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MIKUSIŃSKI FUNCTIONAL EQUATION ON A HEXAGON

KÁROLY LAJKÓ

Dedicated to Professor Árpád Varecza on the occasion of his 60th birthday

Abstract. The general solution of the conditional functional equation (M) is described for functions $f\colon (-r,r) \to \mathbb{R}$, where (M) is satisfied for all $(x,y) \in H$, where $H = \{(x,y) \mid x, y, x+y \in (-r,r)\}$ is a hexagon.

1. Introduction

J. Mikusiński (in 1971) mentioned the functional equation

$$f(x+y)\big[f(x+y) - f(x) - f(y)\big] = 0 \tag{M}$$

which since has been named after him.

The authors of [2] find the general solution of (M) for functions $f\colon X \to Y$ where $(X,+)$ and $(Y,+)$ are (not necessarily commutative) groups. In case $X = Y = \mathbb{R}$ they proved the following

Theorem 1. *The only solutions of equation* (M) *for functions* $f\colon \mathbb{R} \to \mathbb{R}$ *are additive functions, i.e. the solutions of Cauchy functional equation*

$$f(x+y) = f(x) + f(y) \qquad (x, y \in \mathbb{R}). \tag{1}$$

The aim of this paper is to present the general solution of (M) for functions $f\colon (-r,r) \to \mathbb{R}$, where (M) is satisfied for all $(x,y) \in H = \{(x,y) \mid x, y, x+y \in (-r,r)\}$ and $(-r,r)$ is an open interval in $\mathbb{R}$.

2. An extension theorem for (M)

Following the ideas of Aczél [1] and Kuczma [3] we prove the following extension theorem for the Mikusiński functional equation (M).

Theorem 2. *If the function* $f\colon (-r,r) \to \mathbb{R}$ *satisfies the Mikusiński functional equation* (M) *for all* $(x,y) \in H$*, where* H *is a hexagon given above, then there exists a unique function* $F\colon \mathbb{R} \to \mathbb{R}$ *satisfying* (M) *for any* $x, y \in \mathbb{R}$ *and*

$$f(x) = F(x), \qquad x \in (-r,r).$$

Proof. a) First we show that

$$f\left(\frac{x}{2^n}\right) = \frac{1}{2^n} f(x), \qquad x \in (-r,r),\ n \in N. \tag{2}$$

If $f(x) = 0$, then it is easy to see that $f(2^n x) = 0$ $(2^n x \in (-r,r))$, which implies (2).

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If $f(x) \neq 0$, then replacing both x and y by $\frac{x}{2}$, we get from (M)

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \neq 0, \qquad x \in (-r, r). \tag{3}$$

Thus (2) holds for $n = 1$.
Using (3) repeatedly completes the statement.

b) On the other hand, for every $u \in \mathbb{R}$ there exists an $n \in N \cup \{0\}$ such that $x = \frac{u}{2^n} \in (-r, r)$. We define the function F by

$$F\colon \mathbb{R} \to \mathbb{R}, \quad F(u) = 2^n f\left(\frac{u}{2^n}\right) \quad \left(\frac{u}{2^n} \in (-r, r)\right). \tag{4}$$

This definition is correct and (4) gives

$$f(x) = F(x), \qquad x \in (-r, r).$$

c) We must verify that F satisfies (M) for all $x, y \in \mathbb{R}$.
If $x, y \in \mathbb{R}$ are arbitrary, then there exists an $n \in N \cup \{0\}$ such that

$$\frac{x}{2^n}, \ \frac{y}{2^n}, \ \frac{x+y}{2^n} \in (-r, r).$$

Now

$$f\left(\frac{x}{2^n} + \frac{y}{2^n}\right)\left[f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{x}{2^n} + \frac{y}{2^n}\right)\right] = 0,$$

that is

$$2^n f\left(\frac{x+y}{2^n}\right)\left[2^n f\left(\frac{x}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{x+y}{2^n}\right)\right] = 0.$$

This implies that the function F, defined by (4) satisfies (M) for all $x, y \in \mathbb{R}$.

d) To prove the uniqueness, suppose that a function $G\colon \mathbb{R} \to \mathbb{R}$ satisfies (M) in $\mathbb{R}$ and fulfills the condition

$$G(x) = f(x), \qquad x \in (-r, r). \tag{5}$$

Similarly as in a) one can get that

$$G\left(\frac{x}{2^n}\right) = \frac{1}{2^n} G(x), \qquad x \in \mathbb{R}, \ n \in N \cup \{0\}. \tag{6}$$

Take an arbitrary $x \in \mathbb{R}$. There exists an $n \in N \cup \{0\}$ such that $\frac{x}{2^n} \in (-r, r)$.
Thus we have by (4), (5) and (6)

$$G(x) = 2^n G\left(\frac{x}{2^n}\right) = 2^n f\left(\frac{x}{2^n}\right) = F(x).$$

Consequently $G = F$ in $\mathbb{R}$. □

3. The general solution of (M) on a hexagon

Using Theorems 1 and 2 we obtain

Theorem 3. *If the function $f\colon (-r, r) \to \mathbb{R}$ satisfies the Mikusiński functional equation* (M) *for all $(x, y) \in H$, then there exists a unique additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = A(x), \quad x \in (-r, r). \tag{7}$$

Proof. Theorem 2 shows that there exists a unique function $F\colon \mathbb{R} \to \mathbb{R}$ satisfying (M) for all $x, y \in \mathbb{R}$ and $f(x) = F(x)$ $x \in (-r, r)$.

Because of Theorem 1 F is an additive function.

It is easy to see that all additive functions A fulfill also (M) for all $(x, y) \in H$. □

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Institute of Mathematics and Informatics,
University of Debrecen,
H-4010 Debrecen, Pf. 12, Hungary
E-mail address: lajko@math.klte.hu