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ORBIT-CONE CORRESPONDENCE FOR THE PROALGEBRAIC COMPLETION OF NORMAL TORIC VARIETIES

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Abstract. We prove that there is an orbit-cone correspondence for the proalgebraic completion of normal toric varieties, which is analogous to the classical orbit-cone correspondence for toric varieties.

1. Introduction

Let N be a finite rank free abelian group (also called a lattice) and $\mathcal{F}$ a fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. It is well-known that there is a toric variety $X^{\mathcal{F}}$ associated to $\mathcal{F}$. Moreover a classical result states that if T_N is the torus of $X^{\mathcal{F}}$ then there is a bijective correspondence between the T_N-orbits in $X^{\mathcal{F}}$ and the cones σ in $\mathcal{F}$. (See for example [2], Theorem 3.2.6).

Now consider $(I, \leq)$ a directed poset and a projective system of complex algebraic varieties

$$f_{ij} \colon V_j \to V_i\,, \quad i, j \in I, \quad i \leq j\,.$$

This system is called **proalgebraic variety**, and as pointed out in [1] this concept plays an important role in dynamical systems. In the cases that we will study, we can take projective limit and obtain **laminations**, (also called **solenoidal manifolds** in [6]). These have been used, for example in [4] and [6], to obtain results in conformal dynamics.

In this paper we consider a projective system of toric varieties. Namely, given a normal toric variety X, for each $m, n \in \mathbb{N}$ such that $n \mid m$, we can define

$$p_{n,m} \colon \mathbb{C} \to \mathbb{C}$$

by $z \mapsto z^{m/n}$, and these induce

$$q_{n,m} \colon X \to X,$$

which define a projective system. The projective limit, denoted by $X_{\mathbb{Q}}$ is called the **proalgebraic completion** of X. The first part of [1] is devoted to the study of the proalgebraic completion of toric varieties. More particularly, in Theorem 1 they

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give a description of its vector bundle category. For a more general treatment of proalgebraic completions, we refer to [3].

The main result of this paper is an analogous result to the one described in the first paragraph. Namely, we will prove that there is a *proalgebraic torus* acting on the proalgebraic completion of the toric variety $X^{\mathcal{F}}$ (defined by the fan $\mathcal{F}$), and there is a bijective correspondence of its orbits with the cones of $\mathcal{F}$. This correspondence satisfies similar properties to the classical one. See Theorem 3 for the precise statement and Theorem 1 for the comparison with the classical result.

Now let us give an outline of this paper. In Section 2 we recall some concepts regarding toric varieties, as well as the precise statement of the classical theorem that inspires this work (Theorem 1). In section 3 we give the basic definitions regarding proalgebraic completions of normal toric varieties. Namely, we define the algebraic solenoid, which allows to define the proalgebraic torus and finally the proalgebraic completion of a normal toric variety. For this we define it first for an affine toric variety and we use this for the general case. Finally in Section 4 we prove our main result.

2. Preliminaries on toric varieties

Given a lattice N with dual $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and a rational strongly convex polyhedral cone $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$, we have the classical definition as in [5] of the affine normal toric variety

$$X^\sigma = \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C})\,,$$

where $S^\sigma \subset M$ is the semigroup associated to σ. In particular, each point of X^σ is a homomorphism of semigroups $S^\sigma \to \mathbb{C}$. In this setting, the torus of X^σ is the topological group

$$T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)\,.$$

If $\mathcal{F}$ is a fan that lies in the vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, then for every $\sigma_1, \sigma_2 \in \mathcal{F}$ and $\tau = \sigma_1 \cap \sigma_2$ there are natural injective morphisms

$$\mathrm{Hom}_{sg}(S^{\sigma_1}, \mathbb{C}) \hookleftarrow \mathrm{Hom}_{sg}(S^{\tau}, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{sg}(S^{\sigma_2}, \mathbb{C})\,.$$

These define an equivalence relation so the affine toric varieties X^σ, $\sigma \in \mathcal{F}$ glue together to get the normal toric variety

$$X^{\mathcal{F}} = \bigsqcup_{\sigma \in \mathcal{F}} \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C}) / \sim\,,$$

with torus T_N. Recall that the action of T_N on $X^{\mathcal{F}}$ is given by the map

$$T_N \times X^{\mathcal{F}} \to X^{\mathcal{F}}, \qquad \tilde{\gamma} \cdot [\gamma] = [\tilde{\gamma}|_{S^\sigma} \gamma]\,, \tag{1}$$

where $\gamma \colon S^\sigma \to \mathbb{C}$, $\tilde{\gamma} \colon M \to \mathbb{C}^*$.

An important result in toric geometry is that the T_N-orbits of a toric variety

$X^{\mathcal{F}}$ are characterized by the **special points**. Namely, by the semigroup homomorphism (point in $X^\sigma = \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C})$) defined for every $\sigma \in \mathcal{F}$ by

$$m \in S^\sigma \mapsto \begin{cases} 1 & \text{if } m \in S^\sigma \cap \sigma^\perp = \sigma^\perp \cap M, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma^\perp$ is the orthogonal complement of σ on the dual vector space of $N \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 1. *Let $X^{\mathcal{F}}$ be the toric variety associated to the fan $\mathcal{F}$ on $N \otimes_{\mathbb{Z}} \mathbb{R}$. Then*

(a) *There is a bijective correspondence*

$$\{\textit{Cones } \sigma \in \mathcal{F}\} \longleftrightarrow \{T_N\textit{-orbits on } X^{\mathcal{F}}\}$$
$$\sigma \longleftrightarrow T_N \cdot \gamma^\sigma \,.$$

(b) *If $\dim N \otimes_{\mathbb{Z}} \mathbb{R} = r$, then for every cone $\sigma \in \mathcal{F}$ the orbit $T_N \cdot \gamma^\sigma$ is an algebraic torus of dimension $r - \dim \sigma$.*

(c) *For every $\sigma \in \mathcal{F}$ the affine toric variety X^σ is the union of orbits*

$$X^\sigma = \bigcup_{\tau \textit{ face of } \sigma} T_N \cdot \gamma^\tau .$$

The proof is in [2] Theorem 3.2.6. Our main result is a proalgebraic version of this theorem. To get there, we will use the notion of proalgebraic completion of a toric variety from [1].

3. Proalgebraic completion of normal toric varieties

For every $n, m \in \mathbb{N}$ such that $n \mid m$ (n divides m) there is a finite covering map

$$p_{n,m} \colon \mathbb{C}^* \to \mathbb{C}^*, \qquad z \mapsto z^{m/n}. \tag{2}$$

This determines a projective system $\{\mathbb{C}^*, p_{n,m}\}_{n|m}$ of covering spaces whose projective limit is the **algebraic solenoid** $\mathbb{C}^*_{\mathbb{Q}}$ which is a topological group with the initial topology defined by the canonical projections

$$\pi_n \colon \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*, \qquad (z_j)_{j\in\mathbb{N}} \mapsto z_n \,.$$

Let $\widehat{\mathbb{Z}}$ be the profinite completion of the integers. Then the algebraic solenoid is a 2-dimensional **lamination** in the sense of [4] with **transversal space** $\widehat{\mathbb{Z}}$. Furthermore, the **global leaves** of the lamination are just copies of $\mathbb{C}$, and more precisely all the global leaves are obtained by translations of the image of the injective and continuous map

$$\nu \colon \mathbb{C} \to \mathbb{C}^*_{\mathbb{Q}}, \qquad z \mapsto (e^{i(z/j)})_{j\in\mathbb{N}}$$

by elements of $\widehat{\mathbb{Z}}$. This algebraic solenoid is thoroughly studied in [1].

Definition 1. Let N be a lattice with dual $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The group

$$(T_N)_{\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*_{\mathbb{Q}})$$

is called **proalgebraic torus**.

Note that if rank $M = r$, then we have a natural group isomorphisms

$$N \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}} \cong (T_N)_{\mathbb{Q}} \cong (\mathbb{C}^*_{\mathbb{Q}})^r.$$

Hence the proalgebraic torus $(T_N)_{\mathbb{Q}}$ is also a lamination ($2r$-dimensional) with transversal space $\widehat{\mathbb{Z}}^r$. In this case, the global leaves are copies of $\mathbb{C}^r$, and all of them are obtained via translations by elements of $\widehat{\mathbb{Z}}^r$. The maps in 2 used to define the algebraic solenoid extend naturally to the whole complex plane $\mathbb{C}$, so we can consider the limit

$$\mathbb{C}_{\mathbb{Q}} = \{(z_j)_{j\in\mathbb{N}} \mid z_j \in \mathbb{C} \text{ for all } \ j \in \mathbb{N},\ p_{n,m}(z_m) = z_n \text{ whenever } n \mid m\}\ .$$

Definition 2. Let σ be a rational strongly convex polyhedral cone with associated semigroup S^σ. The set

$$X^\sigma_{\mathbb{Q}} = \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})$$

is **the proalgebraic completion** of the affine toric variety $X^\sigma = \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C})$.

It is clear that an element γ in $X^\sigma_{\mathbb{Q}}$ induces a collection $\{\gamma_j \colon S^\sigma \to \mathbb{C}\}_{j\in\mathbb{N}}$ of semigroup homomorphisms satisfying $(\gamma_m(s))^{m/n} = \gamma_n(s)$ for all $s \in S^\sigma$ whenever $n \mid m$, and there is a bijective correspondence

$$X^\sigma_{\mathbb{Q}} \to \varprojlim_{q_{n,m}} \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C})\,, \qquad \gamma \mapsto (\gamma_j)_{j\in\mathbb{N}}\,,$$

where

$$q_{n,m} \colon \mathrm{Hom}_{\mathbb{Z}}(S^\sigma, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(S^\sigma, \mathbb{C})\,, \qquad \gamma \mapsto p_{n,m} \circ \gamma$$

is an algebraic map. Under this correspondence we can endow the proalgebraic completion $X^\sigma_{\mathbb{Q}}$ with a topology.

More generally, let $\mathcal{F}$ be a fan and consider the collection of proalgebraic completions

$$\{X^\sigma_{\mathbb{Q}}\}_{\sigma\in\mathcal{F}} = \{\mathrm{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})\}_{\sigma\in\mathcal{F}}\,.$$

As in the classical case, if $\sigma_1, \sigma_2 \in \mathcal{F}$, and τ is a face of $\sigma_1 \cap \sigma_2$, we have natural injective maps

$$X^{\sigma_1}_{\mathbb{Q}} = \mathrm{Hom}_{sg}(S^{\sigma_1}, \mathbb{C}_{\mathbb{Q}}) \hookleftarrow \mathrm{Hom}_{sg}(S^{\tau}, \mathbb{C}_{\mathbb{Q}}) \hookrightarrow \mathrm{Hom}_{sg}(S^{\sigma_2}, \mathbb{C}_{\mathbb{Q}}) = X^{\sigma_2}_{\mathbb{Q}}\,. \tag{3}$$

These maps allow to define an equivalence relation on

$$\bigsqcup_{\sigma\in\mathcal{F}} \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})$$

as follows: $(\gamma_1 \colon S^{\sigma_1} \to \mathbb{C}_{\mathbb{Q}}) \sim (\gamma_2 \colon S^{\sigma_2} \to \mathbb{C}_{\mathbb{Q}})$ if and only if there exists a face τ of $\sigma_1 \cap \sigma_2$ and a homomorphism $\gamma \colon S^\tau \to \mathbb{C}_{\mathbb{Q}}$ such that $\gamma|_{S^{\sigma_1}} = \gamma_1$, $\gamma|_{S^{\sigma_2}} = \gamma_2$.

Definition 3. Let N be a lattice and let $\mathcal{F}$ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$. We define **the proalgebraic completion** $X^{\mathcal{F}}_{\mathbb{Q}}$ of the toric varierty $X^{\mathcal{F}}$ as the quotient space

$$X^{\mathcal{F}}_{\mathbb{Q}} := \bigsqcup_{\sigma\in\mathcal{F}} \mathrm{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})/\sim\ .$$

As in the affine case, the proalgebraic completion $X^{\mathcal{F}}_{\mathbb{Q}}$ turns out to be a projective limit. More precisely we have the following (see [1] Section 4):

Theorem 2. *The proalgebraic completion $X^{\mathcal{F}}_{\mathbb{Q}}$ of a toric variety $X^{\mathcal{F}}$ is homomorphic to the projective limit over all its toric covers.*

With the previous result it is clear that there is a canonical inclusion

$$(T_N)_{\mathbb{Q}} \subset X^{\mathcal{F}}_{\mathbb{Q}}$$

induced by the inclusion $T_N \subset X^{\mathcal{F}}$. Moreover, the action described in (1) induces the action

$$(T_N)_{\mathbb{Q}} \times X^{\mathcal{F}}_{\mathbb{Q}} \to X^{\mathcal{F}}_{\mathbb{Q}}, \qquad \tilde{\gamma} \cdot [\gamma] = [\tilde{\gamma}|_{S^\sigma} \cdot \gamma]\,, \tag{4}$$

where $\gamma\colon S^\sigma \to \mathbb{C}_{\mathbb{Q}}$, $\tilde{\gamma}\colon M \to \mathbb{C}^*_{\mathbb{Q}}$.

Remark 1. If X is a toric variety defined by a fan $\mathcal{F}$ in $N \otimes_{\mathbb{Z}} \mathbb{R}$, for every $m \in \mathbb{N}$ we take $X_m = X$, and then for every $n \mid m$ we have a map $N \to N$ which is multiplication by the integer $\frac{m}{n}$. Since this map takes cones of $\mathcal{F}$ into cones of $\mathcal{F}$, it defines a map or toric varieties $X_m \to X_n$, and thus a projective system where all the maps involved are algebraic. Moreover, these maps are affine, so we get a scheme structure on $X_{\mathbb{Q}}$.

Remark 2. Note that if X is singular, it is not a manifold, and perhaps a more suitable name for the proalgebraic completion would be solenoidal variety, instead of solenoidal manifold.

4. The orbit-cone correspondence

In order to study the $(T_N)_{\mathbb{Q}}$-orbits of the proalgebraic completion $X^{\mathcal{F}}_{\mathbb{Q}}$ we have the following definition which is the proalgebraic version of the special points in $X^{\mathcal{F}}$.

Definition 4. Let σ be a rational strongly convex polyhedral cone in $N \otimes_{\mathbb{Z}} \mathbb{R}$. We define the **special point** (or distinguished point) in $X^\sigma_{\mathbb{Q}} = \operatorname{Hom}_{sg}(S^\sigma, \mathbb{C}^*_{\mathbb{Q}})$ as the semigroup homomorphism

$$\gamma^\sigma_{\mathbb{Q}}(m) = \begin{cases} (1_j)_j & \text{if } m \in S^\sigma \cap \sigma^\perp = \sigma^\perp \cap M, \\ (0_j)_j & \text{otherwise,} \end{cases}$$

for $m \in S^\sigma$.

As we shall see in the following Lemma, the $(T_N)_{\mathbb{Q}}$-orbits in $X^{\mathcal{F}}_{\mathbb{Q}}$ corresponding to the special points are proalgebraic torus themselves.

Lemma 1. *Let σ be a rational strongly convex polyhedral cone in $N \otimes_{\mathbb{Z}} \mathbb{R}$ and consider the lattice $N(\sigma) = N/N_\sigma$, where N_σ is the sublattice generated by the elements in $\sigma \cap N$. Then there is a bijection*

$$(T_N)_{\mathbb{Q}} \cdot \gamma^\sigma_{\mathbb{Q}} \simeq (T_{N(\sigma)})_{\mathbb{Q}} = \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*_{\mathbb{Q}})\,.$$

Proof. We define the map

$$(T_N)_{\mathbb{Q}} \cdot \gamma^\sigma_{\mathbb{Q}} \to (T_{N(\sigma)})_{\mathbb{Q}}, \qquad \gamma \cdot \gamma^\sigma_{\mathbb{Q}} \mapsto (\gamma \cdot \gamma^\sigma_{\mathbb{Q}})|_{\sigma^\perp \cap M}\,, \tag{5}$$

which is clearly injective by definition of $\gamma^\sigma_{\mathbb{Q}}$. To see that this correspondence is surjective, consider the short exact sequence

$$0 \to N_\sigma \to N \to N(\sigma) \to 0\,.$$

If we tensor with $\mathbb{C}^*_{\mathbb{Q}}$ then we obtain a surjective homomorphism

$$N \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}} \to N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}} \to 0\,,$$

and hence a transitive action of $N \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}}$ on $N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}}$.

Under the natural isomorphisms

$$N \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*_{\mathbb{Q}}) = (T_N)_{\mathbb{Q}}$$

and

$$N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*_{\mathbb{Q}}) = (T_{N(\sigma)})_{\mathbb{Q}}\,,$$

the action of $N \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}}$ on $N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*_{\mathbb{Q}}$ corresponds with the action

$$(T_N)_{\mathbb{Q}} \times (T_{N(\sigma)})_{\mathbb{Q}} \to (T_{N(\sigma)})_{\mathbb{Q}}$$

given by $\gamma \cdot \alpha = \gamma|_{\sigma^\perp \cap M}\alpha$.

Note that by restricting the special point $\gamma^\sigma_{\mathbb{Q}}$ to the group $\sigma^\perp \cap M$ we get an element in $(T_{N(\sigma)})_{\mathbb{Q}}$. Since the previous action is transitive, the map (5) is surjective. $\square$

Remark 3. The group $\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*_{\mathbb{Q}})$ from the previous lemma can be described as the following set of semigroup homomorphisms:

$$\{\gamma \colon S^\sigma \to \mathbb{C}_{\mathbb{Q}} \mid \gamma(m) \neq (0_j)_j \Leftrightarrow m \in \sigma^\perp \cap M\}\,.$$

Indeed, any element of this set can be restricted to obtain an element in

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*_{\mathbb{Q}})\,.$$

Conversely, extending by zero, any element on the group $\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*_{\mathbb{Q}})$ can be seen as an element in the set above.

Theorem 3 (The Orbit-Cone correspondence)**.** *Let $\mathcal{F}$ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$ and let $X^{\mathcal{F}}_{\mathbb{Q}}$ be the proalgebraic completion of the normal toric variety $X^{\mathcal{F}}$. Then*

(a) *There is a bijective correspondence*

$$\begin{aligned}\{\textit{Cones } \sigma \in \mathcal{F}\} &\longleftrightarrow \{(T_N)_{\mathbb{Q}}\textit{-orbits in } X^{\mathcal{F}}_{\mathbb{Q}}\}\\ \sigma &\longleftrightarrow (T_N)_{\mathbb{Q}} \cdot \gamma^\sigma_{\mathbb{Q}}\,.\end{aligned}$$

(b) *If $\dim N \otimes_{\mathbb{Z}} \mathbb{R} = r$, then for every cone $\sigma \in \mathcal{F}$, the orbit $(T_N)_{\mathbb{Q}} \cdot \gamma^\sigma_{\mathbb{Q}}$ is a lamination of dimension $2(r - \dim \sigma)$.*

(c) *For every $\sigma \in \mathcal{F}$ the completion $X^\sigma_{\mathbb{Q}}$ is the union of orbits*

$$X^\sigma_{\mathbb{Q}} = \bigcup_{\tau \textit{ face of } \sigma} (T_N)_{\mathbb{Q}} \cdot \gamma^\tau_{\mathbb{Q}}\,.$$

Proof. For surjectivity in (a), let O be a $(T_N)_{\mathbb{Q}}$-orbit of $X^{\mathcal{F}}_{\mathbb{Q}}$. Since

$$X^{\mathcal{F}}_{\mathbb{Q}} = \bigcup_{\sigma\in\mathcal{F}} X^{\sigma}_{\mathbb{Q}}\,,$$

if $\sigma_1, \sigma_2 \in \mathcal{F}$, the elements of $X^{\sigma_1}_{\mathbb{Q}}$ that are related to the elements of $X^{\sigma_2}_{\mathbb{Q}}$ are precisely the elements of $X^{\sigma_1\cap\sigma_2}_{\mathbb{Q}}$, i.e.,

$$X^{\sigma_1}_{\mathbb{Q}} \cap X^{\sigma_2}_{\mathbb{Q}} = X^{\sigma_1\cap\sigma_2}_{\mathbb{Q}}\,.$$

Moreover each $X^{\sigma}_{\mathbb{Q}}$ is invariant under the action (4) of the torus $(T_N)_{\mathbb{Q}}$. Hence there exists a unique minimal cone $\sigma \in \mathcal{F}$ such that $O \subseteq X^{\sigma}_{\mathbb{Q}}$. We shall prove that $O = (T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma}_{\mathbb{Q}}$. Indeed, let $\gamma \in O$ and consider the set

$$S = \{m \in S^{\sigma} \mid \gamma(m) \neq (0_j)_j\}\,.$$

For every $j \in \mathbb{N}$, let $\pi_j : \mathbb{C}_{\mathbb{Q}} \to \mathbb{C}$ be the canonical projection which is in particular a semigroup homomorphism, and note that

$$S = \{m \in S^{\sigma} \mid \gamma(m) \neq (0_j)_j\} = \{m \in S^{\sigma} \mid (\pi_j \circ \gamma)(m) \neq 0\}$$

for every $j \in \mathbb{N}$. It follows that $S = \Gamma \cap M$ for some face Γ of $\sigma^{\vee}$, and consequently there exists a face τ of σ such that $\Gamma = \sigma^{\vee} \cap \tau^{\perp}$.

Now the equality

$$S = \Gamma \cap M = \sigma^{\vee} \cap \tau^{\perp} \cap M$$

implies $\gamma \in X^{\tau}_{\mathbb{Q}}$, and then $\tau = \sigma$ because of the minimality of σ. By Remark 3,

$$S = \sigma^{\perp} \cap M$$

implies that $\gamma \in (T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma}_{\mathbb{Q}}$ and therefore $O = (T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma}_{\mathbb{Q}}$. This proves surjectivity.

For injectivity, let $\sigma_1 \neq \sigma_2$ be cones in $\mathcal{F}$. If $\dim \sigma_1 \neq \dim \sigma_2$, it follows from Lemma 1 that the orbits are not equal. If $\dim \sigma_1 = \dim \sigma_2$, an easy computation shows that

$$(T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma_1}_{\mathbb{Q}}, (T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma_2}_{\mathbb{Q}} \not\subset X^{\sigma_1\cap\sigma_2}_{\mathbb{Q}}\,,$$

and hence $(T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma_1}_{\mathbb{Q}} \neq (T_N)_{\mathbb{Q}} \cdot \gamma^{\sigma_2}_{\mathbb{Q}}$.

Part (b) follows from Lemma 1 and the fact that any proalgebraic torus is a lamination; part (c) follows directly from part (a). □

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