Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
18 (2002), 33–51
www.emis.de/journals

EXPLODED AND COMPRESSED NUMBERS

I. SZALAY

Abstract. In this paper we introduce the concept of exploded and compressed real number such that the set of exploded real numbers contains the set of real numbers. Moreover, the concept of equality, ordering, neighbourhood and convergence will be extended for the set of exploded real numbers. We will find operations for exploded real numbers such that the set of exploded real numbers will be isomorphic to the set of real numbers. The set of compressed real numbers is a subset of real numbers and a model for the relationship of the sets of real numbers and exploded real numbers, respectively. Introducing the concept of super-function we extend the concept of continuity to functions with one variable. Finally, we say a few words on the repeated explosions and compressions.

Preliminary: the genesis of exploded real numbers

The concept of natural numbers $1, 2, 3, \ldots$ originates from the pre-historical ages. The oldest documents for example the Rhind papyrus or the column of Hammurabi (see [1] pp. 1-3.) contain the operations addition and multiplication. In the modern language of algebra we say that the set of natural numbers with respect to addition forms a commutative semigroup and with respect to multiplication forms a commutative semigroup with the unity element 1 (See [2] pp. 28., 32., 35. and 29.) Moreover, the distributivity

$$(n+m)k = nk + mk$$

is valid. The use of (small) natural numbers was an essential step in the development of human language. The ordering of natural numbers was also known and if $n < m$ then

$$n + k < m + k \quad \text{(monotonity of addition)}$$

and

$$nk < mk \quad \text{(monotonity of multiplication)}$$

were evident.

Some concrete positive rational numbers originate from the ancient times. The set of positive rational numbers with respect to multiplication is already a group ([2] p. 35.) and the former algebraic properties together with the properties of ordering remain valid.

The first number which was discovered in a mathematical way - between 600 and 300 B.C.- in all probability was the zero. It was signed by Hindi mathematicians by 0. On the other hand, in the set of non-negative integer (or rational) numbers the number 0 has not a multiplicative inverse, so this set is only a semigroup again, with

2000 *Mathematics Subject Classification.* 03C30.
Key words and phrases. Exploded numbers, compressed numbers, super-addition, super-multiplication, exploded-function, super-function.

respect to multiplication. The monotonity of addition is true, but the monotonity of multiplication remains valid under the condition

$$k > 0$$

only.

In China about the centuries 2-1 B.C. we can find negative integer numbers. (See [3] p. 236.) The set $\{0, \pm 1, \pm 2, \ldots\}$ is an integrity domain ([2] p. 64.). The monotonity of addition is true and the monotonity of multiplication, with $k > 0$, remains valid.

The set of rational numbers forms a field with respect to addition and multiplication ([2] p. 36. point 7.). Moreover, if a, b and c are rational numbers and $a < b$ then

$$a + c < b + c$$

and

$$ac < bc \quad (c > 0)$$

hold. So, we can see that the set of rational numbers is an ordered field ([2] p. 568.).

Irrational numbers were discovered by the Pythagoreans. The incommensurable magnitudes caused a crisis in the development of Mathematics, which was resolved in the fourth century B.C. (See [1] pp. 10-15.) In geometric algebra "number" is illustrated by a segment of the real axis. The exact introduction of real numbers is from Dedekind in 1872 ([3] p. 281.). In a certain sense, the concept of real number is the top of development of the numbers concept because the set of real numbers is an ordered field such that it is a perfect hull of the ordered field of rational numbers ([2] p. 602.).

Complex numbers originate from the 16th century but the pure algebraic introduction is from Hamilton and J. Bolyai independently from each other in 1837. (See [3] pp. 152-153.) Geometric introduction is from Gauss in 1831. (See [3] p. 131.) By the Gauss-plane we can consider the set of complex numbers as a *transversal extension of the set of real numbers.*

The set of complex numbers is isomorphic with the set of real numbers such that it is an algebraically closed field with respect to the extended addition and multiplication. On the other hand, *the set of complex numbers is not an ordered field.*

The set of exploded real numbers is another - so-called "longitudinal" - extension of the set of real numbers with the following postulates and requirements

POSTULATE OF EXTENSION:

The set of real numbers is a proper subset of the *set of exploded real numbers.* For any real number x there exists one *exploded real number* which is called *exploded* x or the *exploded of* x. Moreover, the set of exploded x is called the set of exploded real numbers.

POSTULATE OF UNAMBIGUITY:

For any pair of real numbers x and y, their *explodeds are equal* if and only if x is equal to y.

POSTULATE OF ORDERING:

For any pair of real numbers x and y, the *exploded* x *is less than exploded* y if and only if x less than y.

POSTULATE OF SUPER-ADDITION: For any pair of real numbers x and y, the *super-sum* of their explodeds is the exploded of their sum.

POSTULATE OF SUPER-MULTIPLICATION:

For any pair of real numbers x and y, the *super-product* of their explodeds is the exploded of their product.

REQUIREMENT OF EQUALITY FOR EXPLODED REAL NUMBERS:
: If x and y are real numbers then x as an exploded real number equals to y as an exploded real number if they are equal in the traditional sense.

REQUIREMENT OF ORDERING FOR EXPLODED REAL NUMBERS:
: If x and y are real numbers then x as an exploded real number is less than y as an exploded real number if x is less than y in the traditional sense.

REQUIREMENT OF MONOTONITY OF SUPER-ADDITION:
: If u and v are arbitrary exploded real numbers and u is less than v then, for any exploded real number w, u superplus w is less than v superplus w.

REQUIREMENT OF MONOTONITY OF SUPER-MULTIPLICATION:
: If u and w are arbitrary exploded real numbers and u is less than v then, for any positive exploded real number w, u super-multiplayed by w is less than v super-multiplayed by w.

In this way, we can find that the set of exploded real number is an ordered field with respect to super-addition and super-multiplication. It is isomorphic with the set of real numbers but *super-operations are not extensions of traditional operations.*

1. The set of exploded numbers

Our starting point is the set of real numbers R with its familiar relations, operations, as equality, ordering, addition and subtraction, multiplication and division. Moreover, we use the concepts of neighbourhood, convergence, monotonity and boundedness in a traditional sense. Our aim is to construct the set of exploded real numbers $\overset{\sqcup}{R}$ with the following requirements.

A. The set R will be the proper subset of $\overset{\sqcup}{R}$, that is

$$R \subset \overset{\sqcup}{R}. \tag{1.1}$$

B. To find an equality-relation for $\overset{\sqcup}{R}$ which extends the concept of familiar equality defined for R.
C. To find an ordering-relation for $\overset{\sqcup}{R}$ which extends the concept of the familiar ordering defined for R.
D. To find a neighbourhood basis for the elements of $\overset{\sqcup}{R}$ which is an extension of a neighbourhood basis defined for R.
E. To find the concept of convergence for the sequences $\{u_n\}_{n=1}^{\infty}$ where $u_n \in \overset{\sqcup}{R}$ which is an extension of concept of convergence used for the sequences of real numbers.
F. To find suitable operations on $\overset{\sqcup}{R}$ for which $\overset{\sqcup}{R}$ has an isomorphic algebraic structure with R.

For any $x \in R$ we denote by $\overset{\sqcup}{x}$ the exploded of x which is a symbol, merely. The set of symbols $\overset{\sqcup}{x}$ is $\overset{\sqcup}{R}$. Moreover, $\overset{\sqcup}{x}$ is called an exploded real number.

Definition 1.2. For any $x, y \in R$, $\overset{\sqcup}{x} \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{y}$ if and only if $x = y$.

Clearly, the relation "$\overset{\overset{\sqcup}{R}}{=}$" is reflexive, symmetrical and transitive. Definition 1.2 means, that for any $u \in \overset{\sqcup}{R}$ there exists a unique $x \in R$ such that

$$u \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{x}. \tag{1.3}$$

Moreover, we say that x is the compressed of u:

$$x = \underset{\sqcup}{u}. \tag{1.4}$$

(1.3) and (1.4) yield the identities

$$\overset{\sqcup}{(\underset{\sqcup}{u})} \overset{\overset{\sqcup}{R}}{=} u, \quad u \in \overset{\sqcup}{R} \tag{1.5}$$

and

$$\underbracket{(\overset{\sqcup}{x})} = x, \quad x \in R. \tag{1.6}$$

Definition 1.7. For any $u, v \in \overbracket{R}$ we say that $u \overset{\overset{\sqcup}{R}}{<} v$ if and only if $\underbracket{u} < \underbracket{v}$. ($\underbracket{u}, \underbracket{v} \in R$.)

Clearly, the relation "$\overset{\overset{\sqcup}{R}}{<}$" is irreflexive, antisymmetric and transitive.

Definition 1.8. The exploded real number u is called positive if $\underbracket{u} > 0$ and it is called negative if $\underbracket{u} < 0$.

Definition 1.9. For any $u_0 \in \overbracket{R}$, with $a, b \in \overbracket{R}$ such that $a \overset{\overset{\sqcup}{R}}{<} u_0 \overset{\overset{\sqcup}{R}}{<} b$ the set

$$I_{u_0}(a,b) = \left\{ u \in \overbracket{R} : a \overset{\overset{\sqcup}{R}}{<} u \overset{\overset{\sqcup}{R}}{<} b \right\} \tag{1.10}$$

is called a super-interval neighbourhood of u_0.

Definitions 1.7 and 1.9 yield

Theorem 1.11. *The exploded real number u belongs to the super-interval neighbourhood $I_{u_0}(a,b)$ if and only if*

$$\underbracket{a} < \underbracket{u} < \underbracket{b} \cdot (\underbracket{a}, \underbracket{u}, \underbracket{b} \in R.) \tag{1.12}$$

Definition 1.7 - 1.9 and Theorem 1.11 give the following properties.

Property 1.13. *For any $u_0 \in \overbracket{R}$ there exists a set $I_{u_0}(a,b)$ such that $u_0 \in I_{u_0}(a,b)$.*

Property 1.14. *If $u_0, v_0 \in \overbracket{R}$ and $u_0 \neq v_0$ then there exists sets $I_{u_0}(a,b)$ and $I_{v_0}(c,d)$ such that*

$$I_{u_0}(a,b) \cap I_{v_0}(c,d) = \emptyset.$$

Property 1.15. *If $I_{u_0}(a_1,b_1)$ and $I_{u_0}(a_2,b_2)$ are arbitrary given then there exists $I_{u_0}(a_3,b_3)$ such that*

$$I_{u_0}(a_3,b_3) \subset \Big(I_{u_0}(a_1,b_1) \cap I_{u_0}(a_2,b_2) \Big).$$

Property 1.16. *If $v_0 \in I_{u_0}(a,b)$ then there exists $I_{v_0}(c,d)$ such that*

$$I_{v_0}(c,d) \subset I_{u_0}(a,b).$$

Definition 1.17. The sequence of exploded real numbers $\{u_n\}_{n=1}^{\infty}$ is called convergent in $\overbracket{R}$ if there exists and exploded real number u_0 such that for any $I_{u_0}(a,b)$ there is a positive real number ν that if $n > \nu$ then $u_n \in I_{u_0}(a,b)$. We say that the sequence $\{u_n\}_{n=1}^{\infty}$ converges to the limit u_0 and we write

$$u_0 = \lim_{\substack{n \overset{\sqcup}{\to} \infty \\ R}} u_n.$$

Property 1.14 yields

Theorem 1.18. *Any convergent sequence of exploded real numbers may have only one limit in $\overbracket{R}$.*

Definition 1.17 and Theorem 1.11 yield

Theorem 1.19. *For any sequence of exploded real numbers*

$$u_0 = \lim_{\substack{n \overset{\sqcup}{\to} \infty \\ R}} u_n \quad \text{if and only if} \quad \underbracket{u_0} = \lim_{n \to \infty} \underbracket{u_n}$$

where the latter convergence is understood in a traditional sense.

Definition 1.20. The sequence of exploded real numbers $\{u_n\}_{n=1}^{\infty}$ is called monotonic increasing in $\overset{\llcorner\lrcorner}{R}$ if

$$u_n \overset{\overset{\llcorner\lrcorner}{R}}{\leq} u_{n+1}, \quad n = 1, 2, 3, \ldots$$

and monotonic decreasing if

$$u_n \overset{\overset{\llcorner\lrcorner}{R}}{\geq} u_{n+1}, \quad n = 1, 2, 3 \ldots.$$

Definitions 1.7 and 1.20 yield

Remark 1.21. The sequence of exploded real numbers $\{u_n\}_{n=1}^{\infty}$ is monotonic increasing or decreasing in $\overset{\llcorner\lrcorner}{R}$ if and only if the sequence $\{\underset{\llcorner\lrcorner}{u_n}\}_{n=1}^{\infty}$ is monotonic increasing or decreasing, respectively. The monotonity of $\{\underset{\llcorner\lrcorner}{u_n}\}_{n=1}^{\infty}$ is understood in a traditional sense.

Definition 1.22. The sequence of exploded real numbers $\{u_n\}_{n=1}^{\infty}$ is bounded in $\overset{\llcorner\lrcorner}{R}$ if there exists a super-interval neighbourhood $I_{\underset{0}{\llcorner\lrcorner}}(a, b)$ such that $u_n \in I_{\underset{0}{\llcorner\lrcorner}}(a, b)$, $n = 1, 2, 3 \ldots$.

Definition 1.22 and Theorem 11 yield

Remark 1.23. The sequence of exploded real numbers $\{u_n\}_{n=1}^{\infty}$ is bounded in $\overset{\llcorner\lrcorner}{R}$ if and only if the sequence $\{\underset{\llcorner\lrcorner}{u_n}\}_{n=1}^{\infty}$ is bounded in a traditional sense.

It is well known that every monotonic and bounded sequence of real numbers is convergent. (Monotonity, boundedness and convergence are understood in a traditional sense.) Applying this with Theorem 1.19 and Remarks 1.21 and 1.23 we have

Theorem 1.24. *Any monotonic and bounded sequence of exploded real numbers is convergent in* $\overset{\llcorner\lrcorner}{R}$.

Referring back to Requirements $B - E$, we gave definitions for equality, ordering, neighbourhood and convergence in $\overset{\llcorner\lrcorner}{R}$. In the following we give definitions for Requirement F.

Definition 1.25. For any pair of exploded real numbers $\overset{\llcorner\lrcorner}{x}$, $\overset{\llcorner\lrcorner}{y}$ the operation $-\!\underset{\wedge}{\overset{\vee}{\oplus}}\!-$ for which

$$\overset{\llcorner\lrcorner}{x} -\!\underset{\wedge}{\overset{\vee}{\oplus}}\!- \overset{\llcorner\lrcorner}{y} \overset{\overset{\llcorner\lrcorner}{R}}{=} \overset{\llcorner\lrcorner}{x+y}, \quad x, y \in R \tag{1.26}$$

is called super-addition where $\overset{\llcorner\lrcorner}{x} -\!\underset{\wedge}{\overset{\vee}{\oplus}}\!- \overset{\llcorner\lrcorner}{y}$ is the super-sum of $\overset{\llcorner\lrcorner}{x}$ and $\overset{\llcorner\lrcorner}{y}$.

Clearly, super-addition is commutative, associative, has the additive unit-element $\overset{\llcorner\lrcorner}{0}$ and every $\overset{\llcorner\lrcorner}{x}$ has the additive inverse-element $\overset{\llcorner\lrcorner}{-x}$.

Using $\overset{\llcorner\lrcorner}{x} \overset{\overset{\llcorner\lrcorner}{R}}{=} u$ and $\overset{\llcorner\lrcorner}{y} \overset{\overset{\llcorner\lrcorner}{R}}{=} v$ and applying (1.4) by (1.26) we have

$$u -\!\underset{\wedge}{\overset{\vee}{\oplus}}\!- v \overset{\overset{\llcorner\lrcorner}{R}}{=} \overset{\llcorner\lrcorner}{\underset{\llcorner\lrcorner}{u} + \underset{\llcorner\lrcorner}{v}}, \quad u, v \in \overset{\llcorner\lrcorner}{R}. \tag{1.27}$$

Definition 1.28. For any pair of exploded real numbers $\overset{\llcorner\lrcorner}{x}$, $\overset{\llcorner\lrcorner}{y}$ the operation $-\!\underset{\wedge}{\overset{\vee}{\ominus}}\!-$ for which

$$\overset{\llcorner\lrcorner}{x} -\!\underset{\wedge}{\overset{\vee}{\ominus}}\!- \overset{\llcorner\lrcorner}{y} \overset{\overset{\llcorner\lrcorner}{R}}{=} \overset{\llcorner\lrcorner}{x} -\!\underset{\wedge}{\overset{\vee}{\oplus}}\!- \overset{\llcorner\lrcorner}{-y}, \quad x, y \in R \tag{1.29}$$

is called super-subtraction where $\overset{\llcorner\lrcorner}{x} -\!\underset{\wedge}{\overset{\vee}{\ominus}}\!- \overset{\llcorner\lrcorner}{y}$ is the super-difference of $\overset{\llcorner\lrcorner}{x}$ and $\overset{\llcorner\lrcorner}{y}$.

By (1.5), (1.29) and (1.27) we can write

$$u \ominus v \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{\underset{\sqcup}{u} - \underset{\sqcup}{v}}, \quad u, v \in R. \tag{1.30}$$

Definition 1.31. For any pair of exploded real numbers $\overset{\sqcup}{x}$, $\overset{\sqcup}{y}$ the operation $\odot$ for which

$$\overset{\sqcup}{x} \odot \overset{\sqcup}{y} \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{x \cdot y}, \tag{1.32}$$

is called super-multiplication where $\overset{\sqcup}{x} \odot \overset{\sqcup}{y}$ is the super-product of $\overset{\sqcup}{x}$ and $\overset{\sqcup}{y}$.

Clearly, the super-multiplication is commutative, associative, has the multiplicative unit-element $\overset{\sqcup}{1}$ and, except for $\overset{\sqcup}{0}$, every $\overset{\sqcup}{x}$ has the multicative inverse-element $\overset{\sqcup}{(\frac{1}{x})}$.

Using $\overset{\sqcup}{x} \overset{\overset{\sqcup}{R}}{=} u$ and $\overset{\sqcup}{y} \overset{\overset{\sqcup}{R}}{=} v$ and applying (1.4) by (1.32) we have

$$u \odot v = \overset{\sqcup}{\underset{\sqcup}{u} \cdot \underset{\sqcup}{v}}, \quad u, v \in \overset{\sqcup}{R}. \tag{1.33}$$

Definition 1.34. Assuming that $\overset{\sqcup}{y} \overset{\overset{\sqcup}{R}}{\neq} \overset{\sqcup}{0}$, for any pair of exploded real numbers $\overset{\sqcup}{x}$, $\overset{\sqcup}{y}$, the operation $\oslash$ for which

$$\overset{\sqcup}{x} \oslash \overset{\sqcup}{y} \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{x} \odot \overset{\sqcup}{\left(\frac{1}{y}\right)}, \quad x, y (\neq 0) \in R \tag{1.35}$$

is called super-division.

By (1.32) we can write

$$\overset{\sqcup}{x} \oslash \overset{\sqcup}{y} \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{\left(\frac{x}{y}\right)}, \quad x, y (\neq 0) \in R. \tag{1.36}$$

Definitions 1.26 and 1.31 show that the distributivity

$$(\overset{\sqcup}{x} \oplus \overset{\sqcup}{y}) \odot \overset{\sqcup}{z} \overset{\overset{\sqcup}{R}}{=} (\overset{\sqcup}{x} \odot \overset{\sqcup}{z}) \oplus (\overset{\sqcup}{y} \odot \overset{\sqcup}{z}), \quad x, y, z \in R \tag{1.37}$$

holds.

Moreover, we have

Theorem 1.38. *The set of exploded real numbers $\overset{\sqcup}{R}$ is a field with operations super-addition and super-multiplication.*

Clearly, the fields R and $\overset{\sqcup}{R}$ are isomorphic by the transformation

$$\begin{aligned} x &\rightarrowtail \overset{\sqcup}{x}, \quad x \in R \\ + &\rightarrowtail \oplus \\ \cdot &\rightarrowtail \odot . \end{aligned}$$

Moreover, Definitions 1.7, 1.8 and Theorem 1.11 with (1.5), (1.27) and (1.33) yield the monotonity of super-addition: *if* $u, v, w \in \overset{\sqcup}{R}$ *and* $u \overset{\overset{\sqcup}{R}}{<} v$ *then* $u \oplus w \overset{\overset{\sqcup}{R}}{<} v \oplus w$. Moreover, *if* $w \overset{\overset{\sqcup}{R}}{>} \overset{\sqcup}{0}$ *then* $u \odot w \overset{\overset{\sqcup}{R}}{<} v \odot w$.

2. The exploder-function

First of all we say that a function f is traditional if its domain $D_f \subseteq R$ and range $R_f \subseteq R$, (where R_f is defined under (4.1)). Moreover, a traditional function σ is called an exploder-function if it has the following properties.

Property 2.1. *The open interval (-1,1) is a subset of the definition-domain of the function σ, that is*

$$(-1,1) \subseteq D_\sigma .$$

Property 2.2. *For any $x \in (-1,1)$ the equation $\sigma(-x) = -\sigma(x)$ holds.*

Property 2.3. *The function σ is continuous on the interval* $(-1,1)$.

Property 2.4. *The function σ is strictly monotonic increasing on the interval [0,1).*

Property 2.5.

$$\lim_{\substack{x \to 1 \\ x<1}} \sigma(x) = \infty .$$

Property 2.6. *For any $x \in (0,1)$ the inequality $x < \sigma(x)$ holds.*

Remark 2.7. Properties 2.2, 2.5 and 2.6 yield

$$\sigma(0) = 0, \tag{2.8}$$

$$\lim_{\substack{x \to -1 \\ x>-1}} \sigma(x) - \infty \tag{2.9}$$

and for any $x \in (-1,0)$

$$\sigma(x) < x. \tag{2.10}$$

We have already mentioned that the exploded real number $\overset{\sqcup}{x}$ $(x \in R)$ is merely a symbol. Now, using the exploder-function σ we give a concrete meaning for the explodeds of real numbers having absolute values less than 1.

Definition 2.11. For any $x \in (-1,1)$ we say that the σ-exploded of x will be

$$\overset{\sqcup}{x}_\sigma = \sigma(x). \tag{2.12}$$

By Property 2.1 we have that for any $x \in (-1,1)$ the exploded real number $\overset{\sqcup}{x}_\sigma$ is a real number, too. Using the Bolzano-Darboux property Properties 2.1, 2.2, 2.3, 2.4 and 2.5 with (2.9) give

$$R_\sigma = R \tag{2.13}$$

which shows that Requirement A is fulfilled.

Considering the exploder-function restricted for the open interval $(-1,1)$ we have its inverse function $\overline{\sigma}$ which is called compressor-function. Clearly,

$$D_{\overline{\sigma}} = R \tag{2.14}$$

and

$$R_{\overline{\sigma}} = (-1,1). \tag{2.15}$$

Moreover, for any $x, y \in (-1,1)$ Definition 2.11 shows that if $\overset{\sqcup}{x}_\sigma = \overset{\sqcup}{y}_\sigma$ then $x = y$. Considering (2.15), (2.14) and Property 2.1 for any $u \in R$

$$u = \sigma(\overline{\sigma}(u)), \quad (u \in R), \tag{2.16}$$

is obtained.

Theorem 2.17. *Assuming that $u, v \in R$ the equality*

$$u \overset{\overset{\sqcup}{R}}{=} v \tag{2.18}$$

is valid if and only if the equality $u = v$ holds.

Proof. Using (2.16) and (2.12) we have for any $u, v \in R$ that

$$u = \overbracket{(\overline{\sigma}(\underset{\sigma}{u}))} \quad \text{and} \quad v = \overbracket{(\overline{\sigma}(\underset{\sigma}{v}))}$$

holds. Definition 1.2 says that the equality (2.18) is valid if and only if $\overline{\sigma}(u) = \overline{\sigma}(v)$. Using Property 2.4 (together with Property 2.2) we have that the compressor-function is strictly increasing, too. Hence, $u = v$. □

By Theorem 2.17 we can see that Requirement B is fulfilled so we may use the traditional sign of equality "=" for elements $u, v \in \overset{\sqcup}{R}$, too

Using (2.12) and (2.16) by (1.3), (1.4) and (1.5) we have that for any $u \in R$ its compressed is

$$\underset{\sqcup\sigma}{u} = \overline{\sigma}(u), \quad u \in R. \tag{2.19}$$

Theorem 2.20. *Assuming that $u, v \in R$ the inequality*

$$u \overset{\overset{\sqcup}{R}}{<} v$$

is valid if and only if the inequality

$$u < v$$

holds.

Using Definition 1.7 instead of Definition 1.2 the proof of Theorem 2.20 is very similar to the proof of Theorem 2.17, so we omit it.

By Theorem 2.20 we can see that Requirement C is fulfilled so we may use the tradition sign "<" for elements $u, v \in R$, too.

Remark 2.21. If u_0, u_n $(n = 1, 2, \ldots)$ are real numbers and

$$\lim_{n\to\infty} u_n = u_0 \tag{2.22}$$

then we have that for any $a, b \in R$ such that $a < u_0 < b$, there is a real number ν such that if $n > \nu$ then $a < u_n < b$. So, by Definition 1.9, $u_n \in I_{u_0}(a, b)$. Hence, with respect to (1.1) Definition 1.17 says that (2.22) implies the convergence

$$\lim_{\underset{R}{\underset{\sqcup}{n\to\infty}}} u_n = u_0$$

too.

In the following we give some exemplas of exploder-functions satisfying Properties 1-6.

Definition 2.23. In the cases

$$\sigma(x) = \operatorname{area\,th} x \quad \text{and} \quad \overline{\sigma}(x) = \operatorname{th} x \tag{2.24}$$

$$\sigma(x) = \frac{2}{\pi}\operatorname{tg}\frac{\pi}{2}x \quad \text{and} \quad \overline{\sigma}(x) = \frac{2}{\pi}\operatorname{arc\,tg}\frac{\pi}{2}x$$

and

$$\sigma(x) = \frac{x}{1-|x|} \quad \text{and} \quad \overline{\sigma}(x) = \frac{x}{1+|x|}$$

we speak of hyperbolic, trigonometric and geometric explosions, respectively.

In the following we will mostly use the hyperbolic explosion and compression. So, with respect to (2.12) and (2.19) for the hyperbolic explosion and compression we write

$$\overbracket{x} = \operatorname{area}\operatorname{th} x, \quad x \in (-1,1) \tag{2.25}$$

and

$$\underbracket{u} = \operatorname{th} u, \quad u \in R \tag{2.26}$$

while for the other special σ-functions given under Definition 2.23 we write

$$\overbracket{x}_T = \frac{2}{\pi}\operatorname{tg}\frac{\pi}{2}x \quad \text{and} \quad \underbracket{u}_T = \frac{2}{\pi}\operatorname{arc}\operatorname{tg}\frac{\pi}{2}u$$

and

$$\overbracket{x}_G = \frac{x}{1-|x|} \quad \text{and} \quad \underbracket{u}_G = \frac{u}{1+|u|},$$

respectively.

Considering (2.8), (2.12), (2.19), (2.25) and (2.26) we can write

$$\overbracket{0} = 0 \quad \text{and} \quad \underbracket{0} = 0, \tag{2.27}$$

so we can say that the 0 *is explosion- and compression-invariant.* This is not fulfilled for the other real numbers. Namely, for any $x \in R^+$ (2.26) yields

$$0 < \underbracket{x} < x, \quad (x \in R^+). \tag{2.28}$$

Hence, the identity (1.5) and Definition 1.7 (with (2.17)) say

$$0 < x < \overbracket{x}, \quad (x \in R^+). \tag{2.29}$$

Similarly, for any $x \in R^-$ we have

$$x < \underbracket{x} < 0, \quad (x \in R^-) \tag{2.30}$$

and

$$\overbracket{x} < x < 0, \quad (x \in R^-). \tag{2.31}$$

(2.25) shows that an exploded real number $\overbracket{x}$ is a real number if and only if $x \in (-1,1)$. We call this kind of exploded real numbers visible real exploded numbers. Otherwise, we say that the exploded real number $\overbracket{x}$ is invisible. The exploded real number $\overbracket{1}$ is the smallest exploded real number which is greater than any element of R. The exploded real number $\overbracket{-1}$ is the greatest invisible exploded real number which is smaller than any element of R. With respect to Definition 1.8 the invisible exploded real numbers $\overbracket{1}$ and $\overbracket{-1}$ are called positive- and negative discriminators, respectively. By Definition 1.7 we can say that $x \in R$ *if and only if* $\overbracket{-1} < x < \overbracket{1}$. We remark, that the traditional expression of this is $-\infty < x < \infty$. Moreover, applying Theorem 1.24 we have that *any monotonic sequence of real numbers is convergent in* $\overbracket{R}$.

Considering an exploded real number $\overbracket{x}$ with $x \in (-1,1)$ its additive inverse element is given by (2.25) such that

$$\overbracket{-x} = -\overbracket{x}, \quad x \in (-1,1).$$

By this identity we extend the use of sign "-".

Definition 2.32. For any exploded real number $\overbracket{x}$ with $x \in R$ its additive inverse element is denoted by $(-\overbracket{x})$, that is

$$\overbracket{-x} = -\overbracket{x}, \quad x \in R. \tag{2.33}$$

Using (1.3), (1.4), and (2.33) the identity (1.6) yields the identity

$$\underbracket{-u} = -\underbracket{u}, \quad u \in \overbracket{R}. \tag{2.34}$$

(We may check identity (2.34) for $u \in R$ by (2.26).)

Using Definition 2.32 and 1.8 we extend the concept of absolute value for exploded real numbers by

Definition 2.35. For any $u \in \overset{\sqcup}{R}$ we say that

$$|u| = \begin{cases} u & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -u & \text{if } u < 0. \end{cases}$$

Applying (1.3), (1.4), (2.27) and (2.33) Definitions 1.7, 1.8 and 2.35 yield the identity

$$|\overset{\sqcup}{x}| = \overset{\sqcup}{|x|}, \quad x \in R. \tag{2.36}$$

Moreover, by (1.3), (1.4) and (1.6) the identity (2.36) yields the identity

$$|\underset{\sqcup}{u}| = \underset{\sqcup}{|u|}, \quad u \in \overset{\sqcup}{R}. \tag{2.37}$$

Using Definition 1.7 by (1.6) and (1.27) the identity (2.37) yields the super-triangle inequality

$$|u \overset{\vee}{\underset{\wedge}{\oplus}} v| \leq |u| \overset{\vee}{\underset{\wedge}{\oplus}} |v|, \quad u, v \in \overset{\sqcup}{R}. \tag{2.38}$$

Using Definition 1.2 by (1.5) and (1.33) the identity (2.37) yields that equality

$$|u \overset{\vee}{\underset{\wedge}{\odot}} v| = |u| \overset{\vee}{\underset{\wedge}{\odot}} |v|, \quad u, v \in \overset{\sqcup}{R}. \tag{2.39}$$

Remark 2.40. Applying Definition2.35 the positive discriminator characterizes the set R such that $u \in R$ *if and only if the inequality*

$$|u| < \overset{\sqcup}{1}, \quad (u \in R), \tag{2.41}$$

is valid. Moreover, (2.41), Definition 1.7 with (1.6), (1.33) with (2.39) show that *the super-multiplication is an operation in* R, too. Considering that the super-multiplication is commutative and associative, *set* R *is a commutative semi-group with respect to the operation super-multiplication.*

For the convergence problems the following theorem will be important.

Theorem 2.42. *An exploded real number* u *belongs to the super-interval neighbourhood* $I_{u_0}(a,b)$ *if and only if there are positive exploded real numbers*

$$\varepsilon_1 = u_0 \overset{\vee}{\underset{\wedge}{\ominus}} a \quad \textit{and} \quad \varepsilon_2 = b \overset{\vee}{\underset{\wedge}{\ominus}} u_0 \tag{2.43}$$

such that

$$u_0 \overset{\vee}{\underset{\wedge}{\ominus}} \varepsilon_1 < u < u_0 \overset{\vee}{\underset{\wedge}{\oplus}} \varepsilon_2 \tag{2.44}$$

is valid.

Proof. By Definition 1.9 we have that $\underset{\sqcup}{a} < \underset{\sqcup}{u_0} < \underset{\sqcup}{b}$. Considering (1.30), Definitions 1.7 and 1.8 say that ε_1 and ε_2 under (2.43) are positive. Considering (1.10) and (2.43) by Theorem 1.38 we have that (2.44) is valid if and only if $a < u < b$. So, Definition 1.9 says that $I_{u_0}(a,b) = I_{u_0}(u_0 \overset{\vee}{\underset{\wedge}{\ominus}} \varepsilon_1, u_0 \overset{\vee}{\underset{\wedge}{\oplus}} \varepsilon_2)$. □

Applying Properties 1.13-1.16 by Theorem 2.42 we can use the super-symmetrical neighbourhoods

$$I_{u_0}(u_0 \overset{\vee}{\underset{\wedge}{\ominus}} \varepsilon,\ u_0 \overset{\vee}{\underset{\wedge}{\oplus}} \varepsilon), \quad \varepsilon > 0 \tag{2.45}$$

instead of $I_{u_0}(a,b)$.

Theorem 2.46. *The sequence of exploded real numbers* $\{u_n\}_{n=1}^{\infty}$ *is convergent in* $\overset{\sqcup}{R}$ *to the exploded real number* u_0 *if and only if for any positive* ε *there exists real number* ν *such that if* $n > \nu$ *then*

$$|u_n \circleddash u_0| < \varepsilon.$$

Proof. Using Definition 1.7, by (2.37), (1.30) and (1.6) we obtain that this inequality is equivalent with the inequality

$$|\underset{\sqcup}{u_n} - \underset{\sqcup}{u_0}| < \underset{\sqcup}{\varepsilon}.$$

Having that for any $\varepsilon^*(= \underset{\sqcup}{\varepsilon})$ we have a real number ν such that if $n > \nu$ then $|\underset{\sqcup}{u_n} - \underset{\sqcup}{u_0}| < \varepsilon^*$ holds, we obtain that

$$\lim_{n\to\infty} \underset{\sqcup}{u_n} = \underset{\sqcup}{u_0}$$

which, by Theorem 1.19, is equivalent with $\lim\limits_{\substack{n\to\infty\\ \sqcup \\ R}} u_n = u_0$. □

Theorem 2.47. *If* $u_0, u_n, n = 1, 2, \ldots$ *are real numbers then* $\lim_{n\to\infty} u_n = u_0$ *if and only if* $\lim\limits_{\substack{n\to\infty\\ \sqcup \\ R}} u_n = u_0$.

Proof. Having Remark 2.21 it is sufficient to show that the condition

$$\lim_{\substack{n\to\infty\\ \sqcup \\ R}} u_n = u_0 \tag{2.48}$$

implies

$$\lim_{n\to\infty} u_n = u_0. \tag{2.49}$$

By Theorem 1.19 the condition (2.48) implies that $\lim_{n\to\infty} \underset{\sqcup}{u_n} = \underset{\sqcup}{u_0}$ which using the compressor-function under (2.26) means that

$$\lim_{n\to\infty} \operatorname{th} u_n = \operatorname{th} u_0.$$

Hence, by the continuity of the exploder-function considered under (2.25) we have that $\lim_{n\to\infty} \operatorname{area\,th}(\operatorname{th} u_n) = \operatorname{area\,th}(\operatorname{th} u_0)$ which shows that (2.49) is true. □

Theorem 2.47 shows that Requirement E is fulfilled. So, we can use "$\lim_{n\to\infty}$" instead of "$\lim\limits_{\substack{n\to\infty\\ \sqcup \\ R}}$" for any cases of sequences $\{u_n\}_{n=1}^{\infty}$. In addition, we have to investigate for sequences of real numbers the cases

$$\lim_{n\to\infty} u_n = \infty, \quad u_n \in R, \quad n = 1, 2, \ldots \tag{2.50}$$

and

$$\lim_{n\to\infty} u_n = -\infty, \quad u_n \in R, \quad n = 1, 2, \ldots. \tag{2.51}$$

Using (2.26), condition (2.50) implies that

$$\lim_{n\to\infty} \underset{\sqcup}{u_n} = 1, \quad u_n \in R, \quad n = 1, 2, \ldots. \tag{2.52}$$

Conversely, using (2.25), condition (2.52) implies (2.50). So, conditions (2.50) and (2.52) are equivalent. On the other hand, by (1.5), Theorem 1.19 says that the condition (2.52) is equivalent with the condition

$$\lim_{n\to\infty} u_n = \overset{\sqcup}{1}, \quad u_n \in R, \quad n = 1, 2, \ldots. \tag{2.53}$$

Hence the conditions (2.50) and (2.53) are equivalent. Similarly using (2.33) the equivalence of conditions (2.51) and

$$\lim_{n\to\infty} u_n = -\overset{\sqcup}{1}, \quad u_n \in R, \quad n = 1, 2, \ldots \tag{2.54}$$

is obtained.

Remark 2.55. Generally, the meaning of the condition

$$\lim_{n\to\infty} u_n = \overset{\sqcup}{1}, \quad (u_n \in \overset{\sqcup}{R}),$$

is not the same as the meaning of (2.50) because (2.50) can be considered merely for the sequences of real numbers, while in the other case, $u_n > \overset{\sqcup}{1}$ is allowed, that is u_n may be an invisible exploded number, too. Similarly, if

$$\lim_{n\to\infty} u_n = -\overset{\sqcup}{1}$$

then the exploded real number u_n may be smaller than the negative discriminator $-\overset{\sqcup}{1}$. So, we can use (2.53) instead of (2.50) (or (2.54) instead of (2.51)) with $u_n \in R$, $n = 1, 2, \ldots$.

Closing Part 2, we give

Definition 2.56. For any subset $S \subset R$ the set of explodeds of elements of S is called the exploded of S and denoted by $\overset{\sqcup}{S}$. Similarly, we use the compressed set of $S \subset \overset{\sqcup}{R}$, denoted by $\underset{\sqcup}{S}$. Clearly, $\underset{\sqcup}{R} = (-1, 1)$.

3. The field $\underset{\sqcup}{R}$

Using the compressor-function considered under (2.26), any real number ξ belongs to the set $\underset{\sqcup}{R}$ if and only if there exists and unambiguously determined real number x such that

$$\xi = \operatorname{th} x (= \underset{\sqcup}{x}) \tag{3.1}$$

is fulfilled. Our aim is to find the operations $\oplus$ and $\odot$ such that for any $x, y \in R$

$$\operatorname{th}(x + y) = \operatorname{th} x \oplus \operatorname{th} y \tag{3.2}$$

and

$$\operatorname{th}(x \cdot y) = \operatorname{th} x \odot \operatorname{th} y \tag{3.3}$$

will be satisfied. Using the identity

$$\operatorname{th}(x + y) = \frac{\operatorname{th} x + \ thy}{1 + \operatorname{th} x \cdot \operatorname{th} y}$$

the suitable definition of the $\oplus$ is clearly: for any $\xi, \eta \in \ \underset{\sqcup}{R}$

$$\xi \oplus \eta = \frac{\xi + \eta}{1 + \xi \cdot \eta}, \quad (\xi, \eta \in (-1, 1)). \tag{3.4}$$

Moreover, by (3.1) we have

$$x = \operatorname{area\,th} \xi \ \ (= \overset{\sqcup}{\xi}), \quad \xi \in (-1, 1), \tag{3.5}$$

so the definition of $\odot$ is: for any $\xi, \eta \in \ \underset{\sqcup}{R}$

$$\xi \odot \eta = \operatorname{th}(\operatorname{area\,th} \xi \cdot \operatorname{area\,th} \eta), \quad \xi, \eta \in (-1, 1). \tag{3.6}$$

The operation $\oplus$ is called sub-addition and the operation $\odot$ is called sub-multi-pli-ca-tion. Clearly, the operations sub-addition and sub-multiplication are commutative and associative. For any $x \in \ \underset{\sqcup}{R}$

$$\xi \oplus 0 = \xi,$$

$$\xi \oplus (-\xi) = 0,$$

$$\xi \odot \underset{\sqcup}{1} = \xi$$

and if $\xi \neq 0$ then

$$\xi \odot \operatorname{th} \frac{1}{\operatorname{area\,th} \xi} = \underset{\sqcup}{1}.$$

Moreover, for any $\xi, \eta, \zeta \in \underbracket{R}$ the distributivity

$$(\xi \oplus \eta) \odot \zeta = (\xi \odot \eta) \oplus (\eta \odot \zeta)$$

holds. So, we have

Theorem 3.7. *The set of compressed real numbers $\underbracket{R}$ is a field with operations sub-addition and sub-multiplication.*

Clearly, the fields R and $\underbracket{R}$ are isomorphic by the transformation

$$\begin{aligned} x &\rightarrowtail \underbracket{x}, \quad x \in R \\ + &\rightarrowtail \oplus, \\ \cdot &\rightarrowtail \odot\,. \end{aligned}$$

By Theorem 3.7 we can use the sub-subtraction

$$\xi \ominus \eta = \frac{\xi - \eta}{1 - \xi \cdot \eta}, \quad \xi, \eta \in (-1, 1)$$

and sub-division

$$\xi \odot \eta = \operatorname{th} \frac{\operatorname{area\,th} \xi}{\operatorname{area\,th} \eta}, \quad \xi, \eta \in (-1, 1) \quad \text{and} \quad \eta \neq 0. \tag{3.8}$$

Using (3.1) and $\eta = \operatorname{th} y = \underbracket{y}$ by (3.2) and(3.3) the identities

$$x + y = \overbracket{\underbracket{x} \oplus \underbracket{y}}, \quad x, y \in R \tag{3.9}$$

and

$$x \cdot y = \overbracket{\underbracket{x} \odot \underbracket{y}}, \quad x, y \in R \tag{3.10}$$

are obtained, respectively. Comparing (1.27) to (3.9) and (1.33) to (3.10) we can see that the field $\underbracket{R}$ with the operations sub-addition and sub-multiplication is a model of the field R with the familiar addition and multiplication. Moreover, the field R with its familiar operations is a model of the field $\overbracket{R}$ with super-addition and super-multiplication. On the real axis we can imagine the relationship of the fields R and $\overbracket{R}$ by the fields $\underbracket{R}$ and R where the "visible" compressed real numbers are in the interval (-1,1) while the "invisible" numbers belong to the set $R \backslash \underbracket{R}$ where 1 and -1 plays the role of positive and negative discriminators, respectively. Considering requirements A-E, we mention that for (1.1) $\underbracket{R} \subset \overbracket{(\underbracket{R})} = R$ is obtained. Moreover, for both $\underbracket{R}$ and R the same concepts of equality, arrangement, neighbourhood, convergence are used. Analogously to Remark 2.40 we have

Remark 3.11. The set $\underbracket{R}$ is a commutative semi-group with respect to the operation of multiplication.

Moreover, we mention

Theorem 3.12. *The sequence of compressed real numbers $\{\xi_n\}_{n=1}^{\infty}$ is convergent to $\xi_0 \in \underbracket{R}$ if and only if for any positive ε there exists a real number ν such that if $n > \nu$ then*

$$|\xi_n \ominus \xi_0| < \varepsilon. \tag{3.13}$$

Proof. Sufficiency is obvious because the inequality $|\xi_n - \xi_0| < 2|\xi_n \ominus \xi_0|$ gives the convergence of sequence $\{\xi_n\}_{n=1}^{\infty}$. On the other hand, if the sequence $\{\xi_n\}_{n=1}^{\infty}$ converges to $\xi_0 \in \underbracket{R}$ then it has a bound $K \in (-1, 1)$ such that the inequality $-|K| \leq \xi_n \leq |K|$ holds for any $n = 1, 2, \ldots$. Hence we get the inequality

$$|\xi_n \ominus \xi_0| \leq \frac{1}{1 - K^2} |\xi_n - \xi_0|$$

which gives the necessity of condition (3.13). $\square$

Moreover, we can observe that every monotonic sequence of compressed real numbers is convergent. If the sequence is bounded in $\underbracket{R}$ then its limit belongs to $\underbracket{R}$. Otherwise, its limit is 1 or -1.

4. Super- and sub-functions

Let f be a given traditional function, that is, its domain $D_f \subseteq R$ and range $R_f \subseteq R$, where

$$R_f = \{y \in R : y = f(x) \quad \text{and} \quad x \in D_f\}. \tag{4.1}$$

For any traditional function f we define its super-function denoted by $\operatorname{spr} f$ as follows:

$$D_{\operatorname{spr} f} = \left\{u \in \overbracket{R} : \underbracket{u} \in D_f\right\} \tag{4.2}$$

and

$$\operatorname{spr} f(u) = \overbracket{f(\underbracket{u})}. \tag{4.3}$$

Hence,

$$R_{\operatorname{spr} f} = \left\{v \in \overbracket{R} : v = \operatorname{spr} f(u) \quad \text{and} \quad u \in D_{\operatorname{spr} f}\right\}. \tag{4.4}$$

Clearly, by Definition 2.56 we obtain

$$D_{\operatorname{spr} f} = \overbracket{D_f} \quad \text{and} \quad R_{\operatorname{spr} f} = \overbracket{R}_f. \tag{4.5}$$

For any function F considered on the Descartes-product $\overbracket{R} \times \overbracket{R}$ that is, its definition-domain $D_F \subseteq \overbracket{R}$ and $R_F \subseteq \overbracket{R}$, where

$$R_F = \left\{v \in \overbracket{R} : v = F(u) \quad \text{and} \quad u \in D_F\right\} \tag{4.6}$$

we define its sub-function, denoted by $\operatorname{sub} F$ as follows:

$$D_{\operatorname{sub} F} = \{x \in R : \overbracket{x} \in D_F\} \tag{4.7}$$

and

$$\operatorname{sub} F(x) = \underbracket{F(\overbracket{x})}. \tag{4.8}$$

Hence,

$$R_{\operatorname{sub} F} = \{y \in R : y = \operatorname{sub} F(x) \quad \text{and} \quad x \in D_{\operatorname{sub} F}\}. \tag{4.9}$$

Clearly,

$$D_{\operatorname{sub} F} = \underbracket{D_F} \quad \text{and} \quad R_{\operatorname{sub} F} = \underbracket{R_F}. \tag{4.10}$$

Theorem 4.11. *Every function F considered on the Descartes-product $\overbracket{R} \times \overbracket{R}$ is the super-function of its sub-function, that is*

$$F = \operatorname{spr}(\operatorname{sub} F). \tag{4.12}$$

Proof. Considering $\operatorname{sub} F$ as a traditional function (see (4.1) (4.9)) by (4.2) and (4.10) we obtain

$$D_{\operatorname{spr}(\operatorname{sub} F)} = \left\{u \in \overbracket{R} : \underbracket{u} \in \underbracket{D_F} \Leftrightarrow u \in D_F\right\},$$

that is $D_{\operatorname{spr}(\operatorname{sub} F)} = D_F$. Moreover, by (4.3) and (4.8)

$$\operatorname{spr}(\operatorname{sub} F(u)) = \overbracket{\operatorname{sub} F(\underbracket{u})} = \overbracket{(\underbracket{F(u)})} = F(u)$$

is obtained so we have (4.12). □

Corollary 4.13. *Every traditional function f is a super-function φ considered on the Descartes-product $\underbracket{R} \times \underbracket{R}$.*

Proof. By (4.12) we can write

$$f = \operatorname{spr}(\operatorname{sub} f).$$

Moreover, by (4.10)

$$D_{\operatorname{sub} f} = \underbracket{D_f} \subseteq \underbracket{R} \quad \text{and} \quad R_{\operatorname{sub} f} = \underbracket{R_f} \subseteq \underbracket{R}$$

hence, $\varphi = \operatorname{sub} f$. □

Theorem 4.14. *Every traditional function f is the sub-function of its super-function, that is*

$$f = \operatorname{sub}(\operatorname{spr} f). \tag{4.15}$$

Proof. By (4.10) and (4.5) we can see that $D_{\operatorname{sub}(\operatorname{spr} f)} = \underbracket{D_{\operatorname{spr} f}} = \underbracket{(\overbracket{D_f})} = D_f$. Moreover, (4.8), (4.3) and (1.6) we obtain for any $x \in D_f$ that $\operatorname{sub}(\operatorname{spr} f(x)) = \underbracket{\operatorname{spr} f(\overbracket{x})} = \underbracket{(\overbracket{f(x)})} = f(x)$. □

In the following we mention some elementary examples for super - and sub-functions of traditional functions. For the sake of simplicity we denote the traditional power-function with the exponent $\alpha \in R$ by p_α, that is

$$p_\alpha(x) = x^\alpha \text{ with } D_{p_\alpha} = R^+ \text{ and } R_{p_\alpha} = R^+. \tag{4.16}$$

In certain special cases D_{p_α} and R_{p_α} may be wider, for example $D_{p_3} = R_{p_3} = R$. Hence, (4.16) and (4.2)-(4.4) yield

$$\operatorname{spr} p_\alpha(u) = \overbracket{(\underbracket{u})^\alpha}, \quad \text{with} \quad D_{\operatorname{spr} p_\alpha} = \overbracket{R^+} \quad \text{and} \quad R_{\operatorname{spr} p_\alpha} = \overbracket{R^+}. \tag{4.17}$$

Moreover, we can write

$$\operatorname{spr} \exp u = \overbracket{(e^{\underbracket{u}})}, \quad \textit{with} \quad D_{\operatorname{spr} \exp} = \overbracket{R} \quad \text{and} \quad R_{\operatorname{spr} \exp} = \overbracket{R^+}, \tag{4.18}$$

$$\operatorname{spr} \ln u = \overbracket{(\ln \underbracket{u})}, \quad \text{with} \quad D_{\,\mathit{spr}\ \ln} = \overbracket{R^+} \quad \text{and} \quad R_{\operatorname{spr} \ln} = \overbracket{R}. \tag{4.19}$$

By (4.19), (1.6), (4.18) and (1.5) we have that $\operatorname{spr}\exp(\operatorname{spr}\ln u) = u, u \in \overbracket{R^+}$. In general, we have that if the traditional function f has the inverse function $\overline{f}$, then

$$\operatorname{spr} f(\operatorname{spr} \overline{f}(u)) = u, \quad u \in D_{\operatorname{spr} \overline{f}} \tag{4.20}$$

because (4.2), (4.3), (1.6) and (1.5) yield

$$\operatorname{spr} f(\operatorname{spr} \overline{f}(u)) = \operatorname{spr} f\,\overbracket{(\overline{f}(\underbracket{u}))} = \overbracket{f(\overline{f}(\underbracket{u}))} = \overbracket{(\underbracket{u})} = u.$$

By (4.6)-(4.9) we can consider the sub-functions. For example

$$\operatorname{sub} \exp \xi = \underbracket{e^{\overbracket{\xi}}}, \quad \xi \in \underbracket{R}.$$

Similarly to (4.20) we have

$$\operatorname{sub} f(\operatorname{sub} \overline{f}(\xi)) = \xi, \quad \xi \in D_{\operatorname{sub} \overline{f}}. \tag{4.21}$$

Especially interesting is the case of the exploder-function. Namely, for any $u \in R$, (4.3) and (2.26) yield

$$\operatorname{spr} \operatorname{area} \operatorname{th} u = \overbracket{u}. \tag{4.22}$$

Moreover, for any $u \in \underbracket{R}$, (4.22) and (2.25) show

$$\operatorname{spr} \operatorname{area} \operatorname{th} u = \operatorname{area} \operatorname{th} u. \tag{4.23}$$

So, in this case we can use the extension of exploder-function

$$\operatorname{area} \operatorname{th} u = \overbracket{u} \quad \text{with} \quad D_{\operatorname{area} \operatorname{th}} = R \quad \text{and} \quad R_{\operatorname{area} \operatorname{th}} = \overbracket{R}. \tag{4.24}$$

Similarly, for any $u \in R$ we have

$$\operatorname{spr}\operatorname{th} u = \operatorname{th} u \tag{4.25}$$

so, we have the extension of compressor-function

$$\operatorname{th} u = \underbracket{u} \quad \text{with} \quad D_{\mathrm{th}} = \overbracket{R} \quad \text{and} \quad R_{\mathrm{th}} = R. \tag{4.26}$$

On the other hand, by (4.8), (2.25) and (1.6) for any $\xi \in (-\underbracket{1}, \underbracket{1})$

$$\operatorname{sub}\operatorname{area}\operatorname{th}\xi = \underbracket{\operatorname{area}\operatorname{th}\overbracket{\xi}} = \underbracket{\overbracket{((\overbracket{\xi}))}} = \overbracket{\xi} = \operatorname{area}\operatorname{th}\xi \tag{4.27}$$

is obtained. Similarly, by (4.8), (2.26) and (1.5) give for any $\xi \in \underbracket{R}$

$$\operatorname{sub}\operatorname{th}\xi = \overbracket{\operatorname{th}\underbracket{\xi}} = \overbracket{\underbracket{((\underbracket{\xi}))}} = \underbracket{\xi} = \operatorname{th}\xi. \tag{4.28}$$

Identities (4.23), (4.25), (4.27) and (4.28) show that *the exploder and compressor-functions are invariant for the construction of super- or sub-functions.*

For any function F considered on the Descartes-product $\overbracket{R} \times \overbracket{R}$, we have

Theorem 4.29. *Let us assume that the interval I is a subset of D_F. The function F is monotonic increasing (or decreasing) on the interval I if and only if the function* $\operatorname{sub} F$ *is monotonic increasing (or decreasing) on the interval $\underbracket{I}$.*

Proof. By Definition 2.56. and (4.7) we have that $\underbracket{I} \subseteq D_{\operatorname{sub} F}$. Let us assume that $u_1, u_2 \in I$ and $u_1 < u_2$ implies $F(u_1) \le F(u_2)$. Denoting $u_1 = \overbracket{x_1}$ and $u_2 = \overbracket{x_2}$ we have that $x_1, x_2 \in \underbracket{I}$. Moreover, Definition 1.7 with (1.6) say that $x_1 < x_2$ if and only if $u_1 < u_2$. Similarly, by the identity (4.8) we have that $F(\overbracket{x_1}) \le F(\overbracket{x_2})$ if and only if $\operatorname{sub} F(x_1) \le \operatorname{sub} F(x_2)$. $\square$

Theorems 4.11, 4.14 and 4.29 yield

Corollary 4.30. *The monotonity is invariant for the construction of super- or sub-functions.*

Now, we turn to the limit of function F considered on the Descartes-product $\overbracket{R} \times \overbracket{R}$. Let us assume that $u_0 \in \overbracket{R}$ and it has a neighbourhood

$$I_{u_0}(u_0 \ominus \varepsilon_1, u_0 \oplus \varepsilon_2)$$

with ε_1, $\varepsilon_2 > 0$ such that

$$I_{u_0}(u_0 \ominus \varepsilon_1,\ u_0 \oplus \varepsilon_2) \backslash u_0 \subset D_F. \tag{4.31}$$

Definition 4.32. The exploded real number v_0 is called the limit of the function F at the point u_0 having (4.31) if for any sequence of exploded real numbers $\{u_n\}_{n=1}^{\infty}$, where $u_n \ne u_0$ and

$$\lim_{n\to\infty} u_n = u_0 \tag{4.33}$$

the

$$\lim_{n\to\infty} F(u_n) = v_0 \tag{4.34}$$

holds. This is denoted by

$$\lim_{u\to u_0} F(u) = v_0. \tag{4.35}$$

Theorem 4.36. *The limit under (4.35) exists if and only if*

$$\lim_{x\to\underbracket{u_0}} \operatorname{sub} F(x) = \underbracket{v_0} \tag{4.37}$$

holds.

Proof. Denoting by $u_n = \overbracket{x_n}, n = 1, 2, \ldots$, and $u_0 = \overbracket{x_0}$ Theorem 1.19 with identity (1.6) says that (4.33) is valid if and only if

$$\lim_{n\to\infty} x_n = x_0.$$

On the other hand, (4.34) is valid if and only if

$$\lim_{n\to\infty} \underbracket{F(\overbracket{x_n})} = \underbracket{v_0}.$$

Hence, (4.8) shows that $\operatorname{sub} F(x_n) = \underbracket{F(\overbracket{x_n})}$, so (4.37) and (4.35) are equivalent. $\square$

Considering that $\operatorname{sub} F$ is a traditional function it is well known that (4.37) is valid if and only if for any $\underbracket{\varepsilon}(> 0)$ there exists a $\underbracket{\delta}(> 0)$ such that if

$$|x - \underbracket{u_0}| < \underbracket{\delta} \quad (x \neq \underbracket{u_0}) \tag{4.38}$$

then

$$|\operatorname{sub} F(x) - \underbracket{v_0}| < \underbracket{\varepsilon}. \tag{4.39}$$

Denoting by $x = \underbracket{u}$ and using (1.6), (1.30) and (2.37), Definition 1.7 says that (4.38) is equivalent with

$$|u \ominus u_0| < \delta \quad (u \neq u_0). \tag{4.40}$$

Similarly, by (4.8), (1.6), (1.30) and (2.37), Definition 1.7 says that (4.39) is equivalent with

$$|F(u) \ominus v_0| < \varepsilon. \tag{4.41}$$

Hence, Theorem 4.36 with Definition 4.32 yields

Corollary 4.42. *The limit under (4.35) exists if and only if for any $\varepsilon(> 0)$ there exists a $\delta(> 0)$ such that if (4.40) is valid then (4.41) holds.*

Remark 4.43. Corollary 4.42 is particularly interesting in the cases of discriminators that is u_0 or v_0 are $-\overbracket{1}$ or $\overbracket{1}$.

For example, if F is a traditional function such that

$$\lim_{u\to\infty} F(u) = \infty \quad (u \in R) \tag{4.44}$$

it means that for any $M \in R$ there exists $D \in R$ such that $u > D$ then $F(u) > M$. On the other hand, Definition 4.32 with (2.50), (2.53) and Remark 2.55 shows that (4.44) is equivalent with

$$\lim_{\substack{u\to\overbracket{1} \\ u<\overbracket{1}}} F(u) = \overbracket{1} \quad , (\text{see } (2.41)). \tag{4.45}$$

Moreover, by (4.40) and (4.41), the limit (4.45) means that for any $\varepsilon(> 0)$ there exists $\delta(> 0)$ such that

$$|u \ominus \overbracket{1}| < \delta, \quad (u < \overbracket{1}), \tag{4.46}$$

implies

$$|F(u) \ominus \overbracket{1}| < \varepsilon, \quad (F(u) < \overbracket{1}). \tag{4.47}$$

Let us observe that (4.46) and (4.47) are equivalent with

$$\operatorname{area\,th}(1 - \operatorname{th}\delta) < u$$

and

$$\operatorname{area\,th}(1 - \operatorname{th}\varepsilon) < F(u),$$

respectively. Hence, with $M = \operatorname{area\,th}(1 - \operatorname{th}\varepsilon)$ and $D = \operatorname{area\,th}(1 - \operatorname{th}\delta)$ the meaning of (4.44) and (4.45) can be compared immediately.

Definition 4.48. A function F is called continuous at the point u_0 having (4.31) if $u_0 \in D_F$ and the limit (4.35) exists with $v_0 = F(u_0)$.

Using Corollary 4.42 we have

Corollary 4.49. *A function F is continuous at the point u_0 if and only if for any $\varepsilon(>0)$ there exists $\delta(>0)$ such that the inequality (4.40) implies that*

$$|F(u) \ominus F(u_0)| < \varepsilon.$$

By Definition 4.32 and Theorem 4.36 we immediately have

Theorem 4.50. *Let us assume that the interval I is a subset of D_F. The function F is continuous on the interval I if and only if the function* $\operatorname{sub} F$ *is continuous on the interval $\underset{\sqcup}{I}$.*

Example 4.51. Let us consider the function F which can be defined by the equation

$$F(u) = (\overset{\sqcup}{2} \odot u) \odot (\overset{\sqcup}{1} \oplus \sup p_2(u)). \tag{4.52}$$

Using (4.17) with $\alpha = 2$ and (1.32) we can see that $\sup p_2(u) = u \odot u$. Hence, $D_F = \overset{\sqcup}{R}$. Having the inequality $|F(u)| \leq \overset{\sqcup}{1}$ we obtain that $R_F \subseteq [-\overset{\sqcup}{1}, \overset{\sqcup}{1}]$. Clearly $F(-\overset{\sqcup}{1}) = -\overset{\sqcup}{1}$, $F(0) = 0$ and $F(\overset{\sqcup}{1}) = \overset{\sqcup}{1}$. On the other hand, (4.52), (4.8), (1.32), (1.26), (1.36) and (1.6) give

$$\operatorname{sub} F(x) = \frac{2x}{1+x^2}, \quad D_{\operatorname{sub} F} = R \quad \text{and} \quad R_{\operatorname{sub} F} = [-1, 1]. \tag{4.53}$$

Hence, by Theorem 4.11 and (4.5) the (4.53) yields that $R_F = [-\overset{\sqcup}{1}, \overset{\sqcup}{1}]$. Clearly, the function $\operatorname{sub} F$ is an odd function, decreasing on the interval $(-\infty, -1]$, increasing on the interval (-1,1) and decreasing on the interval $[1, \infty)$, again. At the points -1 and 1 it has its minimum and maximum, respectively. Moreover, it is concave on the intervals $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and convex on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. Finally, it is continuous on R. Hence, by Theorem 4.29 the function F is decreasing on the sets of exploded real numbers

$$\left\{u \in \overset{\sqcup}{R} : u \leq -\overset{\sqcup}{1}\right\} \quad \text{and} \quad \left\{u \in \overset{\sqcup}{R} : u \geq \overset{\sqcup}{1}\right\}$$

and increasing on R. The latter property can be observed if, using (1.5), (1.27), (1.32), (1.36), (2.25) and (2.26), we compute

$$F(u) = 2u, \quad u \in R. \tag{4.54}$$

Theorem 4.50 shows that the function F given by (4.52) is continuous on $\overset{\sqcup}{R}$. Among others it is continuous at the point $u_0 = \overset{\sqcup}{1}$. Checking this fact by (4.54), (4.44) and (4.45) we can see that $\lim_{\substack{u \to \infty \\ u \in R}} F(u) = \infty$.

5. Repeated explosions and compressions

Choosing as our starting point the set of exploded real numbers $\overset{\sqcup}{R}$ and modifying Requirements A-F such that instead of R and $\overset{\sqcup}{R}$ we write $\overset{\sqcup}{R}$ and $\overset{\sqcup}{(\overset{\sqcup}{R})}$, respectively, we can construct the set $\overset{\sqcup}{(\overset{\sqcup}{R})}$. The extensions mentioned in Requirements B-E can easily be carried out. Requirement F is more complicated. Our exploder-function will be the extended function area th (defined under (4.24)) saying that for any $u \in R$

$$\overbracket{u} = \operatorname{area\,th} u$$

while if $u \in \overbracket{R} \setminus R$ then $\overbracket{u}$ is merely a symbol with the operations super-super addition and super-super-multiplication

$$\overbracket{u} \circledast\!\!\!\!\bigoplus \overbracket{v} = \overbracket{u \bigoplus v}, \quad u, v \in \overbracket{R} \tag{5.1}$$

and

$$\overbracket{u} \circledcirc \overbracket{v} = \overbracket{u \bigodot v}, \quad u, v \in \overbracket{R}, \tag{5.2}$$

respectively.

Clearly, $\overbracket{(\overbracket{R})}$ is a field with the operations given by (5.1) and (5.2). The additive unit element is 0 while the multiplicative unit element is $\overbracket{(\overbracket{1})}$.

On the other hand, considering (4.28), we can compress $\underbracket{R}$ again and have the field $\underbracket{(\underbracket{R})}$ with the operations sub-sub addition and sub-sub multiplication:

$$\xi \oplus \eta = \operatorname{th} \frac{\operatorname{area\,th} \frac{\xi+\eta}{1+\xi\eta}}{\operatorname{area\,th} \frac{\operatorname{th} 1 + \operatorname{th}(\operatorname{area\,th} \xi)(\operatorname{area\,th} \eta))}{1+(\operatorname{th} 1)\cdot \operatorname{th}((\operatorname{area\,th} \xi)(\operatorname{area\,th} \eta))}}, \quad \xi, \eta \in \underbracket{(\underbracket{R})} \tag{5.3}$$

and

$$\xi \circledcirc \eta = \operatorname{th}(\operatorname{th}(\operatorname{area\,th}(\operatorname{area\,th} \xi) \cdot \operatorname{area\,th}(\operatorname{area\,th} \eta))), \quad \xi, \eta \in \underbracket{(\underbracket{R})}, \tag{5.4}$$

respectively.

Clearly, the fields

$$\left(\underbracket{(\underbracket{R})}, \oplus, \circledcirc\right), (\underbracket{R}, \oplus, \odot), (R, +, \cdot), (\overbracket{R}, \bigoplus, \bigodot), \quad \left(\overbracket{(\overbracket{R})}, \bigoplus, \bigodot\right)$$

are isomorphic.

Having that

$$\lim_{n\to\infty} \underbrace{\operatorname{th} \ldots \operatorname{th}}_{n} 1 = 0$$

and following the operations (5.1), (5.2), (5.3) and (5.4) we have the double-infinite sequences of fields

$$\ldots \underbracket{(\underbracket{\mathrm{R}})} \subset \underbracket{R} \subset R \subset \overbracket{R} \subset \overbracket{(\overbracket{\mathrm{R}})} \ldots$$

without the narrowest field.

References

[1] C.H. Edwards, Jr. *The Historical Development of Calculus.* Springer Verlag, 1979.
[2] L. Rédei. *Algebra.* Akadémiai Kiadó, Budapest, 1967.
[3] M. Sain. *Matematikatörténeti ABC.* Tankönyvkiadó, Budapest, 1987. In Hungarian.

Received September 17, 2001.

University of Szeged,
Faculty of Teacher's Training,
Department of Mathematics
H-6720 Szeged, Boldogasszony sgt. 6–8
E-mail address: szalay@jgytf.u-szeged.hu