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SOME PROPERTIES FOR FUNCTIONS OF VMO(2^ω)

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Dedicated to Professor W.R. Wade on his sixtieth birthday

Abstract. A function of bounded mean oscillation (BMO) is said to have vanishing mean oscillation or belong to VMO space if its mean oscillation is locally small in a uniform sense. Though there is an extensive literature on the BMO, very few mention is made on the properties for functions of VMO.

In this note, we discuss the connection between modulus of continuity and the approximation of functions by Walsh polynomials in VMO space on the dyadic group 2^ω, VMO(2^ω), the analogy between VMO(2^ω) and C(2^ω), the estimate for certain type of convolution operators on VMO(2^ω), the decomposition theorem for functions in VMO(2^ω) and the characterization of Walsh series which happen to be the Walsh-Fourier series of a function in VMO(2^ω).

1. Notation

Our results are stated in the situation that the dyadic group 2^ω is the additive subgroup of the ring of integers in the 2-series field K of formal Laurent series in one variable over the finite field GF(2). We need to set some basic notation. It is taken from Taibleson's book [9] where the fundamentals are detailed. For the additive subgroup K^+ of the 2-series field K, we may choose a Haar measure dx. Let $d(\alpha x) = |\alpha| dx,\ \alpha \neq 0$ and call $|\alpha|$ the valuation of α.

Let $P^0 = \{x \in \mathrm{K} : |x| \leq 1\}$ and $P^1 = \{x \in \mathrm{K} : |x| < 1\}$. K is totally disconnected, hence the value is discrete valued. Thus there is an element $\wp$ of P^1 of maximum value. Then an element $x \in \mathrm{K}$ is represented as

$$x = \sum_{k=j}^{\infty} a_k \wp^k,\ a_k \in \mathrm{GF}(2), \tag{1}$$

which can contain a finite number of terms with negative powers of $\wp$. The ring of integers $P^0 = \left\{x = \sum_{k=0}^{\infty} a_k \wp^k\right\}$ coincides with the dyadic group 2^ω as an additive group. For E a measurable subset of K, let $|E| = \int_{\mathrm{K}} \Phi_E(x)dx$, where Φ_E is the characteristic function of E and dx is Haar measure normalized so $|2^\omega| = 1$. Then $|P^1| = |\wp| = 2^{-1}$. Let $P^k = \{x \in \mathrm{K} : |x| \leq 2^{-k}\}$ and Φ_k be its characteristic function. For $x = x_0 + \sum_{k=j}^{-1} a_k \wp^k,\ a_k \in \mathrm{GF}(2), x_0 \in 2^\omega$, set

$$w(\wp^k) = \begin{cases} -1 & k = -1, \\ 1 & k < -1, \end{cases} \qquad w(x_0) = 1. \tag{2}$$

Then w is a character on K^+. For $x, y \in \mathrm{K}$, let $w_y(x) = w(y \cdot x)$. w is constant on cosets of 2^ω and if $y \in P^k$ then w_y is constant on cosets of P^{-k}.

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We assume that all functions are complex valued and measurable. If $f \in L^1(\mathrm{K})$ the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(y) = \int_{\mathrm{K}} f(u) w_y(u) du. \tag{3}$$

Then we have $(2^k\Phi_k)\hat{} = \Phi_{-k}$ and $((2^k\Phi_k)*(2^l\Phi_l))\hat{} = \Phi_{-(k\wedge l)}$, where $k\wedge l = \min(k,l)$.

Let $\{u(n)\}_{n=0}^{\infty}$ be a complete list of distinct coset representatives of 2^ω in K^+. We define $u(0) = 0$, $u(1) = \wp^{-1}$ and for $n = b_0 + b_1\cdot 2 + b_2\cdot 2^2 + \cdots + b_s\cdot 2^s$ ($b_i = 0$ or 1), $u(n) = u(b_0) + \wp^{-1}u(b_1) + \cdots + \wp^{-s}u(b_s)$. Then $\left\{w_{u(n)}|_{P^0}\right\}_{n=0}^{\infty} = \left\{w_{u(n)}\right\}_{n=0}^{\infty}$ is a complete set of characters on 2^ω. This is the Walsh-Paley system.

The Dirichlet kernels are the functions

$$D_n(x) = \sum_{k=0}^{n-1} w_{u(k)}(x),\ n \geq 1,\ D_0(x) \equiv 0.$$

If $f \in L^1(2^\omega)$ the Walsh-Fourier coefficients $\{c_k\}_{k=0}^{\infty} = \{\hat{f}(u(k))\}_{k=0}^{\infty}$ are given by $c_k = \int_{2^\omega} f(x) w_{u(k)}(x)dx$. The Walsh-Fourier series is given by

$$f(x) \sim \sum_{k=0}^{\infty} c_k w_{u(k)}(x).$$

The n-th partial sum of the Walsh-Fourier series of f is denoted by $S_n f(x)$ and is defined as $S_n f(x) = \sum_{k=0}^{n-1} c_k w_{u(k)}(x)$. If $f \in L^1(2^\omega)$, $x \in 2^\omega$, $n \geq 0$, then $S_{2^n} f(x) = 2^n \int_{x+P^n} f(t)dt$, as follows from the fact that $D_{2^n} = 2^n \Phi_n$.

$S(2^\omega)$ is the collection of the test functions on 2^ω. If $\phi \in S(2^\omega)$ then ϕ is a "polynomial", that is, $\phi(x) = \sum_{k=0}^{2^n-1} \hat{\phi}(u(k)) w_{u(k)}(x)$ for some $n \geq 0$. C_i denotes a constant.

2. Properties of VMO(2^ω) functions

Let $f \in \mathrm{L}^1(2^\omega)$. By a ball we mean a set $B = \{y \in 2^\omega : |x-y| \leq 2^{-k}\} = x + P^k$ for some $x \in 2^\omega$ and $k \in \mathbb{N}$. If $f \in \mathrm{L}^1(2^\omega)$, write $f_B = \frac{1}{|B|}\int_B f(x)dx$ for the average of f over B. If

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx = \|f\|_* < \infty, \tag{4}$$

where the supremum is over all balls B, then we say f is of bounded mean oscillation, $f \in \mathrm{BMO}(2^\omega)$. It is clear that $\mathrm{L}^\infty(2^\omega) \subset \mathrm{BMO}(2^\omega)$ and for $f \in \mathrm{L}^\infty(2^\omega)$, $\|f\|_* \leq 2\|f\|_\infty$. BMO($2^\omega$) is the dual space to $\mathrm{H}^1(2^\omega)$. That is, each continuous linear functional ℓ on $\mathrm{H}^1(2^\omega)$ can be realized as a mapping

$$\ell(g) = \int_{2^\omega} f(x) g(x) dx, \quad g \in \mathrm{H}^1(2^\omega), \tag{5}$$

when suitably defined, where f is a function in BMO(2^ω). This pairing allows to realize $\mathrm{H}^1(2^\omega)$ as the dual of VMO(2^ω). (See [7], [8] and [12].)

For $0 < \delta < 1$, write

$$M_\delta(f) = \sup_{|B| \leq \delta} \frac{1}{|B|} \int_B |f(x) - f_B| dx. \tag{6}$$

Then $f \in \mathrm{BMO}(2^\omega)$ if and only if $M_\delta(f)$ is bounded and $\|f\|_* = \lim_{\delta\to 1} M_\delta(f)$. BMO($2^\omega$) is a Banach space with norm $M_1(f) + |\hat{f}(0)|$ or $M_1(f) + \|f\|_1$.

We say that f has vanishing mean oscillation, $f \in \mathrm{VMO}(2^\omega)$, if

$$f \in \mathrm{BMO}(2^\omega), \text{and } M_0 f = \lim_{\delta\to 0} M_\delta(f) = 0. \tag{7}$$

VMO(2^ω) contains every continuous functions on 2^ω, C(2^ω). The unbounded function $\log|x|$ belongs to BMO(2^ω). However, $\log|x|$ is not VMO(2^ω). The function $\log|\log|x||$ is in VMO(2^ω), although that is not immediately evident. VMO(2^ω) is a closed subspace of BMO(2^ω), so it contains the BMO-closure of C(2^ω).

The following theorem shows several characterization of VMO(2^ω). (See [10] for the dyadic group case, [5] and [6] for the classical case).

The space VMO(2^ω) is translation invariant. For $y \in 2^\omega$, we let τ_y denote the operator of translation by y; that is, $(\tau_y f)(x) = f(x-y)$ for any function f on 2^ω.

Theorem 2.1. *For f a function in* BMO(2^ω)*, the following conditions are equivalent:*

(i) *f is in* VMO(2^ω)*;*
(ii) $\lim_{|h|\to 0} \|\tau_h f - f\|_* = 0$*;*
(iii) $\lim_{n\to\infty} \|2^n \Phi_n * f - f\|_* = 0$*;*
(iv) *f is in the* BMO*-closure of* C(2^ω)*.*

Next lemma is a simple but useful fact. (See [6].)

Lemma 2.2 (Inequality of Young type)**.** *If f is a function in* BMO(2^ω) *and ϕ is an integrable function on 2^ω, then $\phi * f$ is in* VMO(2^ω) *and $\|f*\phi\|_* \le \|\phi\|_1\|f\|_*$. If, in addition, ϕ is continuous function on 2^ω, then $\phi * f$ is in continuous function on 2^ω.*

Proof. Put $f*\phi(t) = h(t)$. Then, we have
$$\frac{1}{|B|}\int_B |h(t) - h_B|dt \le \|\phi\|_1 \frac{1}{|B|}\int_B |(\tau_u f)(t) - (\tau_u f)_B|dt.$$
Hence, $\|h\|_* \le \|\phi\|_1\|\tau_u f\|_* = \|\phi\|_1\|f\|_*$.

For any $\varepsilon > 0$, there exists a polynomial T such that $\|\phi - T\|_1 < \varepsilon$. Then, $f*T \in$ C(2^ω) and for small $|B|$,
$$\frac{1}{|B|}\int_B |f*(\phi - T)(t) - (f*(\phi-T))_B|dt \le \|\phi - T\|_1\|f\|_* < \varepsilon\|f\|_*.$$
We obtain, by Theorem 2.1., $f*\phi \in$ VMO(2^ω). □

To study the analogy between VMO(2^ω) and C(2^ω), we introduce the analogue in VMO(2^ω) of the Lipschitz classes. Let $\rho(\delta)$ be a positive, continuous, non-decreasing function on $(0, \infty)$ satisfying $\lim_{\delta\to 0}\rho(\delta) = 0$, and $\rho(2\delta) \le C_1\rho(\delta)$.

A continuous function f on 2^ω is said to belong to the class Lip$\rho(\delta)$ if it satisfies $\omega(f,\delta) = O(\rho(\delta))$, where $\omega(f,\delta) = \sup\{\|\tau_h f - f\|_\infty : |h| \le \delta\}$.

We shall say f in BMO(2^ω) belongs to BMO($\rho(\delta)$) provided $M_\delta(f) = O(\rho(\delta))$. We have VMO($2^\omega$)=$\cup_{\rho(\delta)}$BMO($\rho(\delta)$).

Theorem 2.3. *If ρ satisfies the condition*
$$\int_0^1 \frac{\rho(t)}{t}dt < \infty,$$
then BMO($\rho(\delta)$)$\subset$Lip($\sigma(\delta)$)*, where*
$$\sigma(\delta) = \int_0^\delta \frac{\rho(t)}{t}dt.$$
In particular, BMO(δ^α)=Lip(δ^α)*, $0 < \alpha \le 1$.*

The analogue of this theorem in the classical case was shown S. Spanne ([5]). We omit the proof of this theorem.

We consider the translation invariant singular integrals on VMO(2^ω). G.I. Gaudry and I.R. Inglis proved the next theorem ([3] and [4]), which is obtained without the intervention of the space H^1(2^ω).

Theorem 2.4. *Suppose* $K \in \mathrm{L}^1(2^\omega)$. *If*

(i) $|\hat{K}(u(n))| \le C_2$, *for* $|u(n)| \ge 2^{n-1}$,
(ii) $\int_{2^\omega \setminus P^n} |K(x-y) - K(x)|dx \le C_2$ *for* $|y| \le 2^{-n}$,

then, for all $f \in \mathrm{L}^\infty(2^\omega)$, $\|K * f\|_* \le C_3 \|f\|_*$, *where* C_3 *depends on* C_2 *only.*

Corollary 2.5. *If* $f \in \mathrm{C}(2^\omega)$, *then* $K * f \in \mathrm{VMO}(2^\omega)$.

Proof. For a continuous function f and any $\varepsilon > 0$, there exists a polynomial $T \in \mathrm{S}(2^\omega)$ such that $\|f - T\|_\infty < \varepsilon$. Then $K * T \in \mathrm{S}(2^\omega)$ and $\|K * f - K * T\|_* < C_3 \varepsilon$. Hence, we have, by Theorem 2.1., $K * f \in \mathrm{VMO}(2^\omega)$. □

J.B. Garnett and P.W. Jones ([2]) and J.-A. Chao ([1]) proved the following characterization of BMO regular martingales similar to the construction Carleson's.

Theorem 2.6. *Let* $f \in \mathrm{BMO}(2^\omega)$. *Then there exist a* $g \in \mathrm{L}^\infty(2^\omega)$ *with*

$$\|g\|_\infty \le C_4 \|f\|_*,$$

a sequence of balls $\{B_i\}$ *and a corresponding sequence of complex numbers* $\{b_i\}$ *such that* $\sum_{B_i \subset B} |b_i| \le C_4 \|f\|_* |B|$ *for any given ball B, and*

$$f = g + \sum_i b_i \frac{\Phi_{B_i}}{|B_i|} + C_5$$

for a constant C_5.

Theorem 2.7. *(i) Let* $f \in \mathrm{VMO}(2^\omega)$ *and* $f(0) = 0$. *Then there exist a* $g \in \mathrm{C}(2^\omega)$ *with* $\|g\|_\infty \le C_6 \|f\|_*$, *a sequence of balls* $\{B_i\}$ *and a corresponding sequence of complex numbers* $\{b_i\}$ *such that* $\frac{1}{|B|}\sum_{B_i \subset B} |b_i| \to 0$ *as* $|B| \to 0$ *for any given ball B, and* $f = g + \sum_i b_i \frac{\Phi_{B_i}}{|B_i|}$.

(ii) Let $g \in \mathrm{C}(2^\omega)$ *and* $\{B_i\}$ *be a sequence of balls. Assume that to each* B_i, *there is a associated constant* b_i *satisfying* $\frac{1}{|B|}\sum_{B_i \subset B} |b_i| \to 0$ *as* $|B| \to 0$. *Then, if* $f = g + \sum_i b_i \frac{\Phi_{B_i}}{|B_i|} + C_7$ *for a constant* C_7, $f \in \mathrm{VMO}(2^\omega)$.

Proof. (i) Since $f \in \mathrm{BMO}(2^\omega)$, using Theorem 2.6., write $f = g_0 + \sum_{i_0} b_{i_0} \frac{\Phi_{B_{i_0}}}{|B_{i_0}|}$, where $\|g_0\|_\infty \le C_4 \|f\|_*$, and $\sum_{B_{i_0} \subset B} |b_{i_0}| \le C_4 \|f\|_* |B|$ for any ball B. By Theorem 2.1., there is a ball B_{j_0} such that $\|f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}\|_* < \|f\|_*/2$, Let

$$G_0 = g_0 * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}$$

and

$$B_0 = \sum_{i_0} |b_{i_0}| \frac{\Phi_{B_{i_0}}}{|B_{i_0}|} * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|} = \sum_{i_0} |b_{i_0}| \frac{\Phi_{B_{i_0 \wedge j_0}}}{|B_{i_0 \wedge j_0}|},$$

then

$$G_0 \in \mathrm{C}(2^\omega), \qquad \|G_0\|_\infty \le \|g_0\|_\infty \le C_4 \|f\|_*,$$
$$\|B_0\|_\infty \le \sum_{i_0 : B_{i_0} \subset B_{j_0}} \frac{|b_{i_0}|}{|B_{j_0}|} \le C_4 \|f\|_*,$$

and so that $\|f - G_0 - B_0\|_* < \|f\|_*/2$.

Repeating the above argument with $f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}$, we obtain

$$f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|} = g_1 + \sum_{i_1} b_{i_1} \frac{\Phi_{B_{i_1}}}{|B_{i_1}|},$$

with $\|g_1\|_\infty \le C_4\|f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}\|_* < C_4\|f\|_*/2$, and

$$\sum_{B_{i_1}\subset B} |b_{i_1}| \le C_4\|f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}\|_*|B| < C_4\|f\|_*|B|/2.$$

There exists a ball B_{j_1} such that

$$\|(f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}) - (f - f * \frac{\Phi_{B_{j_0}}}{|B_{j_0}|}) * \frac{\Phi_{B_{j_1}}}{|B_{j_1}|}\|_* < \|f\|_*/2^2.$$

Let $G_1 = g_1 * \frac{\Phi_{B_{j_1}}}{|B_{j_1}|}$ and

$$B_1 = \sum_{i_1} |b_{i_1}| \frac{\Phi_{B_{i_1}}}{|B_{i_1}|} * \frac{\Phi_{B_{j_1}}}{|B_{j_1}|} = \sum_{i_1} |b_{i_1}| \frac{\Phi_{B_{i_1\wedge j_1}}}{|B_{i_1\wedge j_1}|}.$$

Then $G_1 \in \mathrm{C}(2^\omega)$, $\|G_1\|_\infty + \|B_1\|_\infty \le C_4\|f\|_*$ and

$$\|f - G_0 - B_0 - G_1 - B_1\|_* < \|f\|_*/2^2.$$

Iterating we obtain sequences $\{G_n\} \subset \mathrm{C}(2^\omega)$ and $\{B_n\}$ with the following properties:

(a) $$\|G_n\|_\infty + \|B_n\|_\infty \le C_4\|f\|_*/2^{n-1},$$

(b) $$\|f - \sum_1^n (G_k + B_k)\|_* \le \|f\|_*/2^{n+1}.$$

By (a), the function $g = \sum_n G_n \in \mathrm{C}(2^\omega)$ and $\frac{1}{|B|}\sum_{B_i\subset B} |b_i| \to 0$ as $|B| \to 0$ and from (b), $f = g + \sum_i b_i \frac{\Phi_{B_i}}{|B_i|}$.

(ii) Let $b(x) = \sum_i b_i \frac{\Phi_{B_i}}{|B_i|}$ and $B = a + P^l$. We shall show that

$$I = \frac{1}{|B|}\int_B |b(t) - b(a)|dt \to 0$$

as $|B| \to 0$.

$$I = 2^l \int_{a+P^l} \left| \sum_i \frac{bi}{|B_i|}(\Phi_{B_i}(t) - \Phi_{B_i}(a)) \right| dt$$
$$= 2^l \int_{a+P^l} \left| \sum_{i:a+P^l\subset B_i} + \sum_{i:B_i\subset a+P^l} \right| dt.$$

In fact, if B_i and B_j are two nondisjoint balls on 2^ω, then either $B_i \subset B_j$ or $B_j \subset B_i$. If $a + P^l \subset B_i$, then $\Phi_{B_i}(x) = \Phi_{B_i}(a) = 1$ and $I = 0$.

If $B_i \subset a + P^l$, then

$$I = 2^l \int_{a+P^l} | \sum_{i:B_i\subset a+P^l} \frac{b_i}{|B_i|}(\Phi_{B_i}(t) - \Phi_{B_i}(a))|dt$$
$$\le 2^l \sum_{i:B_i\subset a+P^l} \frac{|b_i|}{|B_i|} \int_{B_i} |\Phi_{B_i}(t) - \Phi_{B_i}(a)|dt \le 2^{l+1} \sum_{i:B_i\subset a+P^l} |b_i| \to 0$$

as $l \to \infty$. This proves (ii). □

Let consider the Walsh series $W(x) = \sum_{n=0}^\infty a_n w_{u(n)}(x)$, whose coefficients are arbitrary numbers. We can show those Walsh series $W(x)$ which happen to be the Walsh-Fourier series of a function in VMO(2^ω).

Theorem 2.8. *Let $W(x)$ be a Walsh series. Then $W(x)$ is the Walsh-Fourier series of a* VMO *function if and only if* $\|S_{2^n}(W) - S_{2^m}(W)\|_* \to 0$ *as* $n, m \to \infty$.

Proof. If part. Since $\mathrm{VMO}(2^\omega)$ is a Banach space, any Cauchy sequence is convergent to a limit function f. We shall show $f \in \mathrm{VMO}(2^\omega)$. Since $\mathrm{VMO}(2^\omega)$ is embedded continuously into $\mathrm{L}^2(2^\omega)$, we also have $\|S_{2^n}(W) - S_{2^m}(W)\|_2 \to 0$ as $n, m \to \infty$, so that $W(x)$ is the Walsh-Fourier series of an $\mathrm{L}^2(2^\omega)$ function f and we have $S_{2^n}(W)(x) \to f(x)$ a.e. and $\|S_{2^n}(W) - f\|_2 \to 0$. Consequently, $\int_B S_{2^n}(W)(t)dt \to \int_B f(t)dt$ as $n \to \infty$ for any ball B, that is, $(S_{2^n}(W))_B \to (f)_B$ as $n \to \infty$. An application of Fatou's lemma to $|S_{2^n}(W) - S_{2^m}(W)|$ over the ball B shows

$$\frac{1}{|B|}\int |f - f_B| \le \lim_{n\to\infty} \frac{1}{|B|}\int |S_{2^n}(W)(t) - (S_{2^n}(W))_B|dt.$$

Since, for $|B|$ small, $S_{2^m}(W)(t) = (S_{2^m}(W))_B$, we have

$$\begin{aligned}\frac{1}{|B|}\int |f - f_B| &\le \lim_{n\to\infty} \frac{1}{|B|}\int |S_{2^n}(W)(t) - S_{2^m}(W)(t) - (S_{2^n}(W) - S_{2^m}(W))_B|dt \\ &\le \lim_{n\to\infty} \|S_{2^n}(W) - S_{2^m}(W)\|_*\end{aligned}$$

and we obtain $f \in \mathrm{VMO}(2^\omega)$.

On the other hand, we can use the integral formula

$$S_{2^n}W(t) = \int (\tau_u f)(t) D_{2^n}(u)du.$$

Let $D_{n,m}W(t) = S_{2^n}W(t) - S_{2^m}W(t)$. Then, for $|B|$ small,

$$\begin{aligned}&\frac{1}{|B|}\int_B |(D_{n,m}W)(t) - (D_{n,m}W)_B|dt \\ &\le \frac{1}{|B|}\int_B \int |((\tau_u f)(t) - (\tau_u f)_B)(D_{2^n}(u) - D_{2^m}(u))du|dt \\ &\le \int (D_{2^n}(u) + D_{2^m}(u)) \frac{1}{|B|}\int_B |(\tau_u f)(t) - (\tau_u f)_B|dtdu \to 0\end{aligned}$$

as $|B| \to 0$.

This proves $\|S_{2^n}(W) - S_{2^m}(W)\|_* \to 0$ as $n, m \to \infty$. ☐

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