \in \Sigma(h)$) topological space generated by h.

Let us consider the problem (2.1) where h is replaced by $d \in \Sigma(h)$

$$\begin{cases} \frac{\partial u}{\partial t} = Au + f(u) + d(t,x)\,, \quad (t,x) \in (0,\infty)\times\Omega \\ u|_{\partial\Omega} = 0\,, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{2.14}$$

From the previous arguments we deduce that every solution of (2.14) is global. We denote by

$$S_d(t,\tau,u_\tau)$$

the solution of (2.14) at the moment $t \geq \tau$ with initial data $(\tau,u_\tau) \in [0,\infty)\times X$. Then the family $\{S_d(t,\tau,u_\tau)\}_{d\in\Sigma(h)}$ generates a semiprocess family [19], i.e. $\forall\, t \geq \tau \geq 0\ \ \forall u_\tau \in X\ \ \forall\, d \in \Sigma(h)$

$$\begin{aligned} S_d(\tau,\tau,u_\tau) &= u_\tau\,, \\ S_d(t,s,S_d(s,\tau,u_\tau)) &= S_d(t,\tau,u_\tau)\ \ \forall\, t \geq s \geq \tau\,, \\ S_d(t+p,\tau+p,u_\tau) &= S_{d(\cdot+p)}(t,\tau,u_\tau)\ \ \forall\, p \geq 0\,. \end{aligned}$$

Every semiprocess family satisfies the cocycle property

$$S_d(t+p,0,u) = S_{d(\cdot+p)}\big(t,0,S_d(p,0,u)\big)\,.$$

In particular, for $d \equiv 0$

$$S_0(t+p,0,u) = S_0\big(t,0,S_0(p,0,u)\big)\,,$$

i.e. S_0 is a semigroup.

It is known [8] that under conditions (2.2), (2.3) the semigroup S_0 processes a global attractor $\Theta \subset X$, that is

1) Θ is a compact set;
2) $\Theta = S_0(t,0,\Theta) \ \ \forall\, t \geq 0$;
3) for every bounded set $B \subset X$

$$\sup_{u\in B} \operatorname{dist}_X(S_0(t,0,u),\Theta) \to 0 \quad \text{as} \quad t \to \infty\,.$$

In the sequel we denote for $\Sigma = \Sigma(h)$, $B \subset X$

$$S_\Sigma(t,\tau,B) = \bigcup_{d\in\Sigma} \bigcup_{u\in B} S_d(t,\tau,u)\,.$$

Definition 2.2. A compact set $\Theta_\Sigma \subset X$ is called a uniform attractor of the semiprocess family $\{S_d\}_{d\in\Sigma}$ if for every bounded set $B \subset X$ we have

$$dist_X(S_\Sigma(t,0,B),\Theta_\Sigma) \to 0 \quad \text{as} \quad t \to \infty\,, \tag{2.15}$$

and Θ_Σ is the minimum among all closed sets satisfying (2.15).

The following well known result provides conditions for existence of uniform attractor.

Lemma 2.3 ([3]). *Let $\{S_d\}_{d\in\Sigma}$ be a semiprocess family with a first countable space Σ, and*

1) *there exists a bounded set $B_0 \subset X$ such that for every bounded set $B \subset X$,*

$$\exists T = T(B) \quad \forall\, t \geq T \qquad S_\Sigma(t,0,B) \subset B_0\,;$$

2) $\forall\, d_n \subset \Sigma\ \forall t_n \to \infty\ \forall$ *bounded* $\{u_n\} \subset X$ *the sequence* $\{S_{d_n}(t_n,0,u_n)\}$ *is precompact in X.*

Then $\{S_d\}_{d\in\Sigma}$ has a uniform attractor Θ_Σ.

If, additionally, for all $t \geq 0$ the map

$$X \times \Sigma \ni (u,d) \to S_d(t,0,u) \in X \tag{2.16}$$

is continuous, then Θ_Σ is negatively invariant, i.e.

$$\forall\, t \geq 0 \quad \Theta_\Sigma \subset S_\Sigma(t,0,\Theta_\Sigma)\,. \tag{2.17}$$

Remark 2.4. From (2.17) we get inclusion: $\Theta_\Sigma \subset B_0$.

Assume that

$$h(t,x) = \sum_{j=1}^{K} h_j(t)\varphi_j(x)\,, \tag{2.18}$$

where $K \geq 1$, $h_j \in L^\infty(0,+\infty)$, $\varphi_j \in X$.

Let us put

$$W := cl_{\left(L^{2,w}_{loc}(0,+\infty)\right)^K} \left\{(h_1(\cdot+s),\ldots,h_K(\cdot+s))\; s \geq 0\right\}\,,$$

$$\Sigma = \Sigma(h) = \left\{ \sum_{j=1}^{K} d_j(t)\varphi_j(x) \mid \{d_1,\ldots,d_K\} \in W \right\}. \tag{2.19}$$

It is known [22] that the set

$$W_g := cl_{L^{2,w}_{loc}(0,+\infty)} \{g(\cdot + s) \mid s \geq 0\}$$

is compact in $L^{2,w}_{loc}(0,+\infty) \Leftrightarrow \|g\|_+ := \sup_{t\geq 0} \int\limits_t^{t+1} \|g(s)\|_X^2 ds < \infty$. Moreover, such a set is shift-invariant, and $\forall \xi \in W_g$

$$\|\xi\|_+ \leq \|g\|_+ .$$

Therefore, the set Σ defined by (2.19) is shift-invariant, and

$$\forall d \in \Sigma(h) \quad \|d\|_+ \leq \|h\|_\infty . \tag{2.20}$$

Theorem 2.5. *Assume that conditions* (2.2), (2.3), (2.18) *take place. Then the semiprocess family* $\{S_d\}_{d\in\Sigma}$ *generated by mild solutions of the problem* (2.1), *has a uniform attractor* A_Σ, *which satisfies* (2.17).

Proof. For every d with $\|d\|_+ < \infty$ inequality (2.7) implies

$$\|u(t)\|_X \leq Me^{-\lambda t}\|u_0\|_X + \frac{MC_1}{\lambda} + \|d\|_+^{\frac{1}{2}}(1-e^{-\lambda})^{-\frac{1}{2}} .$$

So, from (2.20) for every $d \in \Sigma(h)$ we get that for all bounded $B \subset X$ $\exists T = T(B)$ $\forall t \geq T$

$$S_\Sigma(t,0,B) \subset B_0 = \{u \in X \mid \|u\|_X \leq 1 + C\} , \tag{2.21}$$

for some positive constant C, which does not depend on B. Therefore, assumption 1) from Lemma 2.3 takes place. Moreover, for every bounded $B \subset X$ and every $u(\cdot)$ with $u(0) = u_0$ there exists $K = K(B)$ such that for all $d \in \Sigma$ and all $u_0 \in B$, $t \geq 0$

$$\|f(u(t))\|_\infty \leq K .$$

Then due to (2.10) for $t > 0$ and $\delta \in \left(\frac{1}{2}, 1\right)$

$$\|u(t)\|_{C^{1+\alpha}} \leq \frac{C}{t^\delta}\|u_0\|_X + \int\limits_0^t \frac{C}{s^\delta} K ds + \int\limits_0^t \frac{C}{s^\delta}\|h\|_\infty ds \leq r(t) . \tag{2.22}$$

Due to compact embedding $C^{1+\alpha} \Subset X$ and inclusions: for $\{d_n\} \subset \Sigma$, $t_n \to \infty$, $\|u_0^n\|_X \leq r$

$$\xi_n = S_{d_n}(t_n, 0, u_0^n) = S_{d_n}(t_n, t_n - 1, S_{d_n}(t_n - 1, 0, u_0^n)) =$$

$$= S_{d_n(\cdot + t_n - 1)}(1, 0, S_{d_n}(t_n - 1, 0, u_0^n)) \subset S_\Sigma(1, 0, B_0)$$

for sufficiently large $n \geq 1$, where B_0 is taken from (2.21). So, we conclude that $\{\xi_n\}$ is precompact in X, and, therefore, semiprocess family $\{S_d\}_{d\in\Sigma}$ possesses a uniform attractor Θ_Σ. □

Let us prove (2.17). For this aim we prove the following result.

Lemma 2.6. *Assume that for* $d^n = (d_1^n, \dots, d_k^n), \quad d = (d_1, \dots, d_k)$

$$d^n \to d \quad in \ \left(L^{2,w}_{\rm loc}(0, +\infty)\right)^K , \quad u_0^n \to u_0 \ in \ X . \tag{2.23}$$

Then for all $t \in [0, T]$ *we have*

$$u_n(t) = S_{d_n}(t, 0, u_0) \to u(t) = S_d(t, 0, u_0) \quad in \ X . \tag{2.24}$$

Proof. Due to (2.21) both $\{u_n\}$ and $\{f(u_n)\}$ are bounded in $\mathbb{C}([0, T]; X)$. Let us consider u_n as a weak solution of (2.11) with right side

$$g_n(t) = f(u_n) + \sum_{j=1}^{K} d_j^n(t)\varphi_j .$$

Then $\{g_n\}$ is bounded in $L^2(0, T; X)$, $\{u_n\}$ is bounded in $L^2(0, T; H_0^1)$, $\{u_{n_t}\}$ is bounded in $L^2(0, T; H^{-1})$. So, due to Aubin-Lions Lemma there exists a function $u \in \mathbb{C}([0, T]; L^2)$ such that up to subsequence:

$$u_n \to u \quad \text{weakly in} \quad L^2(0, T; H^{-1}) ,$$

$$u_n \to u \text{ in } L^2(0, T; L^2) \quad \text{and almost everywhere (a.e.) in} \quad (0, T) \times \Omega ,$$

$$\forall\, t \in [0, T] \ u_n(t) \to u(t) \quad \text{weakly in} \quad L^2 . \tag{2.25}$$

Then $f(u_n(t, x)) \to f(u(t, x))$ a.e. and, therefore,

$$g_n \to g = f(u) + \sum_{j=1}^{K} d_j(t)\varphi \text{ weakly in } L^2(0, T; L^2) .$$

So, u is a weak solution of (2.11) with the right hand side g. Thus, due to the previous arguments we have that u is a mild solution of (2.11) in L^2 and, therefore, a mild solution of (2.1) in L^2. Then u is a mild solution of (2.1) in X. Indeed, due to the (2.22) and (2.25) $\forall\, t \in [0, T] \quad u_n(t) \to u(t) \quad \text{in } X$. Then for all $t \in [0, T]$ $u(t, \cdot) \in X \ \Rightarrow \ f(u(t, \cdot)) \in X \ \Rightarrow g \in L^2(0, T; X) \ \Rightarrow u(t) \in S_d(t, 0, u_0)$. Lemma is proved.

Property (2.24) implies (2.16), and, therefore, (2.17). Theorem is proved. □

3. Application to the robust stability theory

In this section we want to obtain asymptotic gain property (2.6).

Theorem 3.1. *Under conditions* (2.2), (2.3), (2.18) *problem* (2.1) *for* $\|h\|_\infty \le R_0$ *possesses asymptotic gain property w.r.t. global attractor* Θ *of the undisturbed* ($h \equiv 0$) *system.*

Proof. Let us assume that we have the limit property

$$dist\left(\Theta_{\Sigma(h)}, \Theta\right) \to 0 \text{ as } \|h\|_\infty \to 0 . \tag{3.1}$$

Let us prove that (3.1) implies (2.6). Indeed, according to construction $\Sigma(0) = \{0\}$, and $h \in \Sigma(h)$. So, for $u_0 \in X$, $z \in \Theta_{\Sigma(h)}$, $t > 0$, $u(t) = S_h(t, 0, u_0)$ we have: for $\theta \in \Theta$:

$$\|u(t) - \theta\|_X \leq \|u(t) - z\|_X + \|z - \theta\|_X \Rightarrow$$

$$\inf_{\theta\in\Theta} \|u(t) - \theta\|_X \leq \|u(t) - z\|_X + \inf_{\theta\in\Theta} \|z - \theta\|_X \Rightarrow$$

$$\inf_{\theta\in\Theta} \|u(t) - \theta\|_X \leq \inf_{z\in\Theta_{\Sigma(h)}} \|u(t) - z\|_X + \sup_{z\in\Theta_{\Sigma(h)}} \inf_{\theta\in\Theta} \|z - \theta\|_X \Rightarrow$$

$$\|u(t)\|_\Theta \leq \mathrm{dist}_X \left(u(t), \Theta_{\Sigma(h)}\right) + \mathrm{dist}_X \left(\Theta_{\Sigma(h)}, \Theta\right) \Rightarrow$$

$$\|S_h(t, 0, u_0)\|_\Theta \leq \mathrm{dist}_X \left(S_{\Sigma(h)}(t, 0, u_0), \Theta_{\Sigma(h)}\right) + \mathrm{dist}_X \left(\Theta_{\Sigma(h)}, \Theta\right) .$$

The first summand in the right part of this inequality tends to zero for every h. Let us put

$$\gamma(s) := \sup_{\|h\|_\infty \leq s} dist_X(A_{\Sigma(h)}, A) + s\,.$$

Due to (3.1) $\gamma \in K$ and $\mathrm{dist}_X(\Theta_{\Sigma(h)}, \Theta) \leq \gamma(\|h\|_\infty)$, so we have the required result. Let us prove (3.1). Assume that (3.1) does not take place. It means that there exists $h_n \to 0$ in $L^\infty(0, +\infty; X)$, there exist $\varepsilon > 0$ and $z_n \in \Theta_{\Sigma(h_n)}$ such that

$$\mathrm{dist}(z_n, \Theta) \geq \varepsilon\,. \tag{3.2}$$

From Theorem 2.5 we have that $\Theta_{\Sigma(h)} \subset K$, where compact K depends on R_0 (see estimation (2.22)). Then

$$z_n \in \Theta_{\Sigma(h_n)} \subset S_{\Sigma(h_n)}(t, 0, \Theta_{\Sigma(h_n)}) \subset S_{\Sigma(h_n)}(t, 0, K)\,.$$

Therefore, $z_n = u_n(t) = S_{d_n}(t, 0, \xi_n)$, where $\xi_n \to \xi$ in X, $\|d_n\|_+ \leq \|h_n\|_\infty \to 0$. Then from Lemma 2.6

$$u_n(t) \to u(t) = S_0(t, 0, \xi) \subset S_0(t, 0, B_0)\,. \tag{3.3}$$

Due to the uniform attraction we can choose $t > 0$ such that

$$dist_X(S_0(t, 0, B_0), \Theta) < \frac{\varepsilon}{2}\,.$$

Then from (3.3)

$$z_n \to u(t) \in O_{\frac{\varepsilon}{2}}(\Theta),$$

that is a contradiction with (3.2). Theorem is proved. □

References

[1] Asrorov, F., Sobchuk, V., Kurylko, O., *Finding of bounded solutions to linear impulsive systems*, East-Europ. J. Enterprise Technol. **6** (4(102)) (2019), 14–20.

[2] Barabash, O., Dakhno, N., Shevchenko, H., Sobchuk, V., *Unmanned aerial vehicles flight trajectory optimisation on the basis of variational enequality algorithm and projection method*, Proceeding 2019 IEEE 5th International Conference "Actual Problems of Unmanned Aerial Vehicles Developments" (APUAVD), National Aviation University, Kyiv, Ukraine, 2019, pp. 136–139.

[3] Chepyzkov, V.V., Vishik, M.I., *Attractors for equations of mathematical physics*, vol. 49, AMS Colloquium Publications, 2002.

[4] Dashkovskiy, S., Feketa, P., Kapustyan, O., Romaniuk, I., *Invariance and stability of global attractors for multi-valued impulsive dynamical systems*, J. Math. Anal. Appl. **458** (1) (2018), 193–218.

[5] Dashkovskiy, S., Kapustyan, O., Romaniuk, I., *Global attractors of impulsive parabolic inclusions*, Discrete Contin. Dyn. Syst. Ser. B **22** (5) (2017), 1875–1886.

[6] Dashkovskiy, S., Kapustyan, O., Schmid, J., *A local input-to-state stability result w.r.t. attractors of nonlinear reaction-diffusion equations*, Math. Control Signals Systems **32** (3) (2020), 309–326.

[7] Dashkovskiy, S., Mironchenko, A., *Input-to-state stability of infinite-dimensional control systems*, Math. Control Signal Systems **25** (2013), 1–35.

[8] Haraux, A., Kirane, M., *Estimation C^1 pour des problemes paraboliques semi-lineaires*, Ann. Fac. Sci. Toulouse Math. **5** (1983), 265–280.

[9] Kapustyan, O.V., Kapustian, O.A., Gorban, N.V., Khomenko, O.V., *Strong global attractor for the three-dimensional Navier-Stokes system of equations in unbounded domain of channel type*, J. Automat. Inform. Sci. **47** (11) (2015), 48–59.

[10] Kapustyan, O.V., Kasyanov, P.O., Valero, J., *Structure of the global attractor for weak solutions of a reaction-diffusion equation*, Appl. Math. Inform. Sci. **9** (5) (2015), 2257–2264.

[11] Kichmarenko, O., Stanzhytskyi, O., *Sufficient conditions for the existence of optimal controls for some classes of functional-differential equations*, Nonlinear Dyn. Syst. Theory **18** (2) (2018), 196–211.

[12] Manthey, R., Zausinger, T., *Stochastic equations in L^2_ρ*, Stochastic Rep. **66** (1977), 370–373.

[13] Mironchenko, A., Prieur, Ch., *Input-to-state stability of infinite-dimensional systems: recent results and open questions*, SIAM Rev. **62** (2020), 529–614.

[14] Mironchenko, A., Wirtz, F., *Characterization of input-to-state stability for infinite-dimensional systems*, IEEE Trans. Automat. Control **63** (6) (2018), 1602–1617.

[15] Nakonechnyi, A.G., Mashchenko, S.O., Chikrii, V.K, *Motion control under conflict condition*, J. Automat. Inform. Sci. **50** (1) (2018), 54–75.

[16] Nakonechnyi, O.G., Kapustian, O.A., Chikrii, A.O., *Approximate guaranteed mean square estimates of functionals on solutions of parabolic problems with fast oscillating coefficients under nonlinear observations*, Cybernet. Systems Anal. **55** (5) (2019), 785–795.

[17] Pazy, A., *Semigroups of linear operators and applications to PDE*, Springer-Verlag New York, 1983.

[18] Pichkur, V.V., Sobchuk, V.V., *Mathematical models and control design of a functionally stable technological process*, J. Optim. Differ. Equ. Appl. (JODEA) **21** (1) (2021), 1–11.

[19] Robinson, J., *Infinite-dimensional dynamical systems. An introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge University Press, 2001.

[20] Samoilenko, A.M., Stanzhitskii, A.N., *On the averaging of differential equations on an infinite interval*, Differ. Equ. **42** (4) (2006), 505–511.

[21] Schmid, J., Kapustyan, O., Dashkovskiy, S., *Asymptotic gain results for attractors of semilinear systems*, Math. Control Relat. Fields **12** (3) (2022), 763–788.

[22] Sell, G., You, Y., *Dynamics of evolutionary equations*, Springer New York, NY, 2000.

[23] Sontag, E.D., *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control **34** (4) (1989), 435–443.

[24] Sontag, E.D., *Mathematical control theory. Deterministic finite-dimensional systems*, Springer, N.Y., 1998.

[25] Stanzhitskii, A.M., *Investigation of invariant sets of Itô stochastic systems with the use of Lyapunov functions*, Ukrainian Math. J. **53** (2) (2001), 323–327.

[26] Stanzhyts'kyi, O., *Investigation of exponential dichotomy of Ito stochastic systems by using quadratic forms*, Ukrainian Math. J. **53** (11) (2001), 1882–1894.

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WEAK-STRONG UNIQUENESS FOR A CLASS OF DEGENERATE PARABOLIC CROSS-DIFFUSION SYSTEMS

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Abstract. Bounded weak solutions to a particular class of degenerate parabolic cross-diffusion systems are shown to coincide with the unique strong solution determined by the same initial condition on the maximal existence interval of the latter. The proof relies on an estimate established for a relative entropy associated to the system.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$ and outer unit normal $\mathbf{n}$, and assume that the constants a, b, c, and d satisfy

$$(a,\, b,\, c,\, d) \in (0,\infty)^4 \quad \text{and} \quad ad > bc\,. \tag{1.1}$$

We consider the evolution equations

$$\left.\begin{aligned} \partial_t f &= \operatorname{div}\big(f\nabla[af+bg]\big) \\ \partial_t g &= \operatorname{div}\big(g\nabla[cf+dg]\big) \end{aligned}\right\} \text{ in } (0,\infty)\times\Omega\,, \tag{1.2a}$$

supplemented with homogeneous Neumann boundary conditions

$$\nabla f\cdot\mathbf{n} = \nabla g\cdot\mathbf{n} = 0 \ \text{ on } \ (0,\infty)\times\partial\Omega\,, \tag{1.2b}$$

and non-negative initial conditions

$$(f,g)(0) = (f^{in}, g^{in}) \ \text{ in } \ \Omega\,. \tag{1.2c}$$

The porous medium equation [27] as well as the thin film Muskat problem [10] arise as special cases of (1.2a).

We point out that (1.2a) is a quasilinear degenerate parabolic system with a full diffusion matrix, so that the study of its well-posedness is already a challenging issue. On the one hand, owing to its parabolic structure, the system (1.2) fits into the theory developed in [2], from which the local existence and uniqueness of a strong solution starting from an initial condition with positive components can be inferred, see Theorem 2.1 below. However, comparison principles cannot be applied in the context of (1.2) and the degeneracy featured in (1.2a) might lead to the

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breakdown of the positivity of the components in finite time and thus to that of their regularity. As a consequence, strong solutions cannot be extended beyond a finite time in general. On the other hand, non-negative global weak solutions to (1.2), which are also bounded, are constructed in [20, 22], but the uniqueness of such solutions is an open problem, even in dimension $N = 1$. This is in sharp contrast with the porous medium equation for which several uniqueness results for weak solutions are available in the literature, see [1, 3, 7, 21, 24, 25, 27] and the references therein. It is actually the strong coupling in (1.2a) which makes it difficult to generalize the methods from the above references to this two-phase version of the porous medium equation.

The goal of this paper is to prove a weaker result, namely that, given a sufficiently smooth initial condition (f^{in}, g^{in}) with positive components, all bounded weak solutions to (1.2) coincide on the time interval on which the strong solution exists. For that purpose, we shall rely on the availability of a suitable relative entropy functional, an idea which has proved instrumental in several recent works on weak-strong uniqueness/stability results for (systems of) partial differential equations. In particular, this method has been applied in various settings such as: the compressible Navier-Stokes system [11] and the Fourier-Navier-Stokes system [12], the (isentropic) Euler equations [5, 16], hyperbolic-parabolic systems [9], the Navier-Stokes-Korteweg and the Euler-Korteweg systems [6, 15], the Navier-Stokes equation with surface tension [14], (reaction-)cross-diffusion systems [8, 19], entropy-dissipating reaction-diffusion equations [13], energy-reaction-diffusion systems [17], and Maxwell-Stefan systems [18].

Before stating precisely our main result, let us first make precise the meaning of weak and strong solutions to (1.2). Here and below, for $p \in [1, \infty]$, $L_{p,+}(\Omega, \mathbb{R}^2)$ denotes the positive cone of $L_p(\Omega, \mathbb{R}^2)$; that is,

$$L_{p,+}(\Omega, \mathbb{R}^2) := \{(f, g) \in L_p(\Omega, \mathbb{R}^2) \, : \, f \geq 0 \text{ and } g \geq 0 \text{ a.e. in } \Omega\} .$$

Definition 1.1 (Bounded weak solution)**.** Assume (1.1) and let $u^{in} := (f^{in}, g^{in})$ be an element of $L_{\infty,+}(\Omega, \mathbb{R}^2)$. Given $T \in (0, \infty]$, a *bounded weak solution* u to (1.2) on $[0, T)$ is a pair of functions $u = (f, g)$ such that:

(i) for each $t \in (0, T)$,

$$(f, g) \in L_{\infty,+}((0, t) \times \Omega, \mathbb{R}^2) \cap L_2((0, t), H^1(\Omega, \mathbb{R}^2)) \cap W_2^1((0, t), H^1(\Omega, \mathbb{R}^2)') \,; \tag{1.3}$$

(ii) for all $\varphi \in H^1(\Omega)$ and $t \in (0, T)$,

$$\begin{aligned} &\int_\Omega (f(t, x) - f^{in}(x))\varphi(x) \, \mathrm{d}x \\ &\qquad + \int_0^t \int_\Omega f(s, x) \nabla[af + bg](s, x) \cdot \nabla\varphi(x) \, \mathrm{d}x\mathrm{d}s = 0 \end{aligned} \tag{1.4a}$$

and

$$\begin{aligned} &\int_\Omega (g(t, x) - g^{in}(x))\varphi(x) \, \mathrm{d}x \\ &\qquad + \int_0^t \int_\Omega g(s, x) \nabla[cf + dg](s, x) \cdot \nabla\varphi(x) \, \mathrm{d}x\mathrm{d}s = 0 \,. \end{aligned} \tag{1.4b}$$

Observe that the boundedness and weak differentiability required on f and g in (1.3) guarantee that the integrals in (1.4) are finite.

We next turn to strong solutions to (1.2) and first introduce some notation: for $p > N$ and $s \in (1 + N/p, 2]$, we set

$$H^s_{p,\mathcal{B}}(\Omega) := \{z \in H^s_p(\Omega) \,:\, \nabla z \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

where $H^s_p(\Omega)$ denotes the Bessel potential space, see [2, Section 5] for instance, and

$$\mathcal{O}^s_p := \{u = (f,g) \in H^s_{p,\mathcal{B}}(\Omega, \mathbb{R}^2) \,:\, f > 0 \text{ and } g > 0 \text{ in } \Omega\}\,. \tag{1.5}$$

We observe that the continuous embedding of $H^s_p(\Omega)$ in $\mathrm{C}^1(\overline{\Omega})$ for $p > N$ and $s \in (1 + N/p, 2]$ guarantees that $\mathcal{O}^s_p$ is an open subset of $H^s_{p,\mathcal{B}}(\Omega, \mathbb{R}^2)$.

Definition 1.2 (Strong solution)**.** Assume (1.1) and let $p > N$, $s \in (1 + N/p, 2)$, $T \in (0, \infty]$, and $u^{in} = (f^{in}, g^{in}) \in \mathcal{O}^s_p$. A *strong solution* u to (1.2) on $[0, T)$ is a pair $u = (f, g)$ such that

$$u \in \mathrm{C}([0,T), \mathcal{O}^s_p) \cap \mathrm{C}^1((0,T), L_p(\Omega, \mathbb{R}^2)) \cap \mathrm{C}((0,T), H^2_{p,\mathcal{B}}(\Omega, \mathbb{R}^2))\,,$$

which satisfies (1.2) in a strong sense (and in particular a.e. in $(0, T) \times \Omega$).

One may easily check that a strong solution to (1.2) on $[0, T)$ in the sense of Definition 1.2 is also a bounded weak solution on $[0, T)$ in the sense of Definition 1.1. We emphasize here that strong solutions emanate from initial conditions with positive components, while only non-negativity of initial conditions is required for weak solutions.

The aim of this paper is to establish a weak-strong uniqueness result for (1.2) as stated in Theorem 1.3 below. As in [13], the main tool to be used in the proof is the relative entropy functional

$$H(u_1|u_2) := \int_\Omega \left\{ \left[f_1 \ln\left(\frac{f_1}{f_2}\right) - (f_1 - f_2) \right] + \frac{b}{c} \left[g_1 \ln\left(\frac{g_1}{g_2}\right) - (g_1 - g_2) \right] \right\} \mathrm{d}x\,, \tag{1.6}$$

which is well-defined for $u_i = (f_i, g_i) \in L_{2,+}(\Omega, \mathbb{R}^2)$, $i = 1, 2$, provided that f_2 and g_2 are bounded from below by positive constants. It is important to remark that $H(u_1|u_2)$ controls the square of the L_2-norm of $u_1 - u_2$, see (2.14) below, if u_1 and u_2 are additionally bounded functions.

The main step in the proof of Theorem 1.3 is to derive the integral inequality (1.7) which measures the "distance" between a bounded weak solution in the sense of Definition 1.1 and a strong solution in the sense of Definition 1.2. Gronwall's inequality then provides the weak-strong uniqueness property for the evolution problem (1.2).

Theorem 1.3. *Consider $u_1^{in} \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ and $u_2^{in} \in \mathcal{O}^s_p$ for some $s \in (1+N/p, 2)$ and $p > N$. Let $u_2 = (f_2, g_2)$ be the strong solution to* (1.2) *with initial condition u_2^{in} defined on its maximal existence interval $[0, T^+)$, $T^+ \in (0, \infty]$, see Theorem 2.1 below. If $u_1 = (f_1, g_1)$ is a bounded weak solution to* (1.2) *on $[0, T^+)$ with initial condition u_1^{in} and $T \in (0, T^+)$, there exists a positive constant $C = (a, b, c, d, u_1, u_2, T)$*

such that

$$H(u_1(t)|u_2(t)) \le H(u_1^{in}|u_2^{in}) + C\int_0^t H(u_1(s)|u_2(s))\,\mathrm{d}s \quad \textit{for all } t \in [0,T]\,. \tag{1.7}$$

In particular, if $u_1^{in} = u_2^{in} \in \mathcal{O}_p^s$*, then* $u_1(t) = u_2(t)$ *for all* $t \in [0, T^+)$*.*

We emphasize that Theorem 1.3 applies to any pair of initial conditions $u_1^{in} \in L_{\infty,+}(\Omega,\mathbb{R}^2)$ and $u_2^{in} \in \mathcal{O}_p^s$ for some $s \in (1+N/p, 2)$ and $p > N$. Indeed, the existence of a bounded weak solution to (1.2) on $[0,\infty)$ with initial condition u_1^{in} follows from [20, 22], while that of a strong solution to (1.2) on some maximal time interval with initial condition u_2^{in} is provided in Theorem 2.1 below.

2. Proof of the main result

We start this section by considering the evolution problem (1.2) in the setting of strong solutions as specified in Definition 1.2. Using the quasilinear parabolic theory developed in [2], we then prove in Theorem 2.1 that (1.2) is well-posed in this strong setting. The remaining part is then devoted to the proof of Theorem 1.3.

2.1. **Strong solutions to the evolution problem** (1.2)**.**

In order to construct strong solutions to (1.2) we reformulate (1.2a) in a suitable framework. For that purpose, we fix $p > N$ and $s \in (1 + N/p, 2)$ and introduce the mobility matrix

$$M(X) = (m_{jk}(X))_{1\le j,k\le 2} := \begin{pmatrix} aX_1 & bX_1 \\ cX_2 & dX_2 \end{pmatrix}, \qquad X = (X_1, X_2) \in \mathbb{R}^2\,. \tag{2.1}$$

The problem (1.2) can then be recast as

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = \Phi(u(t))[u(t)], \qquad u(0) = u^{in}\,, \tag{2.2}$$

where the quasilinear operator $\Phi : \mathcal{O}_p^s \to \mathcal{L}(H^2_{p,\mathcal{B}}(\Omega,\mathbb{R}^2), L_p(\Omega,\mathbb{R}^2))$ is defined by the relation

$$\Phi(u)[v] := \mathrm{div}(M(u)\nabla v) = \sum_{i=1}^N \partial_i(M(u)\partial_i v), \qquad u \in \mathcal{O}_p^s,\ v \in H^2_{p,\mathcal{B}}(\Omega,\mathbb{R}^2)\,.$$

Observing that, for $u \in \mathcal{O}_p^s$, the matrix-valued function $M(u)$ belongs to $\mathrm{C}^1(\overline{\Omega},\mathbb{R}^{2\times 2})$ and that $M(u(x))$, $x \in \overline{\Omega}$, has its spectrum contained in the right-half plane $\{\mathrm{Re}\, z > 0\}$, we infer from [2, Theorem 4.1 and Example 4.3 (e)] that $\Phi(u)$ is, for each $u \in \mathcal{O}_p^s$, the generator of an analytic semigroup in $\mathcal{L}(L_p(\Omega,\mathbb{R}^2))$. Since

$$[L_p(\Omega), H^2_{p,\mathcal{B}}(\Omega)]_{s/2} = H^s_{p,\mathcal{B}}(\Omega)\,,$$

where $[\cdot,\cdot]$ is the complex interpolation functor, see [26, Theorem 4.3.3], we may now apply to (2.2) the quasilinear parabolic theory presented in [2, Section 12] (see also [23, Remark 1.2 (ii)]) to obtain the following result.

Theorem 2.1. *Let* $p > N$, $s \in (1 + N/p, 2)$*, and assume that* (1.1) *is satisfied. Then, given* $u^{in} \in \mathcal{O}_p^s$*, the problem* (1.2) *has a unique maximal strong solution*

$$u \in \mathrm{C}([0,T^+),\mathcal{O}_p^s) \cap \mathrm{C}^1((0,T^+), L_p(\Omega,\mathbb{R}^2)) \cap \mathrm{C}((0,T^+), H^2_{p,\mathcal{B}}(\Omega,\mathbb{R}^2))\,,$$

where $T^+ \in (0,\infty]$ denotes the maximal existence time.

2.2. **Proof of Theorem 1.3.** Let $T \in (0,T^+)$. Since $\{u_2(t) \ : \ t \in [0,T]\}$ is a compact subset of $\mathcal{O}_p^s$, there is $\sigma \in (0,1)$ (possibly depending on T) such that, for $t \in [0,T]$,

$$
\begin{aligned}
&\sigma \le \min_{x\in\overline{\Omega}} \min\left\{f_2(t,x),\, g_2(t,x)\right\} \qquad \text{and} \\
&\max\left\{\|\nabla f_2(t)\|_\infty, \|\nabla g_2(t)\|_\infty\right\} \le \sigma^{-1}\,.
\end{aligned}
\tag{2.3}
$$

Moreover, since u_1 is a bounded weak solution, we may assume that also

$$
|u_1(t,x)| + |u_2(t,x)| \le \sigma^{-1} \quad \text{a.e. in } (0,T)\times\Omega\,.
\tag{2.4}
$$

Given $\eta \in (0,1)$, let

$$
\begin{aligned}
H_\eta(u_1(t)|u_2(t)) := &\int_\Omega \left[f_1(t)\ln\left(\frac{f_1(t)+\eta}{f_2(t)}\right) - (f_1(t)-f_2(t))\right] \mathrm{d}x \\
&+ \frac{b}{c}\int_\Omega \left[g_1(t)\ln\left(\frac{g_1(t)+\eta}{g_2(t)}\right) - (g_1(t)-g_2(t))\right] \mathrm{d}x, \quad t \in [0,T]\,.
\end{aligned}
$$

As a consequence of (2.4) and of Definition 1.1, we have $u_i(t) \in L_\infty(\Omega,\mathbb{R}^2)$ for $i = 1$, 2 and all $t \in [0,T]$, and the dominated convergence theorem, together with the lower bound in (2.3), yields

$$
\lim_{\eta\to 0} H_\eta(u_1(t)|u_2(t)) = H(u_1(t)|u_2(t))\,, \qquad t \in [0,T]\,.
\tag{2.5}
$$

Furthermore, by virtue of Definition 1.1, Definition 1.2, (2.3), (2.4), and the continuous embedding of $\mathcal{O}_p^s$ in $\mathrm{C}^1(\overline{\Omega},\mathbb{R}^2)$ we have

$$
f_1,\, g_1 \in L_2((0,T),H^1(\Omega)) \cap W_2^1((0,T),H^1(\Omega)')
$$

and

$$
\ln f_2,\, \ln g_2 \in L_2((0,T),H^1(\Omega)) \cap W_2^1((0,T),H^1(\Omega)')\,.
$$

These properties, together with (2.3), (2.4), and suitable versions of the Lions-Magenes lemma, see, e.g., [4, Theorem II.5.12] and Lemma A.1, imply that

$$
[t \mapsto H_\eta(u_1(t)|u_2(t))] \colon [0,T] \to \mathbb{R}
$$

is continuous and

$$
\begin{aligned}
H_\eta(u_1(t)|u_2(t)) &- H_\eta(u_1^{in}|u_2^{in}) \\
&= \int_0^t \left\langle \partial_t f_1, \ln\left(\frac{f_1+\eta}{f_2}\right) + \frac{f_1}{f_1+\eta}\right\rangle_{(H^1)',H^1} \mathrm{d}s \\
&\quad - \int_0^t \left\langle \partial_t f_2, \frac{f_1}{f_2}\right\rangle_{(H^1)',H^1} \mathrm{d}s \\
&\quad + \frac{b}{c}\int_0^t \left\langle \partial_t g_1, \ln\left(\frac{g_1+\eta}{g_2}\right) + \frac{g_1}{g_1+\eta}\right\rangle_{(H^1)',H^1} \mathrm{d}s \\
&\quad - \frac{b}{c}\int_0^t \left\langle \partial_t f_2, \frac{f_1}{f_2}\right\rangle_{(H^1)',H^1} \mathrm{d}s
\end{aligned}
\tag{2.6}
$$

for all $t \in [0,T]$, where $\langle \cdot,\cdot \rangle_{(H^1)',H^1}$ is the duality bracket between $H^1(\Omega)$ and $H^1(\Omega)'$. Reformulating (2.6) with the help of (1.4), we find

$$
\begin{aligned}
& H_\eta(u_1(t)|u_2(t)) - H_\eta(u_1^{in}|u_2^{in}) \\
& = - \int_0^t \int_\Omega f_1 \nabla(af_1+bg_1) \cdot \Big(\frac{\nabla f_1}{f_1+\eta} - \frac{\nabla f_2}{f_2} \Big)\, \mathrm{d}x\mathrm{d}s \\
& \quad - \int_0^t \int_\Omega \Big[f_1 \nabla(af_1+bg_1) \cdot \nabla\Big(\frac{f_1}{f_1+\eta} \Big) - f_2 \nabla(af_2+bg_2) \cdot \nabla\Big(\frac{f_1}{f_2} \Big) \Big]\, \mathrm{d}x\mathrm{d}s \\
& \quad - \frac{b}{c} \int_0^t \int_\Omega g_1 \nabla(cf_1+dg_1) \cdot \Big(\frac{\nabla g_1}{g_1+\eta} - \frac{\nabla g_2}{g_2} \Big)\, \mathrm{d}x\mathrm{d}s \\
& \quad - \frac{b}{c} \int_0^t \int_\Omega \Big[g_1 \nabla(cf_1+dg_1) \cdot \nabla\Big(\frac{g_1}{g_1+\eta} \Big) - g_2 \nabla(cf_2+dg_2) \cdot \nabla\Big(\frac{g_1}{g_2} \Big) \Big]\, \mathrm{d}x\mathrm{d}s\,.
\end{aligned}
$$

Hence,

$$
H_\eta(u_1(t) \mid u_2(t)) - H_\eta(u_1^{in}|u_2^{in}) = T_\eta^1(t) + T^2(t)\,, \tag{2.7}
$$

where

$$
\begin{aligned}
T_\eta^1(t) := {} & \eta^2 \int_0^t \int_\Omega \nabla(af_1+bg_1) \cdot \frac{\nabla f_1}{(f_1+\eta)^2}\, \mathrm{d}x\mathrm{d}s \\
& + \eta^2 \frac{b}{c} \int_0^t \int_\Omega \nabla(cf_1+dg_1) \cdot \frac{\nabla g_1}{(g_1+\eta)^2}\, \mathrm{d}x\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
T^2(t) := & - \int_0^t \int_\Omega \Big[\nabla(af_1+bg_1) \cdot \Big(\nabla f_1 - \frac{f_1}{f_2} \nabla f_2 \Big) - f_2 \nabla(af_2+bg_2) \cdot \nabla\Big(\frac{f_1}{f_2} \Big) \Big]\, \mathrm{d}x\mathrm{d}s \\
& - \frac{b}{c} \int_0^t \int_\Omega \Big[\nabla(cf_1+dg_1) \cdot \Big(\nabla g_1 - \frac{g_1}{g_2} \nabla g_2 \Big) - g_2 \nabla(cf_2+dg_2) \cdot \nabla\Big(\frac{g_1}{g_2} \Big) \Big]\, \mathrm{d}x\mathrm{d}s\,.
\end{aligned}
$$

In view of Definition 1.1 (i), both functions $\nabla(af_1+bg_1) \cdot \nabla f_1$ and $\nabla(cf_1+dg_1) \cdot \nabla g_1$ belong to $L_1((0,t)\times\Omega)$ and

$$
\left.
\begin{aligned}
\lim_{\eta\to 0} \eta^2 \frac{\nabla f_1}{(f_1+\eta)^2} &= 0 \\
\lim_{\eta\to 0} \eta^2 \nabla \frac{g_1}{(g_1+\eta)^2} &= 0
\end{aligned}
\right\} \qquad \text{a.e. in } (0,t)\times\Omega\,,
$$

as $\nabla f_1 = 0$ a.e. on $\{(s,x)\ :\ f_1(s,x)=0\}$ and $\nabla g_1 = 0$ a.e. on $\{(s,x)\ :\ g_1(s,x)=0\}$. The dominated convergence theorem now implies that

$$
\lim_{\eta\to 0} T_\eta^1(t) = 0 \qquad \text{for all } t \in [0,T]\,. \tag{2.8}
$$

Hence, letting $\eta \to 0$ in (2.7), we deduce from (2.5) and (2.8) that

$$
H(u_1(t)|u_2(t)) - H(u_1^{in}|u_2^{in}) = T^2(t) \quad \text{for } t \in [0,T]\,. \tag{2.9}
$$

With respect to $T^2(t)$, we note that

$$
\begin{aligned}
\frac{T^2(t)}{a} =& -\int_0^t\int_\Omega \left[\nabla f_1\cdot\nabla\Big(f_1+\frac{b}{a}g_1\Big)+\frac{b}{a}\nabla g_1\cdot\nabla\Big(f_1+\frac{d}{c}g_1\Big)\right]\mathrm{d}x\mathrm{d}s\\
&+\int_0^t\int_\Omega \left[\frac{f_1}{f_2}\nabla f_2\cdot\nabla\Big(f_1+\frac{b}{a}g_1\Big)+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\cdot\nabla\Big(f_1+\frac{d}{c}g_1\Big)\right]\mathrm{d}x\mathrm{d}s\\
&+\int_0^t\int_\Omega \left[\nabla f_1\cdot\nabla\Big(f_2+\frac{b}{a}g_2\Big)+\frac{b}{a}\nabla g_1\cdot\nabla\Big(f_2+\frac{d}{c}g_2\Big)\right]\mathrm{d}x\mathrm{d}s\\
&-\int_0^t\int_\Omega \left[\frac{f_1}{f_2}\nabla f_2\cdot\nabla\Big(f_2+\frac{b}{a}g_2\Big)+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\cdot\nabla\Big(f_2+\frac{d}{c}g_2\Big)\right]\mathrm{d}x\mathrm{d}s\,.
\end{aligned}
\tag{2.10}
$$

Introducing

$$
T_I^2(t) := -\frac{b(ad-bc)}{ac}\int_0^t\int_\Omega \left[|\nabla g_1|^2-\Big(1+\frac{g_1}{g_2}\Big)\nabla g_1\cdot\nabla g_2+\frac{g_1}{g_2}|\nabla g_2|^2\right]\mathrm{d}x\mathrm{d}s
$$

and $T_{II}^2(t) := T^2(t)-T_I^2(t)$, we note that

$$
\begin{aligned}
T_I^2(t) &= -\frac{b(ad-bc)}{ac}\int_0^t\int_\Omega \left[\Big|\nabla g_1-\frac{1}{2}\Big(1+\frac{g_1}{g_2}\Big)\nabla g_2\Big|^2-\Big|\frac{g_1-g_2}{2g_2}\nabla g_2\Big|^2\right]\mathrm{d}x\mathrm{d}s\\
&\le \frac{b(ad-bc)}{ac}\int_0^t\int_\Omega \Big|\frac{g_1-g_2}{2g_2}\nabla g_2\Big|^2\mathrm{d}x\mathrm{d}s\,,
\end{aligned}
\tag{2.11}
$$

thanks to (1.1). Furthermore, in view of the relation

$$
\frac{d}{c}=\frac{b}{a}+\frac{ad-bc}{ac}\,,
$$

$$
\begin{aligned}
\frac{T_{II}^2(t)}{a} =& -\int_0^t\int_\Omega \Big|\nabla\Big(f_1+\frac{b}{a}g_1\Big)\Big|^2\mathrm{d}x\mathrm{d}s\\
&-\int_0^t\int_\Omega \nabla\Big(f_1+\frac{b}{a}g_1\Big)\cdot\left[\Big(1+\frac{f_1}{f_2}\Big)\nabla f_2+\frac{b}{a}\Big(1+\frac{g_1}{g_2}\Big)\nabla g_2\right]\mathrm{d}x\mathrm{d}s\\
&-\int_0^t\int_\Omega \nabla\Big(f_2+\frac{b}{a}g_2\Big)\cdot\Big(\frac{f_1}{f_2}\nabla f_2+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\Big)\mathrm{d}x\mathrm{d}s\\
=& -\int_0^t\int_\Omega \left|\nabla\Big(f_1+\frac{b}{a}g_1\Big)-\frac{1}{2}\left[\Big(1+\frac{f_1}{f_2}\Big)\nabla f_2+\frac{b}{a}\Big(1+\frac{g_1}{g_2}\Big)\nabla g_2\right]\right|^2\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{4}\int_0^t\int_\Omega \Big|\Big(1+\frac{f_1}{f_2}\Big)\nabla f_2+\frac{b}{a}\Big(1+\frac{g_1}{g_2}\Big)\nabla g_2\Big|^2\mathrm{d}x\mathrm{d}s\\
&-\int_0^t\int_\Omega \nabla\Big(f_2+\frac{b}{a}g_2\Big)\cdot\Big(\frac{f_1}{f_2}\nabla f_2+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\Big)\mathrm{d}x\mathrm{d}s\,.
\end{aligned}
$$

Observing that

$$
\begin{aligned}
&\frac{1}{4}\Big|\Big(1+\frac{f_1}{f_2}\Big)\nabla f_2+\frac{b}{a}\Big(1+\frac{g_1}{g_2}\Big)\nabla g_2\Big|^2-\nabla\Big(f_2+\frac{b}{a}g_2\Big)\cdot\Big(\frac{f_1}{f_2}\nabla f_2+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\Big)\\
&\quad=\frac{1}{4}\Big|\nabla\Big(f_2+\frac{b}{a}g_2\Big)+\frac{f_1}{f_2}\nabla f_2+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\Big|^2-\nabla\Big(f_2+\frac{b}{a}g_2\Big)\cdot\Big(\frac{f_1}{f_2}\nabla f_2+\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\Big)\\
&\quad=\frac{1}{4}\Big|\nabla\Big(f_2+\frac{b}{a}g_2\Big)-\frac{f_1}{f_2}\nabla f_2-\frac{b}{a}\frac{g_1}{g_2}\nabla g_2\Big|^2\\
&\quad=\frac{1}{4}\Big|\Big(1-\frac{f_1}{f_2}\Big)\nabla f_2+\frac{b}{a}\Big(1-\frac{g_1}{g_2}\Big)\nabla g_2\Big|^2\\
&\quad\leq\frac{1}{2}\Big|\frac{f_1-f_2}{f_2}\nabla f_2\Big|^2+\frac{b^2}{2a^2}\Big|\frac{g_1-g_2}{g_2}\nabla g_2\Big|^2,
\end{aligned}
$$

the last estimate resulting from Young's inequality, we are led to

$$
\frac{T_{II}^2(t)}{a}\leq\frac{1}{2}\int_0^t\int_\Omega\Big[\Big|\frac{f_1-f_2}{f_2}\nabla f_2\Big|^2+\frac{b^2}{a^2}\Big|\frac{g_1-g_2}{g_2}\nabla g_2\Big|^2\Big]\,\mathrm{d}x\mathrm{d}s\,. \tag{2.12}
$$

On behalf of (2.9), (2.11), and (2.12) we conclude that

$$
\begin{aligned}
H(u_1(t)|u_2(t))\leq H(u_1^{in}|u_2^{in})&+\frac{b(ad-bc)}{ac}\int_0^t\int_\Omega\Big|\frac{g_1-g_2}{2g_2}\nabla g_2\Big|^2\,\mathrm{d}x\mathrm{d}s\\
&+\frac{a}{2}\int_0^t\int_\Omega\Big[\Big|\frac{f_1-f_2}{f_2}\nabla f_2\Big|^2+\frac{b^2}{a^2}\Big|\frac{g_1-g_2}{g_2}\nabla g_2\Big|^2\Big]\,\mathrm{d}x\mathrm{d}s\,.
\end{aligned}
$$

Recalling (2.3), we deduce that there exists a positive constant $C=C(a,b,c,d)$ such that

$$
H(u_1(t)|u_2(t))\leq H(u_1^{in}|u_2^{in})+C\sigma^4\int_0^t\int_\Omega\Big[|f_1-f_2|^2+\frac{b}{c}|g_1-g_2|^2\Big]\,\mathrm{d}x\mathrm{d}s \tag{2.13}
$$

for all $t\in[0,T]$. In view of the inequality

$$
x\ln\Big(\frac{x}{y}\Big)-(x-y)\geq\frac{1}{2}\frac{|x-y|^2}{\max\{x,y\}}\,,\qquad (x,y)\in[0,\infty)\times(0,\infty)\,, \tag{2.14}
$$

which follows from [18, Lemma 18], it is not difficult to infer from (2.13), by taking also into account the boundedness of u_1 and u_2 in $(0,T)\times\Omega$ provided by (2.4), that

$$
H(u_1(t)|u_2(t))\leq H(u_1^{in}|u_2^{in})+C\int_0^t H(u_1(s)|u_2(s))\,\mathrm{d}s \tag{2.15}
$$

for all $t\in[0,T]$. This completes the proof of (1.7).

Annexe A. A version of the Lions-Magenes lemma

In this section we establish a version of the Lions-Magenes lemma, see Lemma A.1 below, which is used in the proof of Theorem 1.3 when differentiating the mapping

$$\left[t \mapsto \int_\Omega [f(t)\ln(f(t)+\eta) - f(t)]\,\mathrm{d}x\right] : (0,T) \to \mathbb{R}\,, \qquad \text{with } \eta > 0\,,$$

for some appropriate non-negative function f. Before stating the result, we note that the function $\Phi(s) := s\ln(s+\eta) - s$, $s \geq 0$, satisfies $\Phi''(s) = (s+2\eta)/(s+\eta)^2$, $s \geq 0$. Thus

$$\|\Phi''\|_\infty < \infty\,.$$

Lemma A.1 (Lions-Magenes lemma)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $\Phi \in \mathrm{C}^2(\mathbb{R})$ satisfy $\|\Phi''\|_\infty < \infty$. Assume that*

$$f \in L_2((0,T), H^1(\Omega)) \cap W_2^1((0,T), H^1(\Omega)')\,.$$

Then

$$\left[t \mapsto I(t) := \int_\Omega \Phi(f(t))\,\mathrm{d}x\right] \in \mathrm{C}([0,T],\mathbb{R})$$

and for all $0 \leq t_0 \leq t \leq T$ we have

$$\int_\Omega \Phi(f(t))\,\mathrm{d}x - \int_\Omega \Phi(f(t_0))\,\mathrm{d}x = \int_{t_0}^t \Big\langle \partial_t f(\tau), \Phi'(f(\tau)) \Big\rangle_{(H^1)',H^1} \mathrm{d}\tau\,. \tag{A.1}$$

As we are lacking a precise reference for Lemma A.1, we include below a proof for the sake of completeness. As a first step, we establish in Lemma A.2 an auxiliary result which is used in the proof of Lemma A.1.

Lemma A.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $f \in \mathrm{C}^1(\mathcal{I}, L_2(\Omega))$, where $\mathcal{I} \subset \mathbb{R}$ is an interval. Let further $\Phi \in \mathrm{C}^2(\mathbb{R})$ satisfy $\|\Phi''\|_\infty =: L < \infty$. Then,*

$$\left[t \mapsto I(t) := \int_\Omega \Phi(f(t))\,\mathrm{d}x\right] \in \mathrm{C}^1(\mathcal{I},\mathbb{R})$$

and

$$I'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(f(t))\,\mathrm{d}x = \int_\Omega \Phi'(f(t))\partial_t f(t)\,\mathrm{d}x, \qquad t \in \mathcal{I}\,. \tag{A.2}$$

Proof. We may assume without loss of generality that $\Phi(0) = \Phi'(0) = 0$ (as the claim is obvious for affine functions). Then

$$|\Phi(r) - \Phi(s)| \leq L(|r|+|s|)|r-s|\,,\ \ |\Phi'(r) - \Phi'(s)| \leq L|r-s|\,,\ \ (r,s) \in \mathbb{R}^2\,. \tag{A.3}$$

In particular, since $\Phi(0) = \Phi'(0) = 0$,

$$|\Phi(f(t))| \leq L|f(t)|^2 \qquad \text{and} \qquad |\Phi'(f(t))| \leq L|f(t)|\,, \qquad t \in \mathcal{I}\,,$$

and it follows that

$$\Phi(f(t)) \in L_1(\Omega) \qquad \text{and} \qquad \Phi'(f(t))\partial_t f(t) \in L_1(\Omega)\,, \qquad t \in \mathcal{I}\,.$$

Let $t \neq t_0 \in \mathcal{I}$. We then have

$$\begin{aligned}
&\Big|\frac{I(t)-I(t_0)}{t-t_0} - \int_\Omega \Phi'(f(t_0))\partial_t f(t_0)\,\mathrm{d}x\Big| \\
&\le \int_\Omega \Big|\frac{\Phi(f(t))-\Phi(f(t_0))}{t-t_0} - \Phi'(f(t_0))\partial_t f(t_0)\Big|\,\mathrm{d}x \\
&\le \int_\Omega \int_0^1 \Big|\Phi'((1-s)f(t_0)+sf(t))\frac{f(t)-f(t_0)}{t-t_0} - \Phi'(f(t_0))\partial_t f(t_0)\Big|\,\mathrm{d}s\mathrm{d}x \\
&\le J_1(t) + J_2(t)\,,
\end{aligned}$$

where

$$J_1(t) := \int_\Omega \int_0^1 \Big|\Phi'((1-s)f(t_0)+sf(t))\Big[\frac{f(t)-f(t_0)}{t-t_0} - \partial_t f(t_0)\Big]\Big|\,\mathrm{d}s\mathrm{d}x\,,$$

$$J_2(t) := \int_\Omega \int_0^1 \Big|[\Phi'((1-s)f(t_0)+sf(t)) - \Phi'(f(t_0))]\partial_t f(t_0)\Big|\,\mathrm{d}s\mathrm{d}x\,.$$

By (A.3), Hölder's inequality, and the regularity of f,

$$J_1(t) \le L(\|f(t_0)\|_2 + \|f(t)\|_2)\Big\|\frac{f(t)-f(t_0)}{t-t_0} - \frac{df}{dt}(t_0)\Big\|_2 \underset{t\to t_0}{\longrightarrow} 0\,,$$

$$J_2(t) \le L\|f(t)-f(t_0)\|_2 \Big\|\partial_t f(t_0)\Big\|_2 \underset{t\to t_0}{\longrightarrow} 0\,.$$

Therefore, I is differentiable at t_0 and its derivative is given by (A.2). It next readily follows from (A.3) and the regularity of f that $\Phi'(f)$ and $\partial_t f$ both belong to $\mathrm{C}(\mathcal{I}, L_2(\Omega))$, from which we deduce that $I' \in \mathrm{C}(\mathcal{I})$ with the help of Hölder's inequality. □

We now recall a basic property which is used in the proof of Lemma A.1 below. Let X, Y be Banach spaces such that the embedding of X in Y is continuous and dense and let $T > 0$. Then, $\mathrm{C}^\infty([0,T], X)$ is dense in

$$E_2(X,Y) := L_2((0,T),X) \cap W_2^1((0,T),Y)\,,$$

see, e.g., [4, Lemma II.5.10].

Proof of Lemma A.1. Since $\mathrm{C}^\infty([0,T], H^1(\Omega))$ is dense in $E_2(H^1(\Omega), H^1(\Omega)')$, there is a sequence $(f_n)_{n\ge 1} \in \mathrm{C}^\infty([0,T], H^1(\Omega))$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L_2((0,T),H^1(\Omega))} = \lim_{n\to\infty} \|\partial_t f_n - \partial_t f\|_{L_2((0,T),H^1(\Omega)')} = 0\,. \tag{A.4}$$

Moreover, thanks to the continuous embedding of $E_2(H^1(\Omega), H^1(\Omega)')$ in $\mathrm{C}([0,T], L_2(\Omega))$, see, e.g., [4, Theorem II.5.13], we deduce from (A.4) that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|f_n(t) - f(t)\|_2 = 0\,. \tag{A.5}$$

Let $0 \le t_0 \le t \le T$. By Lemma A.2

$$\int_\Omega \Phi(f_n(t))\,\mathrm{d}x - \int_\Omega \Phi(f_n(t_0))\,\mathrm{d}x = \int_{t_0}^t \big\langle \partial_t f_n(\tau), \Phi'(f_n(\tau))\big\rangle_{(H^1)',H^1}\,\mathrm{d}\tau\,. \tag{A.6}$$

On the one hand, we infer from (A.3), (A.5), and Hölder's inequality that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \Phi(f_n(t))\,\mathrm{d}x = \int_\Omega \Phi(f(t))\,\mathrm{d}x \qquad \text{and} \\ &\lim_{n\to\infty} \int_\Omega \Phi(f_n(t_0))\,\mathrm{d}x = \int_\Omega \Phi(f(t_0))\,\mathrm{d}x\,. \end{aligned} \tag{A.7}$$

On the other hand, it readily follows from (A.3) and (A.4) that

$$\lim_{n\to\infty} \int_0^T \|\Phi'(f_n(\tau)) - \Phi'(f(\tau))\|_2^2\,\mathrm{d}\tau = 0\,.$$

Moreover, the boundedness and continuity of Φ'', (A.4), and Lebesgue's dominated convergence theorem entail that

$$\Phi'(f_n) \in L_2((0,T),H^1(\Omega)) \quad \text{with} \quad \nabla\Phi'(f_n) = \Phi''(f_n)\nabla f_n\,, \qquad n\geq 1\,,$$

and

$$\lim_{n\to\infty} \int_0^T \|\Phi''(f_n(\tau))\nabla f_n(\tau) - \Phi''(f(\tau))\nabla f(\tau)\|_2^2\,\mathrm{d}\tau = 0\,.$$

Therefore,

$$\lim_{n\to\infty} \|\Phi'(f_n) - \Phi'(f)\|_{L_2((0,T),H^1(\Omega))} = 0\,.$$

Combining this convergence with (A.4), leads us to

$$\begin{aligned} \lim_{n\to\infty} \int_{t_0}^t &\big\langle \partial_t f_n(\tau), \Phi'(f_n(\tau))\big\rangle_{(H^1)',H^1}\,\mathrm{d}\tau \\ &= \int_{t_0}^t \big\langle \partial_t f(\tau), \Phi(f(\tau))\big\rangle_{(H^1)',H^1}\,\mathrm{d}\tau = 0\,. \end{aligned} \tag{A.8}$$

The identity (A.1) is then a direct consequence of (A.6), (A.7), and (A.8). □

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References

[1] Alt, H.W., Luckhaus, S., *Quasilinear elliptic-parabolic differential equations*, Math. Z. **183** (1983), 311–341.

[2] Amann, H., *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992), Teubner-Texte Math., vol. 133, Teubner, Stuttgart, 1993, pp. 9–126.

[3] Bénilan, Ph., Crandall, M.G., Pierre, M., *Solutions of the porous medium equation in $\mathbb{R}^N$ under optimal conditions on initial values*, Indiana Univ. Math. J. **33** (1984), 51–87.

[4] Boyer, F., Fabrie, P., *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, Applied Mathematical Sciences, vol. 183, Springer, New York, 2013.

[5] Brenier, Y., De Lellis, C., Székelyhidi, Jr., L., *Weak-strong uniqueness for measure-valued solutions*, Comm. Math. Phys. **305** (2011), no. 2, 351–361.

[6] Bresch, D., Gisclon, M., Lacroix-Violet, I., *On Navier-Stokes-Korteweg and Euler-Korteweg systems: application to quantum fluids models*, Arch. Ration. Mech. Anal. **233** (2019), no. 3, 975–1025.

[7] Brézis, H., Crandall, M.G., *Uniqueness of solutions of the initial-value problem for* $u_t - \Delta\varphi(u) = 0$, J. Math. Pures Appl. (9) **58** (1979), 153–163.

[8] Chen, X., Jüngel, A., *Weak-strong uniqueness of renormalized solutions to reaction-cross-diffusion systems*, Math. Models Methods Appl. Sci. **29** (2019), no. 2, 237–270.

[9] Christoforou, C., Tzavaras, A.E., *Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity*, Arch. Ration. Mech. Anal. **229** (2018), no. 1, 1–52.

[10] Escher, J., Matioc, A.-V., Matioc, B.-V., *Modelling and analysis of the Muskat problem for thin fluid layers*, J. Math. Fluid Mech. **14** (2012), 267–277.

[11] Feireisl, E., Jin, B.J., Novotný, A., *Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system*, J. Math. Fluid Mech. **14** (2012), no. 4, 717–730.

[12] Feireisl, E., Novotný, A., *Weak-strong uniqueness property for the full Navier-Stokes-Fourier system*, Arch. Ration. Mech. Anal. **204** (2012), no. 2, 683–706.

[13] Fischer, J., *Weak-strong uniqueness of solutions to entropy-dissipating reaction-diffusion equations*, Nonlinear Anal. **159** (2017), 181–207.

[14] Fischer, J., Hensel, S., *Weak-strong uniqueness for the Navier-Stokes equation for two fluids with surface tension*, Arch. Ration. Mech. Anal. **236** (2020), no. 2, 967–1087.

[15] Giesselmann, J., Lattanzio, C., Tzavaras, A.E., *Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics*, Arch. Ration. Mech. Anal. **223** (2017), no. 3, 1427–1484.

[16] Gwiazda, P., Świerczewska-Gwiazda, A., Wiedemann, E., *Weak-strong uniqueness for measure-valued solutions of some compressible fluid models*, Nonlinearity **28** (2015), no. 11, 3873–3890.

[17] Hopf, K., *Weak-strong uniqueness for energy-reaction-diffusion systems*, Math. Models Methods Appl. Sci. **32** (2022), 1015–1069.

[18] Huo, X., Jüngel, A., Tzavaras, A.E., *Weak-strong uniqueness for Maxwell-Stefan systems*, SIAM J. Math. Anal. **54** (2022), no. 3, 3215–3252.

[19] Jüngel, A., Portisch, S., Zurek, A., *Nonlocal cross-diffusion systems for multi-species populations and networks*, Nonlinear Anal. **219** (2022), Paper No. 112800, 1–26.

[20] Laurençot, Ph., Matioc, B.-V., *Bounded weak solutions to a class of degenerate cross-diffusion systems*, arXiv: 2201.06479.

[21] Laurençot, Ph., Matioc, B.-V., *The porous medium equation as a singular limit of the thin film Muskat problem*, arXiv:2108.09032, to appear in Asymptot. Anal.

[22] Laurençot, Ph., Matioc, B.-V., *Bounded weak solutions to the thin film Muskat problem via an infinite family of Liapunov functionals*, Trans. Amer. Math. Soc. **375** (2022), no. 8, 5963–5986.

[23] Matioc, B.-V., Walker, Ch., *On the principle of linearized stability in interpolation spaces for quasilinear evolution equations*, Monatsh. Math. **191** (2020), no. 3, 615–634.

[24] Otto, F., *L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations*, J. Differential Equations **131** (1996), no. 1, 20–38.

[25] Pierre, M., *Uniqueness of the solutions of* $u_t - \Delta(\phi(u)) = 0$ *with initial datum a measure*, Nonlinear Anal. **6** (1982), 175–187.

[26] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[27] Vázquez, J.L., *The Porous Medium Equation*, Clarendon Press, Oxford, 2007.

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FINITE-TIME BLOW-UP IN A TWO-SPECIES CHEMOTAXIS-COMPETITION MODEL WITH SINGLE PRODUCTION

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Abstract. This paper is concerned with blow-up of solutions to a two-species chemotaxis-competition model with production from only one species. In previous papers there are a lot of studies on boundedness for a two-species chemotaxis-competition model with productions from both two species. On the other hand, finite-time blow-up was recently obtained under smallness conditions for competitive effects. Now, in the biological view, the production term seems to promote blow-up phenomena; this implies that the lack of the production term makes the solution likely to be bounded. Thus, it is expected that there exists a solution of the system with single production such that the species which does not produce the chemical substance remains bounded, whereas the other species blows up. The purpose of this paper is to prove that this conjecture is true.

1. Introduction and main result

In this paper we deal with the two-species chemotaxis-competition model with single production,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u (1 - u^{\kappa_1 - 1} - a_1 v^{\lambda_1 - 1}) \,, \\
\dfrac{\partial v}{\partial t} = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v (1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1}) \,, \\
0 = d_3 \Delta w + \alpha u - \gamma w \,, \\
(\nabla u \cdot \nu)|_{\partial\Omega} = (\nabla v \cdot \nu)|_{\partial\Omega} = (\nabla w \cdot \nu)|_{\partial\Omega} = 0 \,, \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x)
\end{cases}
\tag{1.1}
$$

in a ball $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \geq 3, R > 0)$. Here, ν is the outward normal vector to $\partial\Omega$; d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$; u_0, $v_0 \in C^0(\overline{\Omega})$ are nonnegative and radially symmetric. This system describes a situation in which multi species move toward higher concentrations of the signal substance (which is produced by the spesies), and compete with each other.

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In a two-species chemotaxis-competition model obtained on replacing the third equation in (1.1) by

$$0 = d_3\Delta w + \alpha u + \beta v - \gamma w \quad (\beta > 0)\,,$$

boundedness and stabilization in the case $\kappa_1 = \kappa_2 = \lambda_1 = \lambda_2 = 2$ were established under smallness conditions for χ_1 and χ_2 in [2, 5, 7, 8]; more related works can be found in [1, 9]. On the other hand, a result on finite-time blow-up in the two-species chemotaxis system was recently obtained in [6, Theorem 4.1] under the condition

$$\max\{\kappa_1, \lambda_1, \kappa_2, \lambda_2\} < \begin{cases} \frac{7}{6} & \text{if } \ n \in \{3, 4\}\,, \\ 1 + \frac{1}{2(n-1)} & \text{if } \ n \geq 5\,. \end{cases}$$

Now, in the biological view, the production term seems to promote blow-up phenomena; this implies that the lack of the production term makes the solution likely to be bounded. Thus, since the third equation in (1.1) lacks the production term βv, it is expected that there exists a solution of (1.1) such that v remains bounded, whereas u blows up. The purpose of this paper is to prove that this conjecture is true.

The main results read as follows. The first theorem gives blow-up in (1.1).

Theorem 1.1. *Let d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$. Assume that κ_1 and λ_1 satisfy that*

$$\max\{\kappa_1, \lambda_1\} < \begin{cases} \frac{7}{6} & \text{if } \ n \in \{3, 4\}\,, \\ 1 + \frac{1}{2(n-1)} & \text{if } \ n \geq 5\,. \end{cases} \tag{1.2}$$

Then, for all $L > 0$, $M_0 > 0$ and $\widetilde{M_0} \in (0, M_0)$ there exists $r_\star \in (0, R)$ with the following property: If

$$u_0,\ v_0 \in C^0(\overline{\Omega}) \text{ are nonnegative and radially symmetric} \tag{1.3}$$

and

$$\int_\Omega (u_0(x) + v_0(x))\, dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\, dx \geq \widetilde{M_0} \tag{1.4}$$

as well as

$$u_0(x) + v_0(x) \leq L|x|^{-n(n-1)} \quad \text{for all } \ x \in \Omega, \tag{1.5}$$

then there exist $T^* < \infty$ and exactly one triplet (u, v, w) of (1.1) which blows up in finite time in the sense that

$$\lim_{t \nearrow T^*} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \tag{1.6}$$

Remark 1.2. This result means that whether blow-up in (1.1) occurs or not can be determined by the parameters which come only from the first equation.

Theorem 1.1 gives existence of a constant $T^* > 0$ and a classical solution (u, v, w) of (1.1) on $[0, T^*)$ such that $\lim_{t \nearrow T^*}(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty$. Then we consider the next question

whether $\lim_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ *and* $\lim_{t \nearrow T^*} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ *hold.*

The second theorem is concerned with simultaneous blow-up in (1.1).

Theorem 1.3. *Let d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$. Then the following holds*:

(i) *Assume that u_0, $v_0 \in C^0(\overline{\Omega})$ are nonnegative. Let $T \in (0, \infty]$ and let (u, v, w) be a classical solution of (1.1) on $[0, T)$. Then (u, v, w) satisfies that*

$$\textit{if } \lim_{t \nearrow T} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty, \textit{ then } \lim_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

(ii) *Assume that κ_1 and λ_1 satisfy (1.2). Moreover, suppose that $\lambda_2 \geq 2$ and*

$$0 < \chi_2 < \begin{cases} \frac{a_2 d_3 \mu_2}{\alpha} & \textit{if } \lambda_2 = 2\,, \\ \infty & \textit{if } \lambda_2 > 2\,. \end{cases} \tag{1.7}$$

Then there are initial data $u_0, v_0 \in C^0(\overline{\Omega})$ and $T^* < \infty$ such that the corresponding solution (u, v, w) of (1.1) on $[0, T^*)$ satisfies

$$\lim_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \quad \textit{and} \quad \sup_{t \in (0, T^*)} \|v(\cdot, t)\|_{L^\infty(\Omega)} < \infty\,.$$

Remark 1.4. This theorem means that if v blows up at time T then u also blows up at T, and moreover there is a solution such that u blows up at T but v is bounded in $\Omega \times (0, T)$; thus this result gives a positive answer to the conjecture.

This paper is organized as follows. In order to show Theorem 1.1, we will derive a differential inequality for some moment-type function in Section 2. Section 3 is devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.1

We first state a result on local existence of solutions to (1.1).

Lemma 2.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball with some $R > 0$, and let d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$. Assume that $u_0, v_0 \in C^0(\overline{\Omega})$ are nonnegative. Then there exist $T_{\max} \in (0, \infty]$ and a unique triplet (u, v, w) of functions*

$$u,\ v,\ w \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\,,$$

which solves (1.1) classically. Moreover, u, $v \geq 0$ in $\Omega \times (0, T_{\max})$ and

$$\textit{if } T_{\max} < \infty\,, \quad \textit{then} \quad \lim_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \tag{2.1}$$

Also, if u_0, v_0 are radially symmetric, then so are u, v, w for any $t \in (0, T_{\max})$.

Proof. This lemma is shown by a standard fixed point argument as in [3, 7]. □

In this section we assume that u_0, $v_0 \in C^0(\overline{\Omega})$ are nonnegative and radially symmetric and that (u, v, w) is a classical solution of (1.1) on $[0, T_{\max})$ given by Lemma 2.1. Moreover, we regard $u(x, t)$, $v(x, t)$ and $w(x, t)$ as functions of $r := |x|$ and t. Also, we introduce the functions U, V and W as

$$U(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\, d\rho \quad \text{and} \quad V(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho, t)\, d\rho$$

as well as

$$W(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} w(\rho,t)\, d\rho$$

for $s \in [0, R^n]$ and $t \in [0, T_{\max})$, and define ϕ_U and ψ_U as

$$\phi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s) U(s,t)\, ds$$

and

$$\psi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s) U(s,t) U_s(s,t)\, ds$$

for $t \in [0, T_{\max})$ with some $s_0 \in (0, R^n)$ and $b \in (0,1)$. We note that ϕ_U belongs to $C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$. To obtain the differential inequality for ϕ_U, we first give the following lemma.

Lemma 2.2. *Let $s_0 \in (0, R^n)$ and $b \in (0,1)$. Then*

$$\begin{aligned} \phi_U'(t) \geq{}& d_1 n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0 - s) U_{ss}\, ds \\ &+ \frac{\alpha\chi_1 n}{d_3} \psi_U(t) - \frac{\gamma\chi_1 n}{d_3} \int_0^{s_0} s^{-b}(s_0 - s) U_s W\, ds \\ &- \mu_1 n^{\kappa_1 - 1} \int_0^{s_0} s^{-b}(s_0 - s) \left(\int_0^s U_s^{\kappa_1}(\sigma,t)\, d\sigma \right) ds \\ &- a_1 \mu_1 n^{\lambda_1 - 1} \int_0^{s_0} s^{-b}(s_0 - s) \left(\int_0^s U_s(\sigma,t) V_s^{\lambda_1 - 1}(\sigma,t)\, d\sigma \right) ds \\ =:{}& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{2.2}$$

for all $t \in (0, T_{\max})$.

Proof. By straightforward computations we can derive (2.2) (see [6, (4.17)]). □

We next estimate the third term on the right-hand side of (2.2).

Lemma 2.3. *Let $b \in (0, \min\{1, 2 - \frac{4}{n}\})$. For all $L > 0$ and all $M_0 > 0$ there exists $C > 0$ such that if u_0, v_0 satisfy* (1.3) *and $\int_\Omega (u_0(x) + v_0(x))\, dx = M_0$ as well as* (1.5)*, then*

$$I_3 \geq -C s_0^{\frac{2}{n}} \psi_U(t) - C s_0^{1-b+\frac{2}{n}} \tag{2.3}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, \min\{1, T_{\max}\})$.

Proof. As in [10, estimate (4.5)], by integration by parts we have

$$I_3 \geq -(b+1) \frac{\gamma\chi_1 n}{d_3} s_0 \int_0^{s_0} s^{-b-1} U W\, ds$$

for all $t \in (0, T_{\max})$. Furthermore, by the structure of the third equation in (1.1), a result similar to [10, Lemma 4.8] is established, so that we attain (2.3). □

With regard to Lemma 2.3, by virtue of the structure of the third equation in (1.1), a term including $\psi_V(t)$ does not appear unlike [6, Lemma 4.4], where $\psi_V(t) := \int_0^{s_0} s^{-b}(s_0 - s)V(s,t)V_s(s,t)\,ds$. Thus we derive a differential inequality for only ϕ_U to show blow-up.

Lemma 2.4. *Assume that $\kappa_1 > 1$ and $\lambda_1 > 1$ satisfy* (1.2). *Then there exists $b \in \big(1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\}\big)$ with the property that for all $L > 0$ and $M_0 > 0$ one can find $C_1 > 0$, $C_2 > 0$ and $s_1 \in (0, R^n)$ such that if u_0, v_0 satisfy* (1.3) *and $\int_\Omega (u_0(x) + v_0(x))\,dx = M_0$ as well as* (1.5), *then*

$$\phi_U'(t) \geq C_1 s_0^{-(3-b)} \phi_U^2(t) - C_2 s_0^{1-b+\frac{2}{n}} \tag{2.4}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{1, T_{\max}\})$.

Proof. Let us fix $\varepsilon > 0$ such that

$$2\varepsilon \leq 1 - \frac{2}{n}\,. \tag{2.5}$$

Moreover, we can take $b \in \big(1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\}\big)$ such that

$$(n-1)(\max\{\kappa_1, \lambda_1\} - 1) < \frac{b}{2}\,, \tag{2.6}$$

because (1.2) ensures that $(n-1)(\min\{\kappa_1, \lambda_1\} - 1) < \frac{1}{3} = \frac{1}{2}\big(2 - \frac{4}{n}\big)$ if $n = 3$, and that $(n-1)(\min\{\kappa_1, \lambda_1\} - 1) < \frac{1}{2}$ if $n \geq 4$. Noting that (1.3), (1.5) and the condition $\int_\Omega (u_0(x) + v_0(x))\,dx = M_0$, from [6, Lemma 4.2] we can find c_1, $c_2 > 0$ such that

$$I_1 \geq -c_1 s_0^{\frac{3-b}{2} - \frac{2}{n}} \sqrt{\psi_U(t)}$$

and

$$I_4 \geq -c_2 s_0^{-(n-1)(\kappa_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)}$$

for all $t \in (0, \min\{1, T_{\max}\})$. Moreover, thanks to [6, Lemma 4.5], there exists $c_3 > 0$ satisfying

$$I_5 \geq -c_3 s_0^{-(n-1)(\lambda_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)}$$

for all $t \in (0, \min\{1, T_{\max}\})$. Hence, plugging these inequalities and Lemma 2.3 into (2.2) entails that

$$\begin{aligned}
\phi_U'(t) \geq{}& \frac{\alpha \chi_1 n}{d_3} \psi_U(t) - c_4 s_0^{\frac{2}{n}} \psi_U(t) - c_4 s_0^{1-b+\frac{2}{n}} \\
&- c_1 s_0^{\frac{3-b}{2} - \frac{2}{n}} \sqrt{\psi_U(t)} \\
&- c_2 s_0^{-(n-1)(\kappa_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)} \\
&- c_3 s_0^{-(n-1)(\lambda_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)}
\end{aligned}$$

for all $t \in (0, \min\{1, T_{\max}\})$ with some $c_4 > 0$. By Young's inequality we infer that

$$\phi_U'(t) \geq c_5\psi_U(t) - c_4 s_0^{\frac{2}{n}}\psi_U(t) \\ - c_6 s_0^{1-b+\frac{2}{n}}\left(s_0^{2-\frac{6}{n}} + 1 + s_0^{2-\frac{2}{n}-2(n-1)(\kappa_1-1)-2\varepsilon} + s_0^{2-\frac{2}{n}-2(n-1)(\lambda_1-1)-2\varepsilon}\right)$$

for all $t \in (0, \min\{1, T_{\max}\})$ with some $c_5 > 0$ and $c_6 > 0$. Now let us choose $s_1 \in (0, R^n)$ such that $c_4 s_1^{\frac{2}{n}} \leq \frac{c_5}{2}$. Noting from (2.5) and (2.6) that

$$2 - \frac{2}{n} - 2(n-1)(\min\{\kappa_1, \lambda_1\} - 1) - 2\varepsilon > 1 - b > 0\,,$$

we have from the relation $2 - \frac{6}{n} \geq 0$ that

$$\phi_U'(t) \geq \frac{c_5}{2}\psi_U(t) - c_7 s_0^{1-b+\frac{2}{n}} \tag{2.7}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{1, T_{\max}\})$ with some $c_7 > 0$, where we have used the relations $c_4 s_0^{\frac{2}{n}} < c_4 s_1^{\frac{2}{n}} \leq \frac{c_5}{2}$ and $s_0 < R^n$. Now from [10, Lemma 4.4] there exists $c_8 > 0$ satisfying that $\psi_U(t) \geq c_8 s_0^{-(3-b)}\phi_U^2(t)$ for all $t \in (0, T_{\max})$, which together with (2.7) yields (2.4). □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to Lemma 2.4, there exist $c_1 > 0$, $c_2 > 0$ and $s_1 \in (0, R^n)$ such that

$$\phi_U'(t) \geq c_1 s_0^{-(3-b)}\phi_U^2(t) - c_2 s_0^{1-b+\frac{2}{n}}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{1, T_{\max}\})$. Let us pick $s_0 \in (0, s_1)$ fulfilling

$$\sqrt{\frac{c_2}{c_1}}s_0^{\frac{1}{n}} + \frac{2}{c_1}s_0 \leq \frac{\widetilde{M}_0}{2^{3-b}\omega_n}\,.$$

Then it follows that

$$\frac{\widetilde{M}_0}{2^{3-b}\omega_n}s_0^{2-b} \geq \sqrt{\frac{c_2}{c_1}}s_0^{2-b+\frac{1}{n}} + \frac{2}{c_1}s_0^{3-b}\,.$$

Moreover, put

$$r_\star := \left(\frac{s_0}{4}\right)^{\frac{1}{n}} \in (0, R)\,,$$

and select initial data u_0, v_0 satisfy (1.3), (1.4) and (1.5). By [10, estimate (5.5)], we can verify that

$$\phi_U(0) \geq \frac{\widetilde{M}_0}{2^{3-b}\omega_n}s_0^{2-b}.$$

As in the proof of [4, Lemma 4.6] (with $d_1(s_0) = c_1 s_0^{-(3-b)}$, $d_2(s_0) = c_2 s_0^{1-b-\frac{2}{n}}$ and $\phi(s_0) = \frac{\widetilde{M}_0}{2^{3-b}\omega_n}s_0^{2-b}$), we can derive that $T_{\max} \leq \frac{1}{2}$. Therefore, from (2.1) we arrive at (1.6), which completes the proof. □

3. Proof of Theorem 1.3

In the following, we let $T \in (0, \infty]$ and let (u, v, w) be a classical solution of (1.1) on $[0, T)$ with $u_0, v_0 \in C^0(\overline{\Omega})$ being nonnegative. Now we put

$$\mathcal{L}\widetilde{v} := d_2 \Delta \widetilde{v} - \chi_2 \nabla \widetilde{v} \cdot \nabla w$$

for $\widetilde{v} \in C^2(\overline{\Omega})$. Then we note from the second and third equations in (1.1) that

$$\begin{aligned}\frac{\partial v}{\partial t} &= \mathcal{L}v - \chi_2 v \Delta w + \mu_2 v(1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1}) \\ &= \mathcal{L}v + \frac{\alpha\chi_2}{d_3} uv - \frac{\gamma\chi_2}{d_3} vw + \mu_2 v(1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1}) \\ &\le \mathcal{L}v + \frac{\alpha\chi_2}{d_3} uv + \mu_2 v - a_2\mu_2 u^{\lambda_2 - 1} v - \mu_2 v^{\kappa_2} \end{aligned} \tag{3.1}$$

for all $x \in \Omega$ and $t \in (0, T)$. By using this inequality we will show the following two lemmas which play an important role in the proof of Theorem 1.3.

Lemma 3.1. *The solution* (u, v, w) *satisfies that if* $\lim_{t\nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty$, *then* $\lim_{t\nearrow T} \|v(\cdot, t)\|_{L^\infty(\Omega)} < \infty$.

Proof. Assume that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \le c_1$ for all $t \in (0, T)$ with some $c_1 > 0$. Then, from (3.1) we see that

$$\frac{\partial v}{\partial t} \le \mathcal{L}v + \Big(\frac{\alpha\chi_2}{d_3} c_1 + \mu_2\Big) v - \mu_2 v^{\kappa_2}$$

for all $x \in \Omega$ and $t \in (0, T)$. Let us next choose $\overline{v} \in (0, \infty)$ such that $\|v_0\|_{L^\infty(\Omega)} \le \overline{v}$, and denote by $y\colon [0, \infty) \to \mathbb{R}$ the function solving

$$\begin{cases} y'(t) = \left(\dfrac{\alpha\chi_2}{d_3} c_1 + \mu_2\right) y(t) - \mu_2 y^{\kappa_2}(t), & t > 0, \\ y(0) = \overline{v}. \end{cases}$$

Then, by a comparison principle, we can observe that for all $x \in \Omega$ and $t \in (0, T)$,

$$v(x, t) \le y(t) \le \max\left\{ \left(\frac{\frac{\alpha\chi_2}{d_3} c_1 + \mu_2}{\mu_2} \right)^{\frac{1}{\kappa_2 - 1}}, \overline{v} \right\} =: c_2$$

holds, which implies that $\|v(\cdot, t)\|_{L^\infty(\Omega)} \le c_2$ for all $t \in (0, T)$. $\square$

Lemma 3.2. *Assume that* $\lambda_2 \ge 2$ *and* χ_2 *satisfies* (1.7). *Then*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le C \tag{3.2}$$

holds for all $t \in (0, T)$ *with some* $C > 0$.

Proof. When $\lambda_2 = 2$, by (3.1) and the fact $a_2\mu_2 - \frac{\alpha\chi_2}{d_3} > 0$ (from (1.7)) we have

$$\begin{aligned}\frac{\partial v}{\partial t} &\le \mathcal{L}v + \mu_2 v - \mu_2 v^{\kappa_2} - \left(a_2\mu_2 - \frac{\alpha\chi_2}{d_3}\right) uv \\ &\le \mathcal{L}v + \mu_2 v - \mu_2 v^{\kappa_2}\end{aligned}$$

for all $x \in \Omega$ and $t \in (0, T)$. Thus a comparison principle yields (3.2). On the other hand, in the case that $\lambda_2 > 2$, Young's inequality enables us to find some constant $c_1 > 0$ satisfying $\frac{\partial v}{\partial t} \le \mathcal{L}v + (c_1 + \mu_2)v - \mu_2 v^{\kappa_2}$ for all $x \in \Omega$ and $t \in (0, T)$. Similarly, a comparison principle yields (3.2), which concludes the proof. □

Proof of Theorem 1.3. Lemma 3.1 directly entails Theorem 1.3 (i). We next show Theorem 1.3 (ii). Theorem 1.1 asserts that there are initial data $u_0, v_0 \in C^0(\overline{\Omega})$ and $T^* < \infty$ such that the corresponding solution (u, v, w) of (1.1) on $[0, T^*)$ satisfies that $\lim_{t \nearrow T^*}(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty$. Then, noticing from Lemma 3.2 with $T = T^*$ that $\sup_{t\in(0,T^*)} \|v(\cdot, t)\|_{L^\infty(\Omega)} < \infty$ holds, we see that $\lim_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ holds, which means that Theorem 1.3 (ii) holds. □

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References

[1] Baldelli, L., Filippucci, R., *A priori estimates for elliptic problems via Liouville type theorems*, Discrete Contin. Dyn. Syst. Ser. S **13** (7) (2020), 1883–1898.

[2] Black, T., Lankeit, J., Mizukami, M., *On the weakly competitive case in a two-species chemotaxis model*, IMA J. Appl. Math. **81** (5) (2016), 860–876.

[3] Cieślak, T., Winkler, M., *Finite-time blow-up in a quasilinear system of chemotaxis*, Nonlinearity **21** (5) (2008), 1057–1076.

[4] Fuest, M., *Approaching optimality in blow-up results for Keller-Segel systems with logistic-type dampening*, NoDEA Nonlinear Differential Equations Appl. **28** (16) (2021), 17 pp.

[5] Mizukami, M., *Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type*, Math. Methods Appl. Sci. **41** (1) (2018), 234–249.

[6] Mizukami, M., Tanaka, Y., Yokota, T., *Can chemotactic effects lead to blow-up or not in two-species chemotaxis-competition models?*, Z. Angew. Math. Phys. **73** (239) (2022), 25 pp.

[7] Stinner, C., Tello, J.I., Winkler, M., *Competitive exclusion in a two-species chemotaxis model*, J. Math. Biol. **68** (7) (2014), 1607–1626.

[8] Tello, J.I., Winkler, M., *Stabilization in a two-species chemotaxis system with a logistic source*, Nonlinearity **25** (5) (2012), 1413–1425.

[9] Tu, X., Qiu, S., *Finite-time blow-up and global boundedness for chemotaxis system with strong logistic dampening*, J. Math. Anal. Appl. **486** (1) (2020), 25 pp.

[10] Winkler, M., *Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation*, Z. Angew. Math. Phys. **69** (69) (2018), 40 pp.

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EXISTENCE OF BLOW-UP SOLUTIONS FOR A DEGENERATE PARABOLIC-ELLIPTIC KELLER–SEGEL SYSTEM WITH LOGISTIC SOURCE

Yuya Tanaka

Abstract. This paper deals with existence of finite-time blow-up solutions to a degenerate parabolic–elliptic Keller–Segel system with logistic source. Recently, finite-time blow-up was established for a degenerate Jäger–Luckhaus system with logistic source. However, blow-up solutions of the aforementioned system have not been obtained. The purpose of this paper is to construct blow-up solutions of a degenerate Keller–Segel system with logistic source.

1. Introduction and main result

In this paper we consider the quasilinear degenerate Keller–Segel system with logistic source,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u^m - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa\,, & x \in \Omega,\ t > 0\,,\\
0 = \Delta v - v + u\,, & x \in \Omega,\ t > 0\,,\\
\dfrac{\partial u^m}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0\,, & x \in \partial\Omega,\ t > 0\,,\\
u(x,0) = u_0(x)\,, & x \in \Omega\,,
\end{cases}
\tag{1.1}
$$

where $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball with some $R > 0$; $m \geq 1$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$; ν is the outward normal vector to $\partial\Omega$; $u_0 \in L^\infty(\Omega)$ is nonnegative and radially symmetric. This system describes a situation such that a cellular slime moves towards higher concentrations of the chemical substance.

In the case $m = 1$, Winkler [10] obtained initial data leading to finite-time blow-up under a smallness condition for $\kappa > 1$ in three- or higher-dimensional cases. In the case $m \in \left[1, 2 - \frac{2}{n}\right)$, for the system such that the diffusion term is replaced with $\nabla \cdot ((u+1)^{m-1} \nabla u)$, Black, Fuest and Lankeit showed that solutions blow up in finite time under the condition that $\kappa < 1 + \min\left\{\frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)}\right\}$ in [1, Theorem 1.2 (ii)]. On the other hand, a difficulty is caused in (1.1) by the degenerate diffusion term Δu^m because in the case of nondegenerate diffusion

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classical solutions can be considered, whereas in the case of degenerate diffusion classical solutions are not always obtained. In such circumstances, it had not been clear whether blow-up of solutions to (1.1) occurs.

Regarding this difficulty, existence of blow-up solutions was recently established in [8] for the following Jäger–Luckhaus system with $\varepsilon = 0$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u+\varepsilon)^m - \chi\nabla\cdot(u\nabla v) + \lambda u - \mu u^\kappa\,, & x\in\Omega,\ t>0\,,\\ 0 = \Delta v - \overline{M}(t) + u\,, & x\in\Omega,\ t>0\,, \end{cases}$$

where $\overline{M}(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$. This system was studied in [1,3,7,9]; in the case $m=1$ and $\varepsilon=0$, finite-time blow-up was shown under smallness conditions for κ in the three- and higher-dimensional cases in [1,9] (in the case $\overline{M}(t)=v$, see [10]); these conditions were improved in [3]; in the case $m\neq 1$, the condition $\kappa<\min\left\{2,\frac{n}{2}\right\}$ in [3] was generalized to the condition that $\kappa<\min\left\{2,(2-m)\frac{n}{2}\right\}$ if $m\geq 0$ or $\kappa<\min\{2,n\}$ if $m<0$ in [7]. After that, in the case of degenerate diffusion ($\varepsilon=0$), finite-time blow-up solutions was constructed in a framework of weak solutions in [8].

In contrast, for the degenerate Keller–Segel system with logistic source there is no result on blow-up. The purpose is to prove existence of blow-up solutions to (1.1) in a framework of weak solutions under the same condition as in [1, Theorem 1.2 (ii)]. Referring to the method in [8], we introduce *moment solutions* as follows.

Definition 1.1. Let $T\in(0,\infty]$. A pair (u,v) of nonnegative and radially symmetric functions defined on $\Omega\times(0,T)$ is called a *moment solution* of (1.1) on $[0,T)$ if

(i) $u\in C^0_{\mathrm{w}-\star}([0,T);L^\infty(\Omega))\cap L^\infty_{\mathrm{loc}}([0,T);L^\infty(\Omega))$,
$u^m\in L^2(0,T;H^1(\Omega))$ if $T<\infty$; $u^m\in L^2_{\mathrm{loc}}([0,T);H^1(\Omega))$ if $T=\infty$,
$v\in L^\infty_{\mathrm{loc}}([0,T);H^1(\Omega))$,

(ii) for all $\varphi\in L^2(0,T;H^1(\Omega))\cap W^{1,1}(0,T;L^2(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot)\subset[0,T)$ (a.a. $x\in\Omega$),

$$\begin{aligned} &\int_0^T\int_\Omega(\nabla u^m\cdot\nabla\varphi-\chi u\nabla v\cdot\nabla\varphi-(\lambda u-\mu u^\kappa)\varphi-u\varphi_t)\,dxdt\\ &\qquad=\int_\Omega u_0(x)\varphi(x,0)\,dx,\\ &\int_0^T\int_\Omega(\nabla v\cdot\nabla\varphi+v\varphi-u\varphi)\,dxdt=0, \end{aligned}$$

(iii) (u,v) satisfies the following moment inequality:

$$\phi(t)-\phi(0)\geq K\int_0^t\phi^2(\tau)\,d\tau\quad\text{for all } t\in(0,T),$$

where

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds \quad \text{for } t \in (0,T),$$

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,d\rho \quad \text{for } s \in [0,R^n] \text{ and } t \in (0,T)$$

with some $s_0 \in (0,R^n)$, $\gamma \in (0,1)$ and $K = K(R,m,\chi,\mu,\kappa,\gamma,s_0) > 0$.

We next define *maximal moment solutions*, which are ensured by Zorn's lemma as in the proof of [6, Lemma 2.4].

Definition 1.2. Define the set $\mathcal{S}$ as

$$\mathcal{S} := \{(T,u,v) \mid T \in (0,\infty],\ (u,v) \text{ is a moment solution of (1.1) on } [0,T)\},$$

which is not empty as shown in the proof of Theorem 1.3, with the order relation $\preceq$ given by

$$(T_1,u_1,v_1) \preceq (T_2,u_2,v_2) \ :\Longleftrightarrow\ T_1 \le T_2,\ u_2|_{(0,T_1)} = u_1,\ v_2|_{(0,T_1)} = v_1\,.$$

Then Zorn's lemma assures some maximal element $(T_{\max},u,v) \in \mathcal{S}$, and (u,v) is called a *maximal moment solution* of (1.1) on $[0,T_{\max})$.

Now we state the main theorem, in which (1.2) is the same condition in [1, Theorem 1.2 (ii)].

Theorem 1.3. *Let* $m \in \left[1, 2-\frac{2}{n}\right)$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ *and* $\kappa > 1$. *Assume that*

$$\kappa < 1 + \min\left\{\frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)}\right\}. \tag{1.2}$$

Then for all $M_0 > 0$ *and* $L > 0$ *there exist* $\sigma_0 > 0$, $\eta_0 \in (0,M_0)$ *and* $r_\star \in (0,R)$ *with the following property*: *If*

$$u_0 \in L^\infty(\Omega) \text{ is nonnegative and radially symmetric} \tag{1.3}$$

and

$$\int_\Omega u_0(x)\,dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \ge M_0 - \eta_0 \tag{1.4}$$

as well as

$$u_0(x) \le L|x|^{-p} \quad \text{for a.a. } x \in \Omega\,, \tag{1.5}$$

where $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, *then there exists a moment solution of* (1.1) *on* $[0,T_{\max})$ *which blows up at* $T_{\max} < \infty$ *in the sense that*

$$\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty\,.$$

In order to prove Theorem 1.3, we will construct a moment solution. To this end, we derive a moment inequality for a solution of a problem approximate to (1.1). The key to obtaining the inequality is to establish a pointwise estimate for an approximate solution (Lemma 2.1).

2. Proof of Theorem 1.3

To show finite-time blow-up of solutions to (1.1), for the present we focus on the following approximate problem:

$$
\begin{cases}
\dfrac{\partial u_\varepsilon}{\partial t} = \Delta(u_\varepsilon+\varepsilon)^m - \chi\nabla\cdot(u_\varepsilon\nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa\,, & x\in\Omega,\ t>0\,,\\
0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon\,, & x\in\Omega,\ t>0\,,\\
\dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0\,, & x\in\partial\Omega,\ t>0\,,\\
u_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, & x\in\Omega\,,
\end{cases}
\tag{2.1}
$$

where $\varepsilon\in(0,1)$, and $u_{0\varepsilon} := (\rho_\varepsilon * \overline{u_0})|_{\overline{\Omega}}$ with

$$
\overline{u_0}(x) := \begin{cases} u_0(x) & \text{if } x\in\Omega\,,\\ 0 & \text{otherwise,}\end{cases}
$$

$$
\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\left(\int_{\mathbb{R}^n}\rho(y)\,dy\right)^{-1}\rho\left(\frac{x}{\varepsilon}\right)\,, \quad \rho(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x|<1\,,\\ 0 & \text{if } |x|\ge 1\,.\end{cases}
$$

We note that the solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) on $[0,T_\varepsilon)$ is obtained by a standard fixed point argument (see e.g. [11]), where T_ε is the maximal existence time for the solution $(u_\varepsilon, v_\varepsilon)$. We know that ρ_ε is nonnegative and radially symmetric. Thus, for the initial data u_0 satisfying (1.3), $u_{0\varepsilon}$ is nonnegative and radially symmetric. Moreover, we see that $u_{0,\varepsilon}\to u_0$ in $L^1(\Omega)$ as $\varepsilon\searrow 0$ and that on passing to a subsequence if necessary, $u_{0,\varepsilon}\to u_0$ a.a. $x\in\Omega$ as $\varepsilon\searrow 0$. Furthermore, as in [5, Section 2.2] and [8, Lemmas 2.2 and 2.3], we can find $T_0>0$ and $K_0>0$ such that for all $\varepsilon\in(0,1)$,

$$
T_0\le T_\varepsilon \quad\text{and}\quad \sup_{t\in(0,T_0)}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le K_0\,. \tag{2.2}
$$

In order to establish a moment inequality, an estimate for u_ε is a cornerstone. In a degenerate Jäger–Luckhaus system with logistic source the key is radial monotonicity of an approximate solution (see [8, Lemma 2.7]). However, in our case it is difficult to obtain this property due to the structure of the second equation in (2.1). For this reason, instead of monotonicity, based on [10, Lemma 3.3] and [1, lemma 5.2], we show a pointwise estimate for u_ε.

Lemma 2.1. *Let $m\in\left[1,2-\frac{2}{n}\right)$, $\chi>0$, $\lambda>0$, $\mu>0$, $\kappa>1$, $M_0>0$ and $L>0$. Moreover, for any $\sigma_0>0$, set $p:=\frac{n(n-1)}{(m-1)n+1}+\sigma_0$ and assume that u_0 satisfies (1.3), (1.5) and $\int_\Omega u_0(x)\,dx = M_0$ and that there exist $T_0>0$ and $K_0>0$ fulfilling (2.2). Then there exist $\varepsilon_0\in(0,1)$ and $L_1>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$,*

$$
u_\varepsilon(x,t)\le L_1|x|^{-p} \tag{2.3}
$$

for all $x\in\Omega$ and $t\in(0,T_0)$.

Proof. Putting $\widetilde{u}_\varepsilon(x,t) := e^{-\lambda t}u_\varepsilon(x,t)$, we can derive from (2.1) that

$$
\begin{cases}
\dfrac{\partial \widetilde{u}_\varepsilon}{\partial t} \le \nabla\cdot\left(m(e^{\lambda t}\widetilde{u}_\varepsilon+\varepsilon)^{m-1}\nabla\widetilde{u}_\varepsilon - \chi\widetilde{u}_\varepsilon\nabla v_\varepsilon\right), & x\in\Omega,\ t>0,\\
(m(e^{\lambda t}\widetilde{u}_\varepsilon+\varepsilon)^{m-1}\nabla\widetilde{u}_\varepsilon - \chi\widetilde{u}_\varepsilon\nabla v_\varepsilon)\cdot\nu = 0\,, & x\in\partial\Omega,\ t>0\,,\\
\widetilde{u}_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, & x\in\Omega\,.
\end{cases} \tag{2.4}
$$

Next, let $\sigma_0 > 0$. We can take $\xi > 0$ small enough and $\varepsilon_0 \in (0,1)$ such that $u_{0,\varepsilon} \le u_0 + \xi$ for a.a. $x\in\Omega$ and all $\varepsilon\in(0,\varepsilon_0)$. By virtue of this inequality, (1.5) and the fact that $|x|\le R$, it follows that

$$
u_{0,\varepsilon} \le L|x|^{-p} + \xi R^p|x|^{-p} = (L+\xi R^p)|x|^{-p} \tag{2.5}
$$

for all $x\in\Omega$ and $\varepsilon\in(0,\varepsilon_0)$. Also, from the condition $\int_\Omega u_0\,dx = M_0$, we obtain that

$$
\int_\Omega u_{0,\varepsilon}\,dx \le M_0 + \xi|\Omega| =: \widetilde{M_0} \tag{2.6}
$$

for all $\varepsilon\in(0,\varepsilon_0)$. On the other hand, integrating the first equation in (2.1) over Ω, we infer that

$$
\frac{d}{dt}\int_\Omega u_\varepsilon\,dx = \lambda\int_\Omega u_\varepsilon\,dx - \mu\int_\Omega u_\varepsilon^\kappa\,dx \le \lambda\int_\Omega u_\varepsilon\,dx\,,
$$

which ensures that

$$
\int_\Omega u_\varepsilon\,dx \le e^{\lambda t}\int_\Omega u_{0,\varepsilon}\,dx \le e^{\lambda T_0}\widetilde{M_0} \tag{2.7}
$$

for all $t\in(0,T_0)$. Moreover, we see from the second equation in (2.1) that

$$
r^{n-1}(v_\varepsilon)_r = \int_0^r \rho^{n-1}v_\varepsilon\,d\rho - \int_0^r \rho^{n-1}u_\varepsilon\,d\rho \le \frac{1}{\omega_n}\left(\int_\Omega v_\varepsilon\,dx + \int_\Omega u_\varepsilon\,dx\right)
$$

for all $r\in(0,R)$ and $t\in(0,T_\varepsilon)$, where $\omega_n := n|B_1(0)|$. Here, since we integrate the second equation in (2.1) over Ω to guarantee that

$$
\int_\Omega u_\varepsilon\,dx = \int_\Omega v_\varepsilon\,dx\,,
$$

the above inequality and (2.7) yields

$$
r^{n-1}(v_\varepsilon)_r \le \frac{2}{\omega_n}e^{\lambda T_0}\widetilde{M_0} =: c_1
$$

for all $r\in(0,R)$ and $t\in(0,T_0)$. Picking $\theta_0 > n$ so large satisfying $m-1 > \frac{1}{\theta_0} - \frac{1}{n}$ and $p = \frac{n(n-1)}{(m-1)n+1} + \sigma_0 > \frac{(n-1)}{(m-1)+\frac{1}{n}-\frac{1}{\theta_0}}$, we have

$$
\begin{aligned}
\int_\Omega |x|^{\theta_0(n-1)}|\nabla v_\varepsilon(x,t)|^{\theta_0}\,dx &= \omega_n\int_0^R r^{(\theta_0+1)(n-1)}|(v_\varepsilon)_r(\rho,t)|^{\theta_0}\,d\rho\\
&\le \frac{1}{n}\omega_n c_1^{\theta_0}R^n
\end{aligned}
$$

for all $t\in(0,T_0)$. From this inequality and (2.4)–(2.6) we therefore can apply [2, Theorem 1.1] to obtain (2.3). □

We next derive a moment inequality for an approximate solution of (2.1).

Lemma 2.2. *Let $m \in \left[1, 2-\frac{2}{n}\right)$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that* (1.2) *is satisfied and that there exist $T_0 > 0$ and $K_0 > 0$ fulfilling* (2.2)*. Then for all $M_0 > 0$ and $L > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property*: *If u_0 satisfies* (1.3)–(1.5) *with some $\sigma_0 > 0$, then there exist $\varepsilon_0 \in (0,1)$ and $K > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq K \int_0^t \phi_\varepsilon^2(\tau)\, d\tau \tag{2.8}$$

for all $t \in (0, T_0)$, where

$$\phi_\varepsilon(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s,t)\, ds \quad \textit{for } t \in (0, T_\varepsilon),$$

$$w_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_\varepsilon(\rho, t)\, d\rho \quad \textit{for } s \in [0, R^n] \textit{ and } t \in (0, T_\varepsilon)$$

with some $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$.

Proof. Let us first put $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, where we choose $\sigma_0 > 0$ sufficiently small fulfilling that $\kappa < 1 + \min\left\{\frac{n}{2p}, \frac{n-2}{p} - (m-1)\right\}$. Furthermore, we select $\gamma \in \left(\max\left\{\frac{2p\kappa}{n}, 1 - \frac{2}{n} - \frac{p}{n}(m-1)\right\}, \min\left\{2 - \frac{4}{n} - \frac{2p}{n}(m-1), 1\right\}\right)$. Also, noting that $u_{0,\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \searrow 0$, we fix $\xi_0 > 0$ small enough and pick $\varepsilon_0 \in (0,1)$ given by Lemma 2.1 satisfying

$$\int_\Omega u_{0,\varepsilon} \geq M_0 - \xi_0$$

for all $\varepsilon \in (0, \varepsilon_0)$. In order to obtain (2.8), we shall show that there exist $c_1 > 0$, $c_2 > 0$, $\theta \in (0,2)$ and $s_1 \in (0, R^n)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $s_0 \in (0, s_1)$,

$$\phi_\varepsilon'(t) \geq c_1 s_0^{\gamma-3} \phi_\varepsilon^2(t) - c_2 s_0^{3-\gamma-\theta} \tag{2.9}$$

for all $t \in (0, T_0)$. By straightforward computations we have from (2.1) and the definitions of w_ε and ϕ_ε that

$$\begin{aligned}
\phi_\varepsilon'(t) \geq{}& mn^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)\left(n(w_\varepsilon)_s + \varepsilon\right)^{m-1} (w_\varepsilon)_{ss}\, ds \\
&+ n\int_0^{s_0} s^{-\gamma}(s_0-s)(w_\varepsilon)_s w_\varepsilon\, ds - n\int_0^{s_0} s^{-\gamma}(s_0-s)(w_\varepsilon)_s z_\varepsilon\, ds \\
&- n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s)\left\{\int_0^s (w_\varepsilon)_s^\kappa\, d\sigma\right\} ds
\end{aligned}$$

for all $t \in (0, T_\varepsilon)$, where $z_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v_\varepsilon(\rho,t)\, d\rho$ for $s \in [0, R^n]$ and $t \in (0, T_\varepsilon)$. Here, we note that we can apply [1, Lemmas 3.5, 3.8 and 3.9] to the second, third and fourth terms on the right-hand side of the above inequality. Thus, in order to derive (2.9), it is sufficient to estimate the first term. To this end, we will find $c_3 > 0$ independent of ε such that

$$(n(w_\varepsilon)_s + \varepsilon)^m \leq c_3 s^{-\frac{p}{n}(m-1)} (w_\varepsilon)_s + c_3 \tag{2.10}$$

for all $s \in (0, R^n)$ and $t \in (0, T_0)$, which is used after integration by parts in estimating the first term. By means of (2.3), it follows that for any $\varepsilon \in (0, \varepsilon_0)$, $w_\varepsilon(s,t) = \frac{1}{n} u_\varepsilon(s^{\frac{1}{n}}, t) \le c_4 s^{-\frac{p}{n}}$ for all $s \in (0, R^n)$ and $t \in (0, T_0)$, where $c_4 := \frac{L_1}{n}$. From this inequality and the fact that $s \le R^n$ as well as $\varepsilon < 1$, we have

$$\begin{aligned}(n(w_\varepsilon)_s + \varepsilon)^m &\le 2^{m-1}(n^m (w_\varepsilon)_s^m + \varepsilon^m) \\ &\le 2^{m-1} n^m c_4^{m-1} s^{-\frac{p}{n}(m-1)} (w_\varepsilon)_s + 2^{m-1}\end{aligned}$$

for all $s \in (0, R^n)$ and $t \in (0, T_0)$, which means that (2.10) holds. Therefore, by [1, Lemmas 3.5, 3.6 (i), 3.8, 3.9 and 3.11] we can take $c_5 > 0$, $c_6 > 0$, $\theta \in (0, 2)$ and $s_1 \in (0, R^n)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $s_0 \in (0, s_1)$,

$$\phi_\varepsilon'(t) \ge c_5 s_0^{\gamma-3} \phi_\varepsilon^2(t) - c_6 s_0^{3-\gamma-\theta}$$

for all $t \in (0, T_0)$. Furthermore, arguing as in [8, Proof of Proposition 2], we pick $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ such that for any u_0 satisfying (1.3)–(1.5), the inequality $\phi_\varepsilon'(t) \ge \frac{c_5}{2} s_0^{\gamma-3} \phi_\varepsilon^2(t)$ holds for all $\varepsilon \in (0, \varepsilon_0)$, $s_0 \in (0, s_1)$ and $t \in (0, T_0)$, which implies (2.8). □

We are now in the position to show Theorem 1.3.

Proof of Theorem 1.3. We can derive results similar to [8, Lemmas 2.4 and 2.5] since the second equation in (2.1) entails that $\Delta v_\varepsilon = v_\varepsilon - u_\varepsilon \ge -u_\varepsilon$. Thus, as in the proof of [4, Lemma 5.3] we can choose subsequence $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that $u \in L^\infty(0, T_0; L^\infty(\Omega))$, $u^m \in L^2(0, T_0; H^1(\Omega))$, $v \in L^\infty(0, T_0; W^{1,\infty}(\Omega))$ and

$$u_{\varepsilon_k} \to u \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)), \tag{2.11}$$

$$u_{\varepsilon_k} \to u \quad \text{in } C^0([\delta, T_0]; L^q(\Omega)) \quad \text{for all } \delta \in (0, T_0) \text{ and } q \in [1, \infty)\,, \tag{2.12}$$

$$\nabla(u_{\varepsilon_k} + \varepsilon)^m \to \nabla u^m \quad \text{weakly in } L^2(0, T_0; L^2(\Omega))\,, \tag{2.13}$$

$$\nabla v_{\varepsilon_k} \to \nabla v \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)) \tag{2.14}$$

as $k \to \infty$. Moreover, thanks to Lemma 2.2, we can take the initial data u_0 leading to (2.8). Thus, by (2.11)–(2.14), we can show that (u, v) fulfills (i)–(iii) with $T = T_0$ in Definition 1.1 as in [8, Proof of Proposition 1]. Hence, from Definition 1.2 there exists a maximal moment solution (u, v) on $(0, T_{\max})$. In particular, we have

$$\phi(t) - \phi(0) \ge K \int_0^t \phi^2(\tau)\, d\tau$$

for all $t \in (0, T_{\max})$ with some $K > 0$. Putting $\Phi(t) := \int_0^t \phi^2(\tau)\, d\tau + \frac{\phi(0)}{K}$ for $t \in (0, T_{\max})$, we see that $\Phi \in C^0([0, T_{\max}) \cap C^1((0, T_{\max}))$ and from the above inequality that $\Phi'(t) \ge K^2 \Phi^2(t)$ for all $t \in (0, T_{\max})$, which yields

$$t \le \frac{1}{K^2}\left(-\frac{1}{\Phi(t)} + \frac{1}{\Phi(0)}\right) \le \frac{1}{K^2 \Phi(0)}$$

for all $t \in (0, T_{\max})$. This proves $T_{\max} \le \frac{1}{K^2 \Phi(0)} < \infty$. By an extension argument as in [8, Proof of Theorem 1.1] we can obtain $\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$, which concludes the proof. □

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References

[1] Black, T., Fuest, M., Lankeit, J., *Relaxed parameter conditions for chemotactic collapse in logistic-type parabolic-elliptic Keller-Segel systems*, Z. Angew. Math. Phys. **72** (96) (2021), 23 pp.

[2] Fuest, M., *Blow-up profiles in quasilinear fully parabolic Keller-Segel systems*, Nonlinearity **33** (5) (2020), 2306–2334.

[3] Fuest, M., *Approaching optimality in blow-up results for Keller-Segel systems with logistic-type dampening*, NoDEA Nonlinear Differential Equations Appl. **28** (16) (2021), 17 pp.

[4] Ishida, S., Yokota, T., *Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type*, J. Differential Equations **252** (2) (2012), 1421–1440.

[5] Ishida, S., Yokota, T., *Blow-up in finite or infinite time for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type*, Discrete Contin. Dyn. Syst. Ser. B **18** (10) (2013), 2569–2596.

[6] Lankeit, J., *Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion*, J. Differential Equations **262** (7) (2017), 4052–4084.

[7] Tanaka, Y., *Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production*, J. Math. Anal. Appl. **506** (2022), 29 pp., no. 125654.

[8] Tanaka, Y., Yokota, T., *Finite-time blow-up in a quasilinear degenerate parabolic–elliptic chemotaxis system with logistic source and nonlinear production*, Discrete Contin. Dyn. Syst. Ser. B **28** (1) (2023), 262–286.

[9] Winkler, M., *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl. **384** (2) (2011), 261–272.

[10] Winkler, M., *Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation*, Z. Angew. Math. Phys. **69** (69) (2018), 40 pp.

[11] Winkler, M., Djie, K.C., *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect*, Nonlinear Anal. **72** (2) (2010), 1044–1064.

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STABILITY WITH RESPECT TO DOMAIN OF THE LOW MACH NUMBER LIMIT OF COMPRESSIBLE HEAT-CONDUCTING VISCOUS FLUID

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Abstract. We investigate the asymptotic limit of solutions to the Navier-Stokes-Fourier system with the Mach number proportional to a small parameter $\varepsilon \to 0$, the Froude number proportional to $\sqrt{\varepsilon}$ and when the fluid occupies large domain with spatial obstacle of rough surface varying when $\varepsilon \to 0$. The limit velocity field is solenoidal and satisfies the incompressible Oberbeck–Boussinesq approximation. Our studies are based on weak solutions approach and in order to pass to the limit in a convective term we apply the spectral analysis of the associated wave propagator (Neumann Laplacian) governing the motion of acoustic waves.

1. Introduction and formulation of the problem

The Oberbeck-Boussinesq approximation is a mathematical model of a stratified flow, where the fluid is assumed to be incompressible and yet convecting a diffusive quantity creating positive and negative buoyancy force. Then the system of equations reads:

$$\mathrm{div}_x\, \boldsymbol{U} = 0\,, \tag{OB^1}$$

$$\overline{\varrho}\left(\partial_t \boldsymbol{U} + \mathrm{div}_x\,(\boldsymbol{U}\otimes\boldsymbol{U})\right) + \nabla_x P = \mu\Delta\boldsymbol{U} + r\nabla_x F\,, \tag{OB^2}$$

$$\overline{\varrho}c_p\left(\partial_t\Theta + \mathrm{div}_x\,(\boldsymbol{U}\Theta)\right) - \kappa(\overline{\vartheta})\Delta\Theta - \overline{\varrho}\overline{\vartheta}\alpha\mathrm{div}_x\,(F\boldsymbol{U}) = 0\,, \tag{OB^3}$$

$$r + \overline{\varrho}\alpha\Theta = 0\,, \tag{OB^4}$$

where $\boldsymbol{U}$ denotes the velocity of the fluid, Θ stands for the deviation of the temperature, P is the pressure, constants μ, κ, $\overline{\varrho}$, c_p, α are positive (will be defined later). Here F stands for potential of a driving force (e.g. gravitational potential) acting on the fluid. Let us note that the density is constant in the

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Oberbeck-Boussinesq approximation except in the buoyancy force, where it is interrelated in the temperature deviation through Boussinesq relation (OB^4), (see Zeytounian [12]). Let us notice that in the OB approximation Θ is a deviation of temperature from the equilibrium rather then the temperature itself and the temperature deferences are not caused by the flow, but exists independent of the flow. Our aim is to derive the above system on an exterior domain $\mathbb{R}^3 \setminus O$ with no-slip boundary condition on the bounded obstacle O. Therefore we study stability of the rescaled compressible Navier-Stokes-Fourier system when a Mach number is proportional to a small parameter, i.e. $Ma = \varepsilon$ and $\varepsilon \to 0$, and a Froude number $Fr = \sqrt{\varepsilon}$. About other characteristic numbers like Strouhal, Reynolds, Péclet number we assume they are equal one.

We are motivated by a similar asymptotic analysis of barotropic compressible fluid flow, described by the Navier–Stokes system with a low Mach number on varying domains provided in [5]. Our aim is to extend this result to the case of heat-conducting fluids by methods developed in [3, 7]. The asymptotic analysis of complete fluid system on varying domains (but in different way then here) and with a small Mach number is considered in [11], where the author justify OB system on whole $\mathbb{R}^3$ space with concentric gravitation force.

Following [2,5] we introduce a class of admissible domains with rough (oscillating) boundaries of some obstacle. It was observed that such a choice may give rise to the no-slip boundary condition for the asymptotic limit of velocity field. In particular we assume that the given family of domains $\{\Omega_\varepsilon\}_\varepsilon$ satisfies the following hypothesis:

- D1) $\Omega_\varepsilon \subset \mathbb{R}^3$ is bounded domain with C^2 boundary for each $\varepsilon \in (0,1)$ and $\partial\Omega_\varepsilon = \partial O_\varepsilon \cup \mathcal{S}_\varepsilon$;
- D2) for simplicity we assume that the outer part of boundary $\mathcal{S}_\varepsilon$ consists of a sphere centred in the origin and of a radius $\frac{1}{\varepsilon^\delta}$ with $\delta > 0$ (i.e. the domain is sufficiently "large");
- D3) the boundary of the obstacle ∂O_ε is such that for all $\varepsilon \in (0,1)$ $O_\varepsilon \subset B_r(0) \subset B_{1/\varepsilon^\delta}(0)$ with some fixed $r > 0$;
- D4) $\mathbb{R}^3 \setminus O_\varepsilon$ satisfies the uniform α-cone condition with $\alpha > 0$ independent of ε. Namely for any $x_0 \in \partial O_\varepsilon$ there exists a unit vector $\xi_{x_0} \in \mathbb{R}^3$ s.t. $C(x,\gamma,\alpha,\xi_{x_0}) \subset (\mathbb{R}^3 \setminus O_\varepsilon)$ whenever $x \in \mathbb{R}^3 \setminus O_\varepsilon$, $|x - x_0| < \alpha$, where $C(x,\gamma,\alpha,\xi) = \{y \in \mathbb{R}^3 \,|\, 0 < |y-x| \le \alpha, (y-x)\cdot\xi > \cos(\gamma)|y-x|\}$ with vertex at x, aperture $2\gamma < \pi$, height α, and orientation given by a unit vector ξ;
- D5) for each $x_0 \in \partial O_\varepsilon$, there are two open balls $B_r(x_i) \subset \Omega_\varepsilon$, $B_r(x_j) \subset O_\varepsilon$ of radius $r > c_b\varepsilon^\beta$ (the radius r may change but sufficiently "slow") such that $\overline{B_r(x_i)} \cap \overline{B_r(x_j)} = x_0$ with $c_b > 0$, $\beta > 0$ independent of ε;
- D6) after translation and rotation of the coordinate system, a part $\Gamma \subset \partial O$ can be described by a graph of function $\gamma \in W^{1,\infty}(U)$, $U \subset \mathbb{R}^3$ and $\Gamma = \{x \in \mathbb{R}^3 \ : \ (x_1,x_2) \in U,\ x_3 = \gamma(x_1,x_2)\}$ while $\Gamma_\varepsilon = \partial O_\varepsilon \cap U \times \mathbb{R}$ are represented by $\Gamma_\varepsilon = \{x \in \mathbb{R}^3 \ : \ (x_1,x_2) \in U,\ x_3 = \gamma_\varepsilon(x_1,x_2)\}$, where

$\{\gamma_\varepsilon\}_\varepsilon$ is a bounded sequence in $W^{1,\infty}(U)$, $\gamma_\varepsilon \to \gamma$ in $C(\overline{U})$. Moreover Γ_ε are oscillating for $\varepsilon \to 0$. Namely, when we introduce a Young measure $\nu[y]$, $y \in U$, associated to the sequence $\{\nabla_y \gamma_\varepsilon\}_\varepsilon$, we suppose that supp $[\nu[\mathrm{y}]]$ contains two independent vectors in $\mathbb{R}^2$ for a.a. $y \in U$.

In certain sense $\Omega_\varepsilon \to \mathbb{R}^3 \setminus O$. We give here a mathematical justification of the Oberbeck-Boussinesq approximation of a incompressible flow on exterior domain $\Omega = \mathbb{R}^3 \setminus O$ with no-slip boundary condition on the obstacle by asymptotic analysis of weak solutions to the compressible Navier-Stokes-Fourier system in a low Mach number regime: $Ma = \varepsilon \to 0$, on a family of domain Ω_ε varying with $\varepsilon > 0$.

2. Primitive system

In the beginning of this section let us introduce some standard notation. We denote by $\langle \cdot, \cdot \rangle$ duality pairing. By $L^p(B)$ we mean the space of Lebesgue measurable functions g, where $|g|^p$ is integrable over set B. The Sobolev space of functions which derivatives are integrable up to order k in L^p we denote by $W^{k,p}$. By $\mathcal{D}^{k,p}(B)$ we set homogenous Sobolev spaces i.e. $\mathcal{D}^{k,p}(B) = \{g \in L^1_{\text{loc}}(B) \,:\, D^\alpha g \in L^p(B),\ |\alpha| = k\}$, where $k \geq 0$ and $p \geq 1$. In the whole paper c will denote generic constant which may change from line to line.

We start our considerations with a "primitive system" – the rescaled Navier-Stokes-Fourier system with a small Mach and Froude number which consists of: the continuity equation (conservation of mass), the momentum equation, the entropy balance and the total energy balance respectively

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x\, (\varrho_\varepsilon \boldsymbol{u}_\varepsilon) = 0\,, \tag{NSF$^1_\varepsilon$}$$

$$\begin{aligned} &\partial_t(\varrho_\varepsilon \boldsymbol{u}_\varepsilon) + \mathrm{div}_x\, (\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho_\varepsilon, \vartheta_\varepsilon) \\ &\qquad = \mathrm{div}_x \mathbf{S}(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon) + \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x F_\varepsilon\,, \end{aligned} \tag{NSF$^2_\varepsilon$}$$

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}_x\, (\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \boldsymbol{u}_\varepsilon) + \mathrm{div}_x \left(\frac{\boldsymbol{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon)}{\vartheta_\varepsilon} \right) = \sigma_\varepsilon\,, \tag{NSF$^3_\varepsilon$}$$

$$\frac{d}{dt} \int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_\varepsilon |\boldsymbol{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{\varepsilon} \varrho_\varepsilon F_\varepsilon \right) dx = 0\,. \tag{NSF$^4_\varepsilon$}$$

Where the viscous stress tensor satisfies the Newton rheological law and the heat flux is determined by the Fourier law:

$$\mathbf{S}(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon) = \mu(\vartheta_\varepsilon) \left(\nabla_x \boldsymbol{u}_\varepsilon + \nabla_x^T \boldsymbol{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x\, \boldsymbol{u}_\varepsilon \mathbf{Id} \right) + \eta(\vartheta_\varepsilon) \mathrm{div}_x\, \boldsymbol{u}_\varepsilon \mathbf{Id}\,,$$

$$\boldsymbol{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) = -\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon$$

with a positive heat coefficient κ and for the entropy production rate holds:

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon} \left(\varepsilon^2 \mathbf{S}_\varepsilon(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{u}_\varepsilon - \frac{\boldsymbol{q}_\varepsilon(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right). \tag{2.1}$$

The unknowns are the fluid mass density $\varrho_\varepsilon = \varrho_\varepsilon(t,x)$, the velocity field $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(t,x)\colon (0,T)\times\Omega_\varepsilon \to \mathbb{R}^3$ and absolute temperature $\vartheta_\varepsilon = \vartheta_\varepsilon(t,x)\colon (0,T)\times\Omega_\varepsilon \to \mathbb{R}$. The pressure p, the specific internal energy e and the specific entropy s are given scalar valued functions of ϱ and ϑ which are related through Gibbs' equation $\vartheta Ds = De + pD(1/\varrho)$. The system is supplemented with complete slip boundary conditions for velocity field and the boundary of physical space is thermally isolated, i.e.

$$\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0\,, \quad [\mathbf{S}(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon)\boldsymbol{n}] \times \boldsymbol{n} = 0\,, \qquad \boldsymbol{q}\cdot\boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0\,. \tag{2.2}$$

Small parameter ε in the system (NSF$^1_\varepsilon$)–(NSF$^4_\varepsilon$) results from dimensionless form of a Navier-Stokes-Fourier system and corresponds to small Mach and Froude number (Ma= ε, Fr= $\sqrt{\varepsilon}$), see [6], Klein at al. [9], Zeytounian [13]. Smallness of Mach number physically means that characteristic speed of the flow is dominated by the speed of the sound in the medium under consideration. Assumption that Fr >> Ma means that external sources of mechanical energy are small and $\frac{\mathrm{Ma}}{\mathrm{Fr}} \to 0$, what corresponds to low stratification).

2.1. **Structural restrictions.** In order to be able to use the existence result of [6] and later to build uniform estimates, we need to impose structural restrictions on the thermodynamical functions p, e, s as well as on the transport coefficients μ, η, κ. Following [6] (where the reader can find more detailed description and physical motivations) we set

(2.3)
$$p(\varrho_\varepsilon, \vartheta_\varepsilon) = \vartheta_\varepsilon^{5/2} P\left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}\right) + \frac{a}{3}\vartheta_\varepsilon^4\ ,\ a > 0\,,\ \text{where } P \in C^1[0,\infty) \cap C^2(0,\infty)\,,$$
$$P(0) = 0\,, \quad P'(Z) > 0 \ \text{ for all } \ Z \geq 0\,,$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0, \quad \lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0, \tag{2.4}$$
$$\text{and } \ \partial_\varrho p(\varrho, \vartheta) > 0\,.$$

Accordingly to Gibbs' relation, the specific internal energy and the entropy can be written in the following forms

$$e(\varrho, \vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \quad \partial_\vartheta e(\varrho,\vartheta) > 0 \tag{2.5}$$
is positive and bounded,

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \quad S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \tag{2.6}$$
for all $Z > 0$.

The transport coefficients: μ - shear viscosity, η - bulk viscosity and κ - heat conductivity are assumed to be continuously differentiable functions of the temperature

$\vartheta \in [0,\infty)$ satisfying the following growth conditions for all $\vartheta \geq 0$ and some positive constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\eta}$, $\underline{\kappa}$, $\overline{\kappa}$:

$$
\begin{aligned}
&0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \underline{\mu}(1+\vartheta)\,, \qquad 0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta)\,,\\
&0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3)\,.
\end{aligned}
\tag{2.7}
$$

2.2. **Equilibrium state and ill-prepared initial data.** Let us assume that outer force F is defined on whole space $\mathbb{R}^3$ and is independent of ε. The so-called equilibrium state (static state) for each scaled NSF$_\varepsilon$ system consist of static density $\tilde{\varrho}_\varepsilon$ and constant temperature distribution $\overline{\vartheta}$ satisfying (for a convenience we consider a static density $\tilde{\varrho}_\varepsilon$ defined on the whole space $\mathbb{R}^3$)

$$
\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\varrho}_\varepsilon \nabla_x F_\varepsilon \text{ in } \mathbb{R}^3 \qquad \text{where } \lim_{|x|\to\infty} \tilde{\varrho}_\varepsilon(x) = \overline{\varrho}\,.
$$

Hence we have

$$
\begin{aligned}
&\tilde{\varrho}_\varepsilon - \overline{\varrho} = \frac{\varepsilon}{P'(\overline{\varrho})} F + \varepsilon^2 h_\varepsilon F_\varepsilon, \text{ with } P'(\varrho) = \frac{1}{\varrho}\partial_\varrho p(\varrho, \overline{\vartheta}),\ \|h_\varepsilon\|_{L^\infty(\mathbb{R}^3)} < c\\
&\text{and} \quad |\nabla_x \tilde{\varrho}_\varepsilon(x)| \leq \varepsilon c |\nabla_x F_\varepsilon(x)| \quad \text{for } x \in \mathbb{R}^3
\end{aligned}
\tag{2.8}
$$

(notice that the above properties gives closeness of static density $\tilde{\varrho}_\varepsilon$ and constant state $\overline{\varrho}$). Since we work with weak solutions based on energy estimates and control of entropy production rate we need to assume that initial data are close to the equilibrium state. Namely initial density and initial temperature are of the following form

$$
\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon \varrho^{(1)}_{0,\varepsilon}\,, \qquad \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}
\tag{2.9}
$$

where $\overline{\vartheta} > 0$ is positive constants characterising the static distribution of the absolute temperature and

$$
\begin{aligned}
&\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty\cap L^2(\Omega_\varepsilon)} \leq c\,,\ \int \varrho^{(1)}_{0,\varepsilon}\,\mathrm{d}x = 0\,,\ \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^\infty\cap L^2(\Omega_\varepsilon)} \leq c\,,\\
&\int \vartheta^{(1)}_{0,\varepsilon}\,\mathrm{d}x = 0\,,\ \|\boldsymbol{u}_{0,\varepsilon}\|_{L^\infty\cap L^2(\Omega_\varepsilon)} \leq c \quad \text{for all } \varepsilon \in (0,1]\,.
\end{aligned}
\tag{2.10}
$$

The above uniform bounds will allow to control right hand side of total dissipation balance which is a source of uniform estimates needed to perform the limit system. Nevertheless, such a choice allow to consider nontrivial dynamics but on the other hand it causes oscillations in acoustic equation. Those will be eliminated by dispersive estimates.

2.3. **Main result.** We say that functions $\boldsymbol{U}$, Θ and r are a weak solution to the Oberbeck-Boussinesq approximation (OB) if holds: $\boldsymbol{U} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, $\Theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$,

$r \in L^\infty(0,T;L^{5/3}_{\rm loc}(\Omega))$ and

$$\mathrm{div}_x\, \boldsymbol{U} = 0 \text{ a.e. on } (0,T)\times\Omega\,,$$

$$\int_0^T \int_\Omega (\overline{\varrho}(\boldsymbol{U}\cdot\partial_t\varphi + (\boldsymbol{U}\otimes\boldsymbol{U}) : \nabla_x\varphi))\,\mathrm{d}x\,\mathrm{d}t \tag{2.11}$$
$$= -\int_\Omega \overline{\varrho}\boldsymbol{U}_0\cdot\varphi(0,\cdot)\,\mathrm{d}x + \int_0^T\int_\Omega (\mathbf{S} : \nabla_x\varphi - r\nabla_x F\cdot\varphi)\,\mathrm{d}x\,\mathrm{d}t$$

for any $\varphi \in C^\infty_c([0,T);C^\infty_c(\Omega;\mathbb{R}^3))$, where $\mathrm{div}_x\,\varphi = 0$ and $\mathbf{S} = \mu(\overline{\vartheta})(\nabla_x\boldsymbol{U}+\nabla_x\boldsymbol{U}^T)$. Moreover

$$\begin{aligned} &\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\,[\partial_t\Theta + \mathrm{div}_x\,(\Theta\boldsymbol{U})] - \mathrm{div}_x\,(\kappa(\overline{\vartheta})\nabla_x\Theta) - \overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\mathrm{div}_x\,(F\boldsymbol{U}) = 0 \\ &\qquad\qquad \text{a.e. in } (0,T)\times\Omega\,, \\ &\Theta(0,\cdot) = \Theta_0 \\ &r + \overline{\varrho}\alpha(\overline{\varrho},\overline{\vartheta})\Theta = 0 \quad \text{a.e. in } (0,T)\times\Omega\,. \end{aligned} \tag{2.12}$$

By c_p we mean specific heat at constant pressure and $c_p(\overline{\varrho},\overline{\vartheta}) = \partial_\vartheta e(\overline{\varrho},\overline{\vartheta}) + \alpha(\overline{\varrho},\overline{\vartheta})\frac{\overline{\vartheta}}{\overline{\varrho}}\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})$ by $\alpha > 0$ we mean the coefficient of thermal expansion of the fluid, $\alpha(\overline{\varrho},\overline{\vartheta}) = \frac{1}{\overline{\varrho}}\frac{\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}$, both are evaluated at the reference density $\overline{\varrho}$ and temperature $\overline{\vartheta}$. Then the main result reads as follows:

Theorem 2.1. *Let $\Omega_\varepsilon \subset \mathbb{R}^3$ be a family of domains defined by (D1)–(D5) with $\beta < \frac{1}{4}$ and $\delta > 1$. Assume that p, e, and s satisfy* (2.3)–(2.6)*, the transport coefficients μ, η and κ satisfy growth conditions* (2.7) *and driving force is determined by a scalar potential $F \in W^{1,\infty}(\mathbb{R}^3)$. Let $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the scaled Navier-Stokes-Fourier system* $(\mathrm{NSF}^1_\varepsilon)$–$(\mathrm{NSF}^4_\varepsilon)$*, on the sets $(0,T)\times\Omega_\varepsilon$, supplemented with boundary conditions* (2.2) *and initial data* (2.9) *with $\tilde{\varrho}_\varepsilon > 0$, $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$, and satisfying* (2.10) *for all $\varepsilon \in (0,1)$. Moreover we assume that*

$$\begin{aligned} &\varrho^{(1)}_{0,\varepsilon} \rightharpoonup \varrho^{(1)}_0 \text{ weakly in } L^2(\mathbb{R}^3)\,, \quad \boldsymbol{u}_{0,\varepsilon} \rightharpoonup \boldsymbol{U}_0 \text{ weakly in } L^2(\mathbb{R}^3;\mathbb{R}^3)\,, \\ &\vartheta^{(1)}_{0,\varepsilon} \rightharpoonup \vartheta^{(1)}_0 \text{ weakly in } L^2(\mathbb{R}^3)\,. \end{aligned}$$

Then for suitable subsequence as $\varepsilon \to 0$ we obtain that

$$\varrho_\varepsilon \to \overline{\varrho} \text{ strongly in } L^\infty(0,T;L^{5/3}(K))\,, \qquad \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \rightharpoonup r \text{ weakly in } L^2(0,T;L^2(K))\,,$$
$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \rightharpoonup \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3)),$$
$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{U} \text{ weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3;\mathbb{R}^3))\,,\ \boldsymbol{u}_\varepsilon \to \boldsymbol{U} \text{ strongly in } L^2((0,T)\times K;\mathbb{R}^3)$$

for any compact set $K \subset \Omega$, where functions $\boldsymbol{U}$, Θ is a weak solution of the Oberbeck-Boussinesq approximation (OB^1)–(OB^4) *in $(0,T)\times\mathbb{R}^3$ in the sense specified in* (2.11)–(2.12) *with $\boldsymbol{U}(0,\cdot) = \boldsymbol{H}[\boldsymbol{U}_0]$ and $\Theta_0 = \vartheta^{(1)}_0$. Moreover if (D6) is satisfied, $\boldsymbol{U}|_{\partial O} = 0$.*

Here $\boldsymbol{H}[\cdot]$ denotes the projection on the space of divergence free functions on Ω of Helmholtz decomposition. The rest of the paper is devoted to the proof of the Theorem 2.1 or rather to the sketch of the proof with references where reader can find all details.

3. Proof of the Theorem 2.1

Since for each $\varepsilon \in (0,1)$ the set Ω_ε is sufficiently regular and bounded, in order to provide the existence of the family of weak solutions $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ to the primitive system - compressible Navier-Stokes-Fourier ($\mathrm{NSF}^1_\varepsilon$)–($\mathrm{NSF}^4_\varepsilon$) stated on Ω_ε we use the result of E. Feireisl and A. Novotný [6, Theorem 3.2]. Then the following regularity of solutions can be obtained: $\varrho_\varepsilon \in C_{\mathrm{weak}}(0,T; L^{5/3}(\Omega_\varepsilon))$, $\varrho_\varepsilon \in L^q((0,T)\times \Omega_\varepsilon)$ for a certain $q > \frac{5}{3}$ and $\boldsymbol{u}_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$. Moreover the absolute temperature ϑ_ε is a measurable function $\vartheta_\varepsilon(t,x) > 0$ for a.a. $(t,x) \in (0,T)\times\Omega_\varepsilon$ and $\vartheta_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon)) \cap L^\infty(0,T; L^4(\Omega_\varepsilon))$, $\log\vartheta_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon))$.

3.1. **Uniform bounds.** All uniform bounds stated below may be seen as a direct consequence of total dissipation balance and more detailed reasoning may be found in [6, 7, 11].

To begin with, according to these references, we introduce essential and residual part of a measurable function h as

$$h = [h]_{\mathrm{ess}} + [h]_{\mathrm{res}},\ [h]_{\mathrm{ess}} = \chi(\varrho_\varepsilon,\vartheta_\varepsilon)h\,,\ [h]_{\mathrm{res}} = (1-\chi(\varrho_\varepsilon,\vartheta_\varepsilon))h\,,$$

where $\chi \in C^\infty_c((0,\infty)\times(0,\infty))$, $0 \le \chi \le 1$, $\chi = 1$ on the set $\mathcal{O}_{\mathrm{ess}}$ and $\mathcal{O}_{\mathrm{ess}} = [\overline{\varrho}/2, 2\overline{\varrho}] \times [\overline{\vartheta}/2, 2\overline{\vartheta}]$, $\mathcal{O}_{\mathrm{res}} = (0,\infty)^2 \setminus \mathcal{O}_{\mathrm{ess}}$.

The total dissipation balance reads then

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2\right)(t)\,\mathrm{d}x \\
&\quad + \frac{1}{\varepsilon^2}\left(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon - \tilde{\varrho}_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})\right)(t)\,\mathrm{d}x \\
&\quad + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\left[[0,t]\times\overline{\Omega}_\varepsilon\right] = \int_{\Omega_\varepsilon}\left(\frac{1}{2}\varrho_{0,\varepsilon}|\boldsymbol{u}_{0,\varepsilon}|^2\right)\mathrm{d}x \\
&\quad + \frac{1}{\varepsilon^2}\left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})\right)\mathrm{d}x\,,
\end{aligned} \tag{3.1}$$

where $H_{\overline{\vartheta}}$ is ballistic free energy and $H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho\left(e(\varrho,\vartheta) - \overline{\vartheta}s(\varrho,\vartheta)\right)$.

It is provided (see Lemma 5.1 in [6]) that $H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon - \tilde{\varrho}_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})$ is non-negative, strictly coercive, attain global minimum zero at point $(\tilde{\varrho}_\varepsilon,\overline{\vartheta})$, dominates internal energy ϱe and entropy s far from $(\tilde{\varrho}_\varepsilon,\overline{\vartheta})$. Therefore according to our assumptions we are able to deduce from (3.1) that (for details see [5, 6, 7, 11])

$$\operatorname*{ess\,sup}_{t\in(0,T)}\int_{\Omega_\varepsilon}\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2(t,\cdot)\,\mathrm{d}x \le c, \quad \operatorname*{ess\,sup}_{t\in(0,T)}\|\sqrt{\varrho_\varepsilon}\boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le c$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon}\right]_{\rm ess}(t,\cdot)\right\|_{L^2(\Omega_\varepsilon)} \le c\,, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline\vartheta}{\varepsilon}\right]_{\rm ess}(t,\cdot)\right\|_{L^2(\Omega_\varepsilon)} \le c\,,$$

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\Omega_\varepsilon)} \le \varepsilon^2 c\,,$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} (|[\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}| + |[p(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}| + |[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}|\,{\rm d}x) \le \varepsilon^2 c\,,$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} [\varrho_\varepsilon]^{5/3}_{\rm res}(t,\cdot) + [\vartheta_\varepsilon]^4_{\rm res}(t,\cdot)\,{\rm d}x \le \varepsilon^2 c\,, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} 1\!\!1_{\rm res}(t,\cdot) dx \le \varepsilon^2 c\,,$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon}\right]_{\rm res}\right\|_{L^1(\Omega_\varepsilon)} \le c\varepsilon\,,$$

$$\int_0^T \left\|\frac{\vartheta_\varepsilon - \overline\vartheta}{\varepsilon}\right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} {\rm d}t + \int_0^T \left\|\frac{\log(\vartheta_\varepsilon) - \log(\overline\vartheta)}{\varepsilon}\right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} {\rm d}t < c\,,$$

$$\int_0^T \|\boldsymbol{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)}\,{\rm d}t < c\,.$$

3.2. **Convergence.** The hypotheses stated on the family of $\{\Omega_\varepsilon\}_\varepsilon$ provides us:

- the uniform extension property [8]. Namely there exists an extension operator E_ε s.t. $E_\varepsilon\colon W^{1,p}(\Omega_\varepsilon) \mapsto W^{1,p}(\mathbb{R}^3)$, $E_\varepsilon[v]|_{\Omega_\varepsilon} = v$ and $\|E_\varepsilon[v]\|_{W^{1,p}(\mathbb{R}^3)} \le c\|v\|_{W^{1,p}(\Omega_\varepsilon)}$, where the constant c is independent of $\varepsilon \to 0$.
- there exists bounded domain O s.t. $\mathbb{R}^3 \setminus O$ satisfy the uniform α-cone condition and a suitable subsequence of ε's such that $|(\mathbb{R}^3 \setminus O_\varepsilon) \setminus (\mathbb{R}^3 \setminus O)| \to 0$ as $\varepsilon \to 0$. This property is crucial when studying stability if the spectral properties of the Neumann Laplacian, see [1, 5], to provide decay of acoustic waves. For each $x_0 \in \partial O$ there is $x_{\varepsilon,0} \in \partial O_\varepsilon$ such that $x_{\varepsilon,0} \to x_0$ and $O \subset B_s(0)$ and for any compact $K \subset \Omega$, there exists $\varepsilon(K)$ such that $K \subset \Omega_\varepsilon$ for all $\varepsilon < \varepsilon(K)$.

Since the family of $\{\Omega_\varepsilon\}_\varepsilon$ possesses a uniform extension property we may deduce from uniform estimates that

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{U} \quad \text{weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3;\mathbb{R}^3))\,, \tag{3.2}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\vartheta_\varepsilon(t,\cdot) - \overline\vartheta\|_{L^2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0\,,$$

$$\Theta_\varepsilon = \frac{\vartheta_\varepsilon - \overline\vartheta}{\varepsilon} \rightharpoonup \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3))\,.$$

Following the same procedure as in [7, 11] by uniform estimates and closeness of $\overline\varrho$ and $\tilde\varrho_\varepsilon$ (2.8) we get

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\tilde\varrho_\varepsilon(t,\cdot) - \overline\varrho\|_{L^{5/3}+L^q(\Omega_\varepsilon)} \to 0\,,$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \overline\varrho\|_{L^2+L^{5/3}+L^q(\Omega_\varepsilon)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \qquad \text{for} \quad q > 3\,,$$

$$\frac{\varrho_\varepsilon - \overline\varrho}{\varepsilon} \overset{*}{\rightharpoonup} r \text{ weakly* in } L^\infty(0,T;L^{5/3}(K)) \text{ for any compact } K \subset \Omega \text{ for } \varepsilon \to 0\,.$$

Therefore fluid density becomes constant since $\varepsilon \to 0$, i.e. as the Mach number tends to zero. Then continuity equations provides, that

$$\mathrm{div}_x \boldsymbol{U} = 0 \text{ a.a. in } (0,T) \times \Omega\,.$$

By boundary conditions and properties of Ω_ε the limit velocity field satisfies the impermeability condition $\boldsymbol{U} \cdot \boldsymbol{n}|_{\partial O} = 0$ in a weak sense. Moreover the analysis provided by [2], see also [5, Sec. 6.2], gives that $\boldsymbol{U}|_{\partial O} = 0$ if (D6) is satisfied.

To pass to the limit in rescaled NSF$_\varepsilon$ system one of the most difficult steps is to provide strong convergence of the velocity field in order to control the limit of convective term. Namely we need to show that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{U} \quad \text{strongly in } L^2((0,T) \times K) \quad \text{for any compact} \quad K \subset \mathbb{R}^3 \setminus O\,.$$

The main obstacle here are possible oscillations in time of the momentum, since from momentum equations we do not control its time derivative. Then one can observe that it is sufficient to provide that (see [3, 7, 11])

$$\varrho_\varepsilon \boldsymbol{u}_\varepsilon \to \overline{\varrho}\boldsymbol{U} \quad \text{in } L^2(0,T; W^{-1,2}(K)). \tag{3.3}$$

Then due to (3.2) it is even enough to prove, instead of (3.3), that

$$\left\{ t \to \int_{\mathbb{R}^3} (\varrho_\varepsilon \boldsymbol{u}_\varepsilon)(t,\cdot)\boldsymbol{\varphi}\,\mathrm{d}x \right\} \text{ is precompact in } L^2(0,T)$$

and

$$\left\{ t \to \int_{\mathbb{R}^3} \varrho_\varepsilon \boldsymbol{u}_\varepsilon(\cdot,t) \cdot \boldsymbol{\varphi}\,\mathrm{d}x \right\} \quad \to \quad \left\{ t \to \overline{\varrho} \int_{\mathbb{R}^3} \boldsymbol{U}(\cdot,t) \cdot \boldsymbol{\varphi}\,\mathrm{d}x \right\} \text{ in } L^2(0,T) \tag{3.4}$$

for any fixed $\boldsymbol{\varphi} \in C^\infty(\mathbb{R}^3)$ where $\operatorname{supp} \boldsymbol{\varphi} \subset K$ as $\varepsilon \to 0$.

3.3. **Reformulation to the wave equation. Dispersive estimates – local decay of acoustic wave.** As it was already emphasised, our aim now is to show (3.4). This will be provided by the analyse of Lighthill's acoustic analog (see [10]) of our primitive NSF$_\varepsilon$ system, namely

$$\varepsilon \partial_t S_\varepsilon + \omega \mathrm{div}_x \boldsymbol{V}_\varepsilon = \varepsilon \tilde{f}^1_\varepsilon, \qquad \varepsilon \partial_t \boldsymbol{V}_\varepsilon + \nabla_x S_\varepsilon = \varepsilon \tilde{\boldsymbol{f}}^2_\varepsilon\,, \tag{3.5}$$

with homogenous Neuman boundary condition $\boldsymbol{V}_\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$ where

(3.6)

$$S_\varepsilon = A\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) + B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\right) - \overline{\varrho} F_\varepsilon + \frac{B}{\varepsilon}\Sigma_\varepsilon\,, \qquad \boldsymbol{V}_\varepsilon = \varrho_\varepsilon \boldsymbol{u}_\varepsilon\,,$$

(3.7)

$$\tilde{f}^1_\varepsilon = \mathrm{div}_x \underbrace{B\left(\varrho_\varepsilon \frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \boldsymbol{u}_\varepsilon\right)}_{H^1_\varepsilon} + \mathrm{div}_x \underbrace{B\left(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon}\right)}_{H^2_\varepsilon}$$

$$\tilde{\boldsymbol{f}}^2_\varepsilon = \nabla_x \underbrace{\frac{1}{\varepsilon}\left[A\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) + B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right) - \left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right)\right]}_{G^3_\varepsilon}$$

$$- \operatorname{div}_x \left[\underbrace{(\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon)}_{G^{2,2}_\varepsilon} + \underbrace{\mathbf{S}_\varepsilon}_{G^{2,1}_\varepsilon}\right] + \underbrace{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\nabla_x F_\varepsilon}_{G^4_\varepsilon} + \underbrace{B\frac{1}{\varepsilon^2}\nabla_x \Sigma_\varepsilon}_{\nabla_x G^1_\varepsilon}.$$

Where Σ_ε is a time lifting of σ_ε ([6, 7, 11]) and constants A, B, ω are chosen s.t. $B\overline{\varrho}\partial_\vartheta s(\overline{\varrho},\overline{\vartheta}) = \partial_\vartheta p(\overline{\varrho},\overline{\vartheta})$ and $A + B\partial_\varrho(\varrho s)(\overline{\varrho},\overline{\vartheta}) = \partial_\varrho p(\overline{\varrho},\overline{\vartheta})$, $\omega = \partial_\varrho p(\overline{\varrho},\overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})|^2}{\overline{\varrho}^2 \partial_\vartheta s(\overline{\varrho},\overline{\vartheta})} > 0$ (see e.g. [6, 7, 11]). Notice that ω is bounded due to structural restrictions on p and s.

Let $\nabla_x \Phi_\varepsilon$ denote acoustic potential, i.e.

$$\boldsymbol{V}_\varepsilon = \boldsymbol{H}_\varepsilon[\boldsymbol{V}_\varepsilon] + \nabla_x \Phi_\varepsilon.$$

Accordingly we may rewrite $(3.5)_1$ in the following form

$$\begin{aligned}
&\varepsilon \int_0^T \langle S_\varepsilon(t,\cdot), \partial_t \varphi\rangle \,\mathrm{d}t + \omega \int_0^T \int_{\Omega_\varepsilon} \nabla_x \Phi_\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t \\
&\qquad = \varepsilon \,\langle S_{0,\varepsilon}, \varphi(0,\cdot)\rangle + \varepsilon \int_0^T \int_{\Omega_\varepsilon} (H^1_\varepsilon + H^2_\varepsilon)\cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.8}$$

for all $\varphi \in C^\infty_c([0,T]\times\overline{\Omega}_\varepsilon)$. Next since $\varphi = \nabla_x \Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi]$ is an admissible test function in $(3.5)_2$ (due to slip boundary condtion on $\boldsymbol{u}_\varepsilon$) we obtain by integration by parts that

$$\begin{aligned}
&\varepsilon \int_0^T \int_{\Omega_\varepsilon} \Phi_\varepsilon \cdot \partial_t \varphi \,\mathrm{d}t - \int_0^T \langle S_\varepsilon, \varphi\rangle_{[\mathcal{M},C]} \,\mathrm{d}t = -\varepsilon \int_{\Omega_\varepsilon} V_{0,\varepsilon}\cdot \nabla_x \Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi(0,\cdot)]\,\mathrm{d}x \\
&\qquad - \varepsilon\Big\{ \int_0^T \langle G^1_\varepsilon(t,\cdot), \varphi\rangle \,\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} G^{2,1}_\varepsilon : \nabla^2_x \Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi]\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^T \int_{\Omega_\varepsilon} G^{2,2}_\varepsilon : \nabla^2_x \Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi]\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} G^3_\varepsilon \varphi \,\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^T \int_{\Omega_\varepsilon} G^4_\varepsilon \cdot \nabla_x \Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi]\,\mathrm{d}x\mathrm{d}t \Big\}.
\end{aligned} \tag{3.9}$$

The above equations represent a weak formulation of the acoustic equation for the potential of the gradient part of the momentum with Neumann boundary conditions.

Summarising computation from previous sections, due to uniform estimates obtained in Section 3.1 equations (3.8) and (3.9) can be rewritten in the following more conscious form (see [5, 11])

$$
\begin{aligned}
&\varepsilon \int_0^T \langle S_\varepsilon(t,\cdot), \partial_t \varphi \rangle \,\mathrm{d}t + \omega \int_0^T \int_{\Omega_\varepsilon} \nabla_x \Phi_\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t \\
&= \varepsilon \, \langle S_{0,\varepsilon}, \varphi(0,\cdot)\rangle + \frac{\varepsilon}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J^1_\varepsilon \varphi + J^2_\varepsilon (-\Delta_{\varepsilon,\mathrm{N}})^{3/2}[\varphi] \\
&\quad + J^3_\varepsilon (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] + J^4_\varepsilon(-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.10}
$$

for all $\varphi \in C^\infty_c([0,T] \times \overline{\Omega}_\varepsilon)$ and

$$
\begin{aligned}
&\varepsilon \int_0^T \int_{\Omega_\varepsilon} \Phi_\varepsilon \cdot \partial_t \varphi \,\mathrm{d}t - \int_0^T \langle S_\varepsilon, \varphi\rangle \,\mathrm{d}t = -\varepsilon \int_{\Omega_\varepsilon} \Phi_{0,\varepsilon} \varphi(0,\cdot) \,\mathrm{d}x \\
&- \frac{\varepsilon}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} \Big\{ \tilde{J}^1_\varepsilon \varphi + \tilde{J}^2_\varepsilon (-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi] + \tilde{J}^3_\varepsilon (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] \\
&+ \tilde{J}^4_\varepsilon (-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi] + \tilde{J}^5_\varepsilon (-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \Big\} \,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.11}
$$

for any $\varphi \in C^\infty_c([0,T) \times K)$, K compact subset of $\mathbb{R}^3 \setminus O$, $\nabla_x \varphi \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$, where

$$
\|J^i\|_{L^2((0,T)\times\Omega_\varepsilon)} < c \text{ for } i = 1,\dots,4 \text{ and } \|\tilde{J}^j\|_{L^2((0,T)\times\Omega_\varepsilon)} < c \text{ for } j = 1,\dots,5
$$

and for sufficiently small ε and supplemented with the following initial data

$$
S_{0,\varepsilon} = (-\Delta_{\varepsilon,\mathrm{N}})[\tilde{S}^1_{0,\varepsilon}] + (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\tilde{S}^2_{0,\varepsilon}] + \tilde{S}^3_{0,\varepsilon}\,,
$$

with $\|\tilde{S}^i_{0,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \le c$ and

$$
\Phi_{0,\varepsilon} = (-\Delta_{\varepsilon,\mathrm{N}})^{-1} \mathrm{div}_x\, V_{0,\varepsilon}, \text{ where } \|(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\Phi_{0,\varepsilon}]\|_{L^2(\Omega_\varepsilon)} \le c\,.
$$

Then the Duhamel formula gives as an explicit formulation for acoustic potential, i.e.:

$$
\begin{aligned}
\Phi_\varepsilon(t,\cdot) &= \frac{1}{2} \exp\Big(\pm i \sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}\, \frac{t}{\varepsilon}\Big) \Big[\Phi_{0,\varepsilon} \pm \frac{i}{\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}} [S_{0,\varepsilon}] \Big] \\
&\quad + \varepsilon^{-2\beta} \frac{1}{2} \int_0^T \exp\Big(\pm i \sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}\, \frac{t-s}{\varepsilon}\Big) \Big[\tilde{F}_{2,\varepsilon}(s) \pm \frac{i}{\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}} \tilde{F}_{1,\varepsilon}(s) \Big] \mathrm{d}s\,,
\end{aligned} \tag{3.12}
$$

where

$$
\begin{aligned}
\tilde{F}_{1,\varepsilon} &= J^1_\varepsilon + (-\Delta_{\varepsilon,\mathrm{N}})^{3/2}[J^2_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[J^3_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})[J^4_\varepsilon], \\
\tilde{F}_{2,\varepsilon} &= \tilde{J}^1_\varepsilon + (-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\tilde{J}^2_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\tilde{J}^3_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\tilde{J}^4_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})[\tilde{J}^5_\varepsilon]
\end{aligned}
$$

(see (3.10), (3.11)). Let us remark that the "large" coefficient $\varepsilon^{-2\beta}$ appearing in (3.10), (3.11) and (3.12) is a consequence or roughness of the obstacle O_ε (see (D5)). More precisely, an elliptic estimate employed to derive (3.10), (3.11) depends on ε, i.e. $\|\nabla^2_x \varphi\|_{L^p(\Omega_\varepsilon)} \le c(p) \big(\|\Delta_x \varphi\|_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|\varphi\|_{L^p(\Omega_\varepsilon)} \big)$ for any $\varphi \in C^\infty_c(\overline{\Omega_\varepsilon})$ with $\nabla_x \varphi \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$, with $1 < p < \infty$.

With above formulation at hand and by methods developed in [4] we are able to provide local decay of acoustic wave and consequently to show that

$$\left\{t \to \int_{\Omega_\varepsilon} \Phi_\varepsilon G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\,\mathrm{d}x\right\} \to 0 \quad \text{in} \quad L^2(0,T)\,, \tag{3.13}$$

any $G \in C_c^\infty(0,\infty)$, what in fact is a key point to prove (3.4) and consequently to provide convergence in convective term (see for details [5, 11]). The following lemma gives a local decay of acoustic waves.

Lemma 3.1 ([4, 5])**.** *We have*

$$\int_0^T \left| \left\langle \exp\left(i\sqrt{-\Delta_{\varepsilon,\mathrm{N}}}\frac{t}{\varepsilon}[\Psi], G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\right)\right\rangle_{\widetilde{\Omega}_\varepsilon}\right|^2 \mathrm{d}t \le \varepsilon c(\varphi, G)\|\Psi\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}$$

for any $\varphi \in C_c^\infty(K)$, $\Psi \in L^2(\tilde{\Omega}_\varepsilon)$, and any $G \in C_c^\infty(0,\infty)$, where is s.t. $\overline{K} \subset \mathbb{R}^3 \setminus O_\varepsilon$.

Lemma 3.1 applied to Φ_ε given by formula (3.12) provides (3.13), if $\beta < \frac{1}{4}$, see [4] for details. The explicitly given rate of the decay in Lemma 3.1 allow to compensate exploding coefficient $\varepsilon^{-2\beta}$ which reflects the influence of perturbations of the domain. Moreover, let us remark that in order to provide good properties of the spectrum of Neumann Laplacian $-\Delta_{\varepsilon,\mathrm{N}}$ it is crucial to notice that the outer boundary (the boundary of the sphere $\mathcal{S}_\varepsilon$) is irrelevant for the local analysis (on supports of test functions $\boldsymbol{\varphi}$) and in fact we may consider the operator $-\Delta_{\varepsilon,\mathrm{N}}$ on unbounded domain $\mathbb{R}^3 \setminus O_\varepsilon$. Indeed in (3.5) the speed of propagation is finite and proportional to $\sqrt{\omega}/\varepsilon$ and the boundary $\mathcal{S}_\varepsilon$ is sufficiently "far", since $\delta > 1$. For details see again [5, 11].

References

[1] Arrieta, J.M., Krejčiřík, D., *Geometric versus spectral convergence for the Neumann Laplacian under exterior perturbation of the domain*, Integral Methods in Sciences and Engineering **1** (2010), 9–19.

[2] Bucur, D., Feireisl, E., Nečasová, Š., Wolf, J., *On the asymptotic limit of the Navier-Stokes system in domains with rough boundaries*, J. Differential Equations **244** (2008), 2890–2908.

[3] Feireisl, E., *Incompressible limits and propagation of acoustic waves in large domains with boundaries*, Commun. Math. Phys. **294** (2010), 73–95.

[4] Feireisl, E., *Local decay of acoustic waves in the low mach number limits on general unbounded domains under slip boundary conditions*, Commun. Partial Differential Equations **36** (2011), 1778–1796.

[5] Feireisl, E., Karper, T., Kreml, O., Stebel, J., *Stability with respect to domain of the low Mach number limit of compressible viscous fluids*, Math. Models Methods Appl. Sci. **23** (13) (2013), 2465–2493.

[6] Feireisl, E., Novotný, A., *Singular limits in thermodynamics of viscous fluids*, Birkhäuser, Basel, 2009.

[7] Feireisl, E., Schonbek, M., *On the Oberbeck-Boussinesq approximation on unbounded domains*, Nonlinear partial differential equations, Abel Symposial (Holden, H., Karlsen, K.H., eds.), vol. 7, Springer, Berlin, 2012.

[8] Jones, P.W., *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.

[9] Klein, R., Botta, N., Schneider, T., Munz, C.D., Roller, S., Meister, A., Hoffmann, L., Sonar, T., *Asymptotic adaptive methods for multi-scale problems in fluid mechanics*, J. Engrg. Math. **39** (2001), 261–343.

[10] Lighthill, J., *Waves in Fluids*, Cambridge University Press, 1978.

[11] Wróblewska-Kamińska, A., *Asymptotic analysis of complete fluid system on varying domain: form compressible to incompressible flow*, SIAM J. Math. Anal. **49** (5) (2017), 3299–3334.

[12] Zeytounian, R.K., *Joseph Boussinesq and his approximation: a contemporary view*, C.R. Mecanique **331** (2003), 575–586.

[13] Zeytounian, R.K., *Theory and Applications of Viscous Fluid Flows*, Springer, Berlin, 2004.

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