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RELATIVE INCREMENTS OF PEARSON DISTRIBUTIONS

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Abstract. This paper is a direct continuation of [6] whose results are applied to Pearson distributions, particularly to normal, gamma, beta of the first kind, Pareto of the second kind, chi-square and other specific distributions.

In this paper we investigate the *hazard rate* and *relative increment functions* of Pearson distribution functions.

Let f be a (probability) density function. The corresponding distribution function is denoted by F.

Definition 1. By the *relative increment function* [briefly, RIF] of F we mean the fraction

$$h(x) = [F(x+a) - F(x)]/[1 - F(x)],$$

where a is a positive constant, and $F(x) < 1$ for all x.

Monotone properties of RIFs are important from the points of view of statistics, probability theory, in modelling bounded growth processes in biology, medicine and dental science and in reliability and actuarial theories, where the probability that an individual, having survived to time x, will survive to time $x+a$ is $h(x)$; *"death rate per unit time"* in the time interval $[x, x+a]$ is $h(x)/a$, and the *hazard rate (failure rate or force of mortality)* is defined to be

$$\lim_{a \to 0} h(x)/a = f(x)/[1 - F(x)].$$

(See e.g. [3], Vol. 2, Chap. 33, Sec. 7 or [4], § 5.34 and § 5.38.)

In [3], Vol. 2, Chap. 33, Sec. 7.2, some distributions are classified by their increasing/decreasing hazard rates. In [6], we proved

Lemma 1. *Let F be a twice differentiable distribution function with $F(x) < 1$, $f(x) > 0$ for all x. We define the auxiliary function Ψ as follows:*

$$\Psi(x) := [F(x) - 1] \cdot f'(x)/f^2(x).$$

If $\Psi < (>)1$, then the function h, the RIF of F strictly increases (strictly decreases).

According to Remark 0.1 in [6], there is a connection to reliability theory: a distribution function F has IFR (increasing failure rate) iff $\ln[1 - F(x)]$ is concave down i.e., iff $\Psi(x) \le 1$. Similarly, F has DFR (decreasing failure rate) iff $\ln[1 - F(x)]$ is concave up, i.e., $\Psi(x) \ge 1$.

In [6], we investigated the auxiliary function Ψ. In order to get rid of the inconvenient term $F(x) - 1 = -\int_x^\infty f(t)dt$ in Ψ, all problems were reduced to simple formulae containing f/f' only. This fact suggested to work out special methods for the family of Pearson distributions.

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Definition 2. A probability distribution with the density function f is said to be a *Pearson distribution*, if

$$f'(x)/f(x) = Q(x)/q(x),$$

where $Q(x) = Ax + B,\ q(x) = ax^2 + bx + c$ and A, B, a, b, c are real constants with

$$a^2 + b^2 + c^2 > 0, \qquad A^2 + B^2 > 0.$$

Although $A = 1$ in the original definition, it is more convenient, in many concrete cases, to handle Pearson distributions of this new form. In addition, we include some new distributions, like e.g.,

$$f(x) = C \cdot \exp(\tan^{-1} x), \qquad x \in (0, s) =: I$$

where $0 < s < \infty$ and $C = \int_0^s \exp(\tan^{-1} x)dx$.

Now we have $f'/f = (1 + x^2)^{-1}$, so $A = b = 0$ and $a = B = c = 1$. From the formula 1.216 in [2] we get

$$C \approx (s + s^2/2 + s^3/6 - s^4/24 - 7 \cdot s^5/120)^{-1}.$$

The derivative function $f' = C \cdot \exp(\tan^{-1} x)/(1 + x^2) > 0$ in I, so $m = s$ and Remark 1.1 of [6] applies, thus the RIF strictly increases in I.

The main results of [6] were formulated in theorems 1 and 2:

Theorem 1. *Let f be a probability density function and F be the corresponding distribution function with the following properties.*

$$(1) \qquad \begin{cases} I = (r, s) \subseteq \mathbf{R} \text{ is the possible largest finite or infinite open} \\ \text{interval in which } f > 0 \text{ (i.e., } I \text{ is the open support of } f; \\ r \text{ and } s \text{ may belong to the extended real line} \\ \mathbf{R}^* = \mathbf{R} \cup \{-\infty, \infty\}); \end{cases}$$

(2) *there exists an $m \in I$ at which f' is continuous and $f'(m) = 0$;*

(3) $f' > 0$ *in* (r, m) *and* $f' < 0$ *in* (m, s)

(4) f *is twice differentiable in* (m, s)

(5) $(f/f')' = d/dx[f(x)/f'(x)] > 0$ *in* (m, s).

Then the corresponding continuous RIF h is either strictly increasing in I, or strictly increasing in (r, y) and strictly decreasing in (y, s) for some $y \in I$.

Moreover, if $\Psi(s^-) = \lim_{x \to s^-} \Psi(x) \in \mathbf{R}^*$ exists, then

(a) h strictly increases in I, if $\Psi(s^-) \leq 1$;

(b) h strictly increases in (r, y) and strictly decreases in (y, s) for some y in I, if $\Psi(s^-) > 1$.

Theorem 2. *Let f be a density function with (1), (3–4), $m = r$ and*

$$(6) \qquad (f/f')' < 0 \quad \text{in } (m, s).$$

Then r is finite, and

$$(7) \qquad \text{if } \Psi(r^+) < 1 \text{ or}$$

$$[\Psi(r^+) = 1 \text{ and } \Psi < 1 \text{ in some right neighborhood of } r],$$

then $\Psi < 1$ in I, and the corresponding RIF strictly increases in I;

$$(8) \qquad \text{if } \Psi(r^+) > 1,$$

then

$$\text{(8.1)} \qquad \textit{if } \Psi(s^-) \geq 1, \quad \textit{then } \Psi > 1 \textit{ and the RIF strictly decreases in } I;$$

$$\text{(8.2)} \qquad \textit{if } \Psi(s^-) < 1, \quad \textit{then } \Psi > 1 \textit{ in } (r, y) \textit{ and } \Psi < 1 \textit{ in } (y, s) \textit{ for some } y \in I,$$

thus the RIF strictly decreases first and, after reaching its local minimum, strictly increases.

According to Remark 1.5 in [6], the assumption (5) can be reformulated as follows:

$$\text{(5')} \qquad (\ln f)'' < 0, \quad x \in (m, s).$$

We defined the functions f and g be $\overset{''}{\sim}$-equivalent (we write $f \overset{''}{\sim} g$), if $(\ln f(x))'' = (\ln g(x))''$. We denoted $(\ln f)''$ by ℓ'' [6].

And now, we go back to Pearson distributions and their RIFns/hazard rates.

Theorem 3. *Let f be the density function of a Pearson distribution with (1–4).*

Let $M := b \cdot B - A \cdot c$, $\quad L := a \cdot B^2 - A \cdot M$, $\quad D := a \cdot L$ and assume that the conditions (9) are fulfilled:

$$\text{(9.1)} \qquad \textit{If } a = 0, \textit{ then } M > 0;$$

$$\text{(9.2)} \qquad \begin{cases} \textit{If } a \neq 0 \textit{ and } A = 0, \textit{ then} \\ \textit{if } a \cdot B > 0, \textit{ then } m + b_1 \geq 0; \\ \textit{if } a \cdot B < 0, \textit{ then } s + b_1 \leq 0; \end{cases}$$

$$\text{(9.3)} \qquad \textit{If } a \cdot A \neq 0 \textit{ and } q(-B/A) = 0, \textit{ then } a \cdot A > 0;$$

$$\text{(9.4)} \qquad \begin{cases} \textit{If } a \cdot A > 0 \textit{ and } q(-B/A) \neq 0, \textit{ then either } D < 0, \textit{ or} \\ [D \geq 0 \textit{ and (either } Y(m) \geq 0 \textit{ or } Y(s) \leq 0)]; \end{cases}$$

$$\text{(9.5)} \qquad \begin{cases} \textit{If } a \cdot A < 0 \textit{ and } q(-B/A) \neq 0, \textit{ then} \\ [D \geq 0, \ Y(m) \leq 0 \textit{ and } Y(s) \geq 0], \end{cases}$$

where $Y(v) := a \cdot (A \cdot v + B) + \operatorname{sign}[(v - m) + (v - s)] \cdot D_1$, $\quad v \in \{m, s\}$, $\quad b_1 := b/(2a)$, $\quad D_1 := D^{1/2}$ *and* $\infty - \infty := 0$.

Then h, the corresponding RIF, either strictly increases in I, or there exists y in I such that h strictly increases in (r, y) and strictly decreases in (y, s).

Furthermore,

h strictly increases in I if $\Psi(s^-) = \lim_{x \to s^-} \Psi(x) \leq 1$;

h strictly increases in (r, y) and strictly decreases in (y, s) for some $y \in I$, if $\Psi(s^-) > 1$.

Proof. By Theorem 1, it is sufficient to show that the condition (5) is fulfilled. We distinguish three cases.

Case 1. $A \neq 0$ and $q(-B/A) \neq 0$. In this case, the polynomials q and Q have no common zero. The condition (5) can be written in the form

$$(f/f')' = (q/Q)' = [(2ax + b) \cdot Q - A \cdot q]/Q^2 > 0, \qquad \text{i.e.,}$$

$$\text{(10)} \qquad p(x) := a \cdot A \cdot x^2 + 2aBx + M > 0 \quad \text{in } (m, s) \setminus \{-B/A\}.$$

If $a \neq 0$ and $D \geq 0$, then the roots of p are $x_{1,2} = (-aB \pm D_1)/(a \cdot A)$ with $x_1 \leq x_2$.

Subcase c1.1. $a \cdot A > 0$. Then the parabola p is concave up. The condition (9.4) applies.

If $D < 0$, then p has no real zero, and $p(x) > 0$ for all x.

If $D \geq 0$, then p has real zeros. (9.4) gives that either $D_1 \leq a \cdot A \cdot m + a \cdot B$ or $a \cdot A \cdot s + a \cdot B \leq$ $\leq -D_1$, i.e. either $x_2 \leq m$ or $s \leq x_1$. In both cases, $p(x) > 0$ for every $x \in (m, s)$.

Subcase 1.2. $a = 0$. Then p has the special form $p(x) \equiv M$ and it is positive since (9.1) applies.

Subcase 1.3. $a \cdot A < 0$. Then p is concave down. The condition (9.5) applies. The inequality $D \geq 0$ implies that the zeros x_1 and x_2 of p are real, and

$$x_1 = (-aB + D_1)/(a \cdot A) \leq (-aB - D_1)/(a \cdot A) = x_2.$$

On the other hand, from (9.5) we get $aAm + aB \leq D_1$ and $aAs + aB \geq -D_1$, i.e. $x_1 \leq m$ and $s \leq x_2$. Thus $(m, s) \subseteq (x_1, x_2)$ and $p(x) > 0$, $x \in (m, s)$.

Case 2. $A \neq 0$ and $q(-B/A) = 0$. In this case, q and Q have a common zero $x_1 = -B/A$.

Subcase 2.1. $a \neq 0$. Then

$$q/Q = a \cdot (x - x_2)/A, \quad \text{and}$$

$(f/f')' = a/A$ since the condition (9.3) applies.

Subcase 2.2. $a = 0$.

Then $b \neq 0$ (since $b = 0$ implies $q(x) \equiv c$; from $q(-B/A) = 0$ we obtain $c = 0$ which contradicts the Definition 2.) The common root of q and Q is equal to $x_1 = -B/A = -c/b$, so $A \cdot c = b \cdot B$ and $M = 0$. On the other hand, $f/f' = b/A$, which leads to the exponential distribution (Cf. Remark 1.7 in [6].) The conditions (5) and (9.1) do not apply.

Case 3. $A = 0$. Then $B \neq 0$, and the condition (5) has the form

$$(f/f')' = (2ax + b)/B > 0, \quad x \in (m, s). \tag{11}$$

Subcase 3.1. $B > 0$.

If $a = 0$, then the sufficient condition for (11) is $b > 0$, which follows from (9.1).

If $a > 0$, then (9.2) applies to give $-b_1 \leq m$. Thus, every x in (m, s) will be greater than $-b_1$, i.e. (11) is fulfilled.

Similarly, if $a < 0$, then (9.2) gives $s \leq -b_1$. So every x in (m, s) will be less than $-b_1$. Hence, (11) holds.

Subcase 3.2. $B < 0$. Then (11) has the form

$$2ax < -b, \quad x \in (m, s). \tag{12}$$

If $a = 0$, then (9.1) applies to give $b < 0$, and (12) is fulfilled.

If $a > 0$, then (9.2) gives $s \leq -b_1$. So, for every x in (m, s), we have $x < -b_1$, and (12) holds.

If $a < 0$, then (9.2) applies to give $m \geq -b_1$. Thus, for each x from (m, s), we have $x > -b_1$, and (12) is fulfilled. The proof is complete. □

Remark 3.1. The value of s in (9.5) must be finite, since $a \cdot A < 0$ and $a \cdot A \cdot s + a \cdot B + D_1 \geq 0$.

Remark 3.2. If $m = s$, then Remark 1.1 of [6] applies and no conditions (9) are required. The RIF strictly increases in I.

Theorem 4. *Let f be the density function of a Pearson distribution with $m = r$, (1), (3–4) and M, L, D be defined as in Theorem 3. We assume the following conditions (13) are fulfilled:*

$$\text{(13.1)} \qquad \textit{If } a = 0, \textit{ then } M < 0;$$

$$\text{(13.2)} \qquad \begin{cases} \textit{If } a \neq 0 \textit{ and } A = 0, \textit{ then} \\ \textit{if } a \cdot B > 0, \textit{ then } s + b_1 \leq 0; \\ \textit{if } a \cdot B < 0, \textit{ then } m + b_1 \geq 0; \end{cases}$$

$$\text{(13.3)} \qquad \textit{If } a \cdot A \neq 0 \textit{ and } q(-B/A) = 0, \textit{ then } a \cdot A < 0;$$

$$\text{(13.4)} \qquad \begin{cases} \textit{If } a \cdot A > 0 \textit{ and } q(-B/A) \neq 0, \textit{ then} \\ [D \geq 0,\ Y_1(m) \geq 0 \textit{ and } Y_1(s) \leq 0]; \end{cases}$$

$$\text{(13.5)} \qquad \begin{cases} \textit{If } a \cdot A < 0 \textit{ and } q(-B/A) \neq 0, \textit{ then either } D < 0, \textit{ or} \\ [D \geq 0, \textit{ and } \{\textit{either } Y_1(s) \geq 0 \textit{ or } Y_1(m) \leq 0\}], \end{cases}$$

where $Y_1(v) := a \cdot (A \cdot v + B) - \operatorname{sign}[(v - m) + (v - s)] \cdot D_1$, $v \in \{m, s\}$, $D_1 := D^{1/2}$, $b_1 := b/(2a)$ *and* $\infty - \infty := 0$.

Then r is finite, and all the assertions of Theorem 2 hold.

Proof. By Theorem 2, it is enough to prove that (6) holds. We have three cases.

Case 1. $A \neq 0$ and $q(-B/A) \neq 0$. The condition (6) can be written in the form

$$p(x) < 0 \quad \text{in } (m, s) \setminus \{-B/A\}. \tag{14}$$

Subcase 1.1. $a \cdot A < 0$. Then p is concave down, and (13.5) applies. If $D < 0$, then p has no real zero, and $p < 0$ in $\mathbf{R}$. If $D \geq 0$, then p has the real roots $x_1 \leq x_2$. The requirement {either $Y_1(s) \geq 0$ or $Y_1(m) \leq 0$} is equivalent to {either $s \leq x_1$ or $x_2 \leq m$}, and in both cases, $(m, s) \subset (-\infty, x_1) \cup (x_2, \infty) =: U$. Since $p < 0$ in U, (14) is fulfilled.

Subcase 1.2. $a \cdot A > 0$. Then p is concave up, and (13.4) applies. $D \geq 0$, so p has two real zeros x_1, x_2 with $x_1 \leq x_2$. The requirement $\{Y_1(m) \geq 0$ and $Y_1(s) \leq 0\}$ is equivalent to $\{x_1 \leq m$ and $s \leq x_2\}$, i.e., to $(m, s) \subseteq (x_1, x_2)$, and (14) holds since $p < 0$ in (x_1, x_2).

Subcase 1.3. $a = 0$. Then $p(x) \equiv M$, and (13.1) applies.

Case 2. $A \neq 0$ and $q(-B/A) = 0$. Then $x_1 := -B/A$ is the common zero of Q and q.

Subcase 2.1. $a \neq 0$. Then $q/Q = a \cdot (x - x_2)/A$ and (13.3) applies: $(f/f')' = a/A < 0$.

Subcase 2.2. $a = 0$. Then $b \neq 0$ (since $b = 0$ leads to a contradiction, see the Subcase 2.2 of the proof of Theorem 3), and the common root of q and Q is $x_1 = -B/A = -c/b$, thus $M = 0$. So, the conditions (6) and (13.1) do not apply.

Case 3. $A = 0$. Then $B \neq 0$, and (6) has the simple form

$$(f/f')' = (2ax + b)/B < 0 \quad \text{in } (m, s). \tag{15}$$

subcase 3.1. $B > 0$. Then (15) has the form

$$2ax + b < 0 \quad \text{in } (m, s). \tag{16}$$

If $a = 0$, then (13.1) applies to give $M = b \cdot B < 0$, i.e. $b < 0$, so (16) holds.

If $a > 0$, then (13.2) applies to give $s \leq -b_1$, thus (16) is fulfilled. If $a < 0$, then (13.2) applies: $m \geq -b_1$, and (16) holds.

Subcase 3.2. $B < 0$. Then (15) can be written as follows:

$$2ax + b > 0 \quad \text{in } (m, s). \tag{17}$$

If $a = 0$, then (13.1) applies: $M = b \cdot B < 0$, i.e. $b > 0$ and (17) holds.

If $a > 0$, then (13.2) applies to give $m \geq -b_1$, thus (17) is fulfilled. If $a < 0$, then (13.2) gives $s \leq -b_1$, so (17) holds, and the proof is complete. □

The Remarks following Theorems 1 and 2 in [6] apply to Pearson distributions, too.In this simple case, $\Psi(s^-)$ has special forms as may be seen in the following

Lemma 2. *Let f be the density function of a Pearson distribution with (1) and (3–4).*

I. Let $s = \infty$, and $f_\infty := \lim_{x\to\infty} x \cdot f(x)$.

I.1 If $A = a = f_\infty = 0$ and $b + B \neq 0$, then $\Psi(\infty) = B/(b + B)$;

I.2 If $A \neq 0$ and $a = 0$, then $\Psi(\infty) = 1$;

I.3 If $a \cdot A \cdot (a + A) \neq 0$ and $f_\infty = 0$, then $\Psi(\infty) = A/(a + A)$.

I.4 If $(a = A = 0,\ b \cdot f_\infty \neq 0)$ or $(A = 0,\ a \neq 0)$ or $(a \cdot A \cdot f_\infty \neq 0)$, then $\Psi(\infty) = 0$.

II. Let s be a finite real number, and let $\lim_{x\to s^-} f(x) = 0$.

II.1 If $[A \cdot Q(s) \cdot q(s) \neq 0]$ or $[A \cdot q'(s) \neq 0,\quad Q(s) = q(s) = 0]$ or $[A = 0,\ q(s) \neq 0]$ or $[A \cdot Q(s) \neq 0,\ q(s) = q'(s) = 0]$, then $\Psi(s^-) = 1$.

II.2 If $A \neq 0$, $q(-B/A) \cdot Q(s) \cdot [Q(s) + q'(s)] \neq 0$, $q(s) = 0$, then $\Psi(s^-) = Q(s)/[Q(s) + q'(s)]$.

II.3 If $A \neq 0$, $q(-B/A) \cdot q(s) \neq 0$, $Q(s) = 0$, then $\Psi(s^-) = 0$.

II.4 If $A \neq 0$, $q(-B/A) = q(s) = 0$, $a + A \neq 0$, and [either $Q(s) = q'(s) = 0$ or $Q(s) \neq 0$], then $\Psi(s^-) = A/(a + A)$.

II.5 If $A = a = q(s) = 0$, $q'(s) \cdot (b + B) \neq 0$, then $\Psi(s^-) = B/(b + B)$.

Proof. I. In this part, lim always means $\lim_{x\to\infty}$.

I.1 Now we have $f'/f = B/(bx + c) \neq 0$. If $b \neq 0$, then we apply L'Hospital's rule to get

$$\begin{aligned}\Psi(\infty) &= \lim[(F-1)/(x \cdot f)] \cdot [Bx/(bx + c)] = (B/b) \cdot \lim(F-1)/(x \cdot f) = \\ &= (B/b) \cdot \lim(1 + x \cdot f'/f)^{-1} = (B/b) \cdot \lim[1 + Bx/(bx + c)]^{-1} = \\ &= B/(b + B).\end{aligned}$$

If $b = 0$, then

$$\Psi(\infty) = (B/c) \cdot \lim[(F-1)/f] = (B/c) \cdot \lim(f/f') = (B/c) \cdot (c/B) = 1 = B/(b + B).$$

I.2 We have $f'/f = (Ax + B)/(bx + c)$, thus

$$\begin{aligned}\Psi(\infty) &= \lim[(F-1)/f] \cdot [(Ax + B)/(bx + c)] = \\ &= (A/b) \cdot \lim(f/f') = (A/b) \cdot \lim(bx + c)/(Ax + B) = 1,\end{aligned}$$

provided $b \neq 0$. If $b = 0$, then

$$\begin{aligned}\Psi(\infty) &= \lim[(F-1)/(f/x)] \cdot [(Ax + B)/(cx)] = \\ &= (A/c) \cdot \lim x^2/(x \cdot f'/f - 1) = (A/c) \cdot \lim[(A + B/x)/c - x^{-2}]^{-1} = 1.\end{aligned}$$

1.3 We have

$$\begin{aligned}\Psi(\infty) &= \lim[(F-1)/(x \cdot f)] \cdot [(Ax^2 + Bx)/(ax^2 + bx + c)] = \\ &= (A/a) \cdot \lim f/(f + x \cdot f') = (A/a) \cdot \lim[1 + x \cdot f'/f]^{-1} = \\ &= (A/a) \cdot \lim[1 + (Ax^2 + Bx)/(ax^2 + bx + c)]^{-1} = \\ &= A/(a + A), \text{ provided } q(-B/A) \neq 0. \text{ If } q(-B/A) = 0,\end{aligned}$$

then

$$f'/f = A \cdot (x + B/A)/[a \cdot (x + B/A) \cdot (x - x_2)] = A/[a \cdot (x - x_2)],$$

and

$$\Psi(\infty) = \lim[(F-1)/(x \cdot f)] \cdot Ax/[a \cdot (x - x_2)] = (A/a) \cdot \lim f/(f + x \cdot f') =$$
$$= (A/a) \cdot \lim[1 + Ax/(ax - ax_2)]^{-1} = A/(a + A).$$

I.4 (i) Let $a = A = 0,\ b \neq 0,\ f_\infty \neq 0$. Then $f'/f = B(bx + c)$, and

$$\Psi(\infty) = \lim[(F-1)/(x \cdot f)] \cdot [Bx/(bx + c)] =$$
$$= (B/b) \cdot \lim[(F-1)/(x \cdot f)] = (B/b) \cdot (0/f_\infty) = 0.$$

(ii) Let $A = 0,\ a \neq 0$. Then $f'/f = B/(ax^2 + bx + c)$, and

$$\Psi(\infty) = \lim[(F-1)/(x^2 \cdot f)] \cdot [Bx^2/(ax^2 + bx + c)] =$$
$$= (B/a) \cdot \lim[(F-1)/(x^2 \cdot f)] = 0,$$

if $\lim x^2 \cdot f \neq 0$; otherwise, L'Hospital's rule applies to give

$$\Psi(\infty) = (B/a) \cdot \lim f/(x^2 \cdot f' + 2x \cdot f) =$$
$$= (B/a) \cdot \lim[2x + x^2 \cdot f'/f]^{-1} =$$
$$= (B/a) \cdot \lim[2x + B/(a + b/x + c/x^2)]^{-1} = 0.$$

(iii) Let $a \cdot A \cdot f_\infty \neq 0$. Then

$$\Psi(\infty) = \lim[(F-1)/(x \cdot f)] \cdot [(Ax^2 + Bx)/(ax^2 + bx + c)] =$$
$$= (A/a) \cdot \lim[(F-1)/(x \cdot f)] = (A/a) \cdot (0/f_\infty) = 0.$$

II. In this part, lim always means $\lim_{x \to s^-}$, where $s \in \mathbf{R}$.

II.1 (i) Let $A \cdot Q(s) \cdot q(s) \neq 0$. We have two cases:
Case (i.1): $q(-B/A) \neq 0$.
Then $\Psi(s^-) = \lim[(F-1)/f] \cdot [Q/q] = [Q(s)/q(s)] \cdot \lim(f/f') = 1$.
Case (i.2): $q(-B/A) = 0$. Then $f'/f = Q/q = A \cdot (x + B/A).[a \cdot (x + B/A) \cdot (x - x_2)] =$
$= A/[a \cdot (x - x_2)]$, where $x_2 \neq s$. Thus we have $\Psi(s^-) = A/[a \cdot (s - x_2)] \cdot \lim(f/f') = 1$.

(ii) Let $A \cdot q'(s) \neq 0$ and $q(s) = Q(s) = 0$. Then $s = -B/A$, and $\lim Q/q = \lim Q'/q' = A/(2as + b)$. Hence,

$$\Psi(s^-) = A/(2as + b) \cdot \lim(f/f') = A/(2as + b) \cdot \lim q'/Q' = 1.$$

(iii) Let $A = 0$ and $q(s) \neq 0$. Then we get

$$\Psi(s^-) = B/q(s) \cdot \lim(F-1)/f = B/q(s) \cdot \lim f/f' = 1.$$

(iv) $q = a \cdot (x - s)^2$, since $q(s) = q'(s) = 0$. So we have $\Psi(s^-) = [Q(s)/a] \cdot L_1$, where

$$L_1 = \lim(F-1)/[(x-s)^2 \cdot f] = \lim f/[2 \cdot (x - s) \cdot f + (x - s)^2 \cdot f'] =$$
$$= \lim[2 \cdot (x - s) + Q(x)/a]^{-1} = a/Q(s).$$

Thus we get $\Psi(s^-) = 1$.

II.2. Let $A \neq 0,\ q(-B/A) \cdot Q(s) \cdot [Q(s) + q'(s)] \neq 0$ and $q(s) = 0$. We have two cases.

Case (i): $q'(s) \neq 0$. Then L'Hospital's rule gives

$$\Psi(s^-) = Q(s) \cdot \lim(F-1)/(f \cdot q) = Q(s) \cdot \lim f/(f' \cdot q + f \cdot q') =$$
$$= Q(s) \cdot \lim[q' + q \cdot f'/f]^{-1} = Q(s) \cdot \lim[q' + Q]^{-1} = Q(s)/[Q(s) + q'(s)].$$

Case (ii): $q'(s) = 0$. Then $q(x) = a \cdot (x-s)^2,\ a \neq 0$. We have

$$\Psi(s^-) = \lim[(F-1)/f] \cdot Q(x) \cdot a^{-1} \cdot (x-s)^{-2} = Q(s) \cdot a^{-1} \cdot L_1,$$

where

$$L_1 = \lim(F-1)/[f \cdot (x-s)^2] = \lim f/[f' \cdot (x-s)^2 + 2 \cdot (x-s) \cdot f] =$$
$$= \lim[(x-s)^2 \cdot Q/q + 2 \cdot (x-s)]^{-1} = [Q(s)/a]^{-1},$$

so $\Psi(s^-) = 1 = Q(s)/[Q(s) + q'(s)]$.

II.3 Let $A \cdot q(s) \neq 0$ and $Q(s) = 0$. Then $s = -B/A$, and $Q = A \cdot (x-s)$. Thus $\Psi(s^-) = A/q(s) \cdot L_1$, where $L_1 = \lim(F-1)/[f/(x-s)]$.
If $\lim f/(x-s) \neq 0$, then $L_1 = 0$, so $\Psi(s^-) = 0$.
If $\lim f/(x-s) = 0$, then L'Hospital's rule gives

$$L_1 = \lim f/\{[(x-s) \cdot f' - f] \cdot (x-s)^{-2}\} =$$
$$= \lim(x-s)^2/[A \cdot (x-s)^2/q(x) - 1] = 0,$$

and $\Psi(s^-) = 0$, too.

II.4 Let $A \neq 0,\ q(-B/A) = q(s) = 0,\ a + A \neq 0$.
Case (i): $Q(s) = q'(s) = 0$. Then $s = -B/A$,
$q(x) = a \cdot (x-s)^2,\ a \neq 0$, and $Q(x) = A \cdot (x-s)$.
We have $\Psi(s^-) = A/a \cdot \lim(F-1)/[(x-s) \cdot f] = A/a \cdot \lim[1 + (x-s) \cdot Q/q]^{-1} = A/(a+A)$.
Case (ii): $Q(s) \neq 0$. The equality $q'(s) = 0$ would lead to $q(x) = a \cdot (x-s)^2$ and $s = -B/A$, which contradicts $Q(s) \neq 0$. So we have $q'(s) \neq 0$ and $s \neq -B/A$.
Thus $q(x) = a \cdot (x + B/A) \cdot (x-s)$, and $Q/q = A/[a \cdot (x-s)]$. Hence,

$$\Psi(s^-) = A/a \cdot \lim(F-1)/[(x-s) \cdot f] = A/a \cdot \lim[1 + (x-s) \cdot Q/q]^{-1} = A/(a+A).$$

II.5 Let $A = a = q(s) = 0,\ q'(s) \cdot (b + B) \neq 0$. Then $q = bx + c,\ b \neq 0,\ s = -c/b$ and $Q/q = B/[b \cdot (x-s)]$. Thus $\Psi(s^-) = B/b \cdot \lim(F-1)/[(x-s) \cdot f] = B/b \cdot \lim[1 + (x-s) \cdot Q/q] = B/(b+B)$. The proof is complete □.

Let us see some applications to specific Pearson distributions.

Example 1. Normal distribution: $f = K \cdot E$, where $K = \sigma^{-1} \cdot (2\pi)^{-1/2}$, $E = \exp[-1/2 \cdot \sigma^{-2} \cdot (x-m)^2]$, $\sigma > 0$. We have $A = 1,\ B = -m,\ a = b = 0,\ c = -\sigma^2$.

Theorem 3 applies with $I = (-\infty, \infty)$, $m \in I$, (1–4) are fulfilled, (9.1) applies, since $a = 0$, $M = \sigma^2 > 0$; Lemma 2, I.2 applies, because $f_\infty = k \cdot \lim_{x\to\infty} x \cdot E = 0$, so $\Psi(\infty) = 1$, and the RIF h strictly increases in I. (On the other hand, (5$'$) can be checked immediately, since $f \overset{\prime\prime}{\sim} g = E$ and $l'' = [-\frac{1}{2} \cdot \sigma^{-2} \cdot (x-m)^2]'' = -1/\sigma^2 < 0$ in $\mathbf{R}$). Cf. [1,5].

Example 2. (Special) Gamma distribution:

$$f(x) = K \cdot x^{\alpha-1} \cdot \exp(-\lambda x), \quad K = \lambda^\alpha/\Gamma(\alpha), \quad \lambda > 0,\ \alpha > 1;$$
$$A = -\lambda, \quad B = \alpha - 1, \quad a = c = 0, \quad b = 1;$$

Theorem 3 applies: $I = (0, \infty)$; $m = (\alpha-1)/\lambda \in I$; (1–4) hold; (9.1): $a = 0$, $M = \alpha - 1 > 0$; Lemma 2, I.2 applies, since $f_\infty = K \cdot \lim_{x\to\infty} x^\alpha/\exp(\lambda x) = 0$. Thus, $\Psi(\infty) = 1$ and the RIF strictly increases in I.

Example 3. Chi-square distribution:

$$f(x) = K \cdot x^{n/2-1} \cdot \exp(-x/2), \quad K = 2^{-n/2}/\Gamma(n/2),\ n \text{ is a positive integer},\ I = (0,\infty).$$

If $n > 2$, then we obtain a special case of Example 2 with $\alpha = n/2$, $\lambda = 1/2$. Thus, the RIF strictly increases in I. If $n = 2$, then we get a special exponential distribution, for which the RIF is constant (see Remark 1.7 in [6]). If $n = 1$, then $f = (2\pi \cdot x)^{-1/2} \cdot \exp(-x/2)$; Theorem 4 with (13.1) applies, since $f' < 0$ in I, $m = 0$, $M = -2 < 0$, $f_\infty = (2\pi)^{-1/2} \cdot \lim_{x\to\infty}(x/e^x)^{1/2} = 0$, $A = B = -1/2$, $a = c = 0$, $b = 1$,

$$\Psi(\infty) = 1, \Psi(0^+) = -\frac{1}{2} \cdot \lim_{x\to 0^+}(x \cdot f)^{-1} \cdot \lim_{x\to 0^+}[(F-1)\cdot(x+1)] =$$
$$= \frac{1}{2} \cdot \lim_{x\to 0^+}(x \cdot f)^{-1} = (\pi/2)^{1/2} \cdot \lim_{x\to 0^+} x^{-1/2} = +\infty;$$

(8) and (8.1) apply: the RIF strictly decreases in I.

Example 4. Beta distribution of the first kind: $f(x) = C \cdot x^\alpha \cdot (1-x)^\beta$, where $\alpha, \beta > -1$, $C = \Gamma(\alpha+\beta+2)/[\Gamma(\alpha+1)\cdot\Gamma(\beta+1)]$. Let $\alpha, \beta > 0$.

Theorem 3 applies: $I = (0,1)$; $m \in I$; $A = -(\alpha+\beta)$, $B = \alpha$, $a = -1$, $b = 1$, $c = 0$; $M = \alpha$, $L = \alpha\cdot\beta$; (9.4) applies: $a \cdot A = \alpha+\beta > 0$, $-B/A = \alpha/(\alpha+\beta)$, $q(x) = x - x^2$, $q(-B/A) = \alpha\cdot\beta\cdot(\alpha+\beta)^{-2} \neq 0$, $D = -\alpha\cdot\beta < 0$; Lemma 2, II.2 applies: $Q(x) = \alpha - (\alpha+\beta)\cdot x$, $s = 1$, $Q(1) = -\beta \neq 0$, $q(1) = 0$, $Q(1) + q'(1) = -(1+\beta) \neq 0$, thus $\Psi(1^-) = \beta/(1+\beta) < 1$, and the RIF strictly increases in I.

Example 5. $f(x) = C\cdot(1 - x^2/s^2)^n$, where $C = [s\cdot B(\frac{1}{2}, n+1)]^{-1}$, $s > 0$, n is a positive integer

(Example 6.1 in [4]); $A = 2n$, $B = b = 0$, $a = 1$, $c = -s^2$; $I = (-s,s)$; $m = 0 \in I$; Theorem 3 applies: $M = 2ns^2$; $L = -4n^2s^2$; (9.4) applies: $q(-B/A) = -s^2 \neq 0$, $a\cdot A = 2n > 0$, $D = L < 0$; Lemma 2, II.2 applies: $Q(s) = 2ns \neq 0$, $q(s) = 0$, $q'(s) = 2s \neq 0$, $Q(s) + q'(s) = 2s\cdot(n+1) \neq 0$, so $\Psi(s^-) = n/(n+1) < 1$; thus, the RIF strictly increases in I.

Example 6. $F(x) = 1 - (-x)^k$, $k > 1$ is integer; $A = a = c = 0$, $B = k-1$, $b = 1$; $I = (-1,0)$; $m = -1$; (9.1), Remark 1.2 in [6] and Lemma 2, II.5 apply: $M = k-1 > 0$; $s = 0$,

$q(0) = 0$, $q'(0) = 1$, $b + B = k$, so $\Psi(0^-) = (k-1)/k < 1$, and the RIF strictly increases in I.

Example 7. Pareto distribution of the 2nd kind: $F(x) = 1 - x^{-k}$, $k > 0$ (Cf. Chap. 19 in Vol. 1 of [3]); $A = a = c = 0$, $B = k+1$, $b = -1$; $I = (1,\infty)$; $m = 1$.

Theorem 4 with (13.1), Remark 1.2 of [6] and Lemma 2, I.1 apply: $M = -k-1 < 0$, $f_\infty = \lim_{x\to\infty} k/x^k = 0$, $b + B = k \neq 0$, so $\Psi(\infty) = (k+1)/k[= \Psi(1^+)] > 1$ and the RIF strictly decreases in I. (Actually, $\Psi(x) \equiv (k+1)/k$ in I.)

Remark 4.1. We can say that the distributions in the examples $1-2$, 3 (if $n > 2$), $4-6$ are IHR (increasing hazard rate), while the examples 3 (when $n = 2$) and 7 are DHR (decreasing hazard rate) distributions.

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