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NEIGHBORHOODS OF A CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. The main object of this paper is to prove several inclusion relations associated with the (n, δ) neighborhoods of various subclasses of starlike and convex functions of complex order by making use of the known concept of neighborhoods of analytic functions. Special cases of some of these inclusion relations are shown to yield known results.

1. Introduction

Let $A(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0; \quad n \in \mathrm{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathrm{C}, \quad |z| < 1\}.$$

Following [6, 9], we defined the (n, δ)-neighborhood of a function $f(z) \in A(n)$ by

$$N_{n,\delta}(f) := \left\{ g \in A(n) \colon g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leq \delta \right\}. \tag{1.2}$$

In particular, for the identity function

$$e(z) = z, \tag{1.3}$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in A(n) \colon g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k\,|b_k| \leq \delta \right\}. \tag{1.4}$$

A function $f(z) \in A(n)$ is said to be starlike of complex order $\gamma(\gamma \in \mathrm{C} - \{0\})$, that is, $f \in S_n^*(\gamma)$, if it also satisfies the inequality

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \quad (z \in \Delta;\ \gamma \in \mathrm{C} - \{0\}). \tag{1.5}$$

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Furthermore, a function $f(z) \in A(n)$ is said to be convex of complex order γ ($\gamma \in \mathrm{C} - \{0\}$), that is, $f \in C_n(\gamma)$, if it also satisfies the inequality

$$Re\left\{1 + \frac{1}{\gamma}(\frac{zf'(z)}{f(z)})\right\} > 0, \ (z \in \Delta; \ \gamma \in \mathrm{C} - \{0\}). \tag{1.6}$$

The classes $S_n^*(\gamma)$ and $C_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [8] and Wiatrowski [13], respectively, (see also [5, 11]).

Let $S_n(\gamma, \lambda, \beta)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$\left|\frac{1}{\gamma}\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda zf'(z) + (1-\lambda)f(z)} - 1\right\}\right| < \beta,$$

$$(z \in \Delta; \ \gamma \in \mathrm{C} - \{0\}; \ 0 \leq \lambda \leq 1; \ 0 < \beta \leq 1).$$

Let $\Re_n(\gamma, \lambda, \beta)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$\left|\frac{1}{\gamma}\left\{f'(z) + \lambda zf''(z) - 1\right\}\right| < \beta,$$

$$(z \in \Delta; \ \gamma \in \mathrm{C} - \{0\}; \ 0 \leq \lambda \leq 1; \ 0 < \beta \leq 1).$$

Neighborhoods of the classes $S_n(\gamma, \lambda, \beta)$ and $\Re_n(\gamma, \lambda, \beta)$ investigated by Altıntaş, Özkan and Srivastava [3].

Let A be class of functions $f(z)$ of the form $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$. For $f(z)$ belong to A, Salagean [10] has introduced the following operator called the Salagean operator:

$$D^0 f(z) = f(z), \ D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D(D^{n-1} f(z)) \ (n \in \mathrm{N} = \{1, 2, 3, \ldots\}).$$

Note that $D^n f(z) = z + \sum\limits_{k=2}^{\infty} k^n a_k z^k$, $n \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$.

Let $A(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1}, \quad (a_k \geq 0; \ n \in \mathrm{N} := \{1, 2, 3, \ldots\}) \tag{1.7}$$

which analytic in the open unit disk

$$\Delta = \{z : z \in \mathrm{C}, \ |z| < 1\}.$$

We define the (n, δ)-neighborhood of a function $f(z) \in A(n)$ by

$$N_{n,\delta}(f) := \left\{g \in \Delta(n) : g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \wedge \sum_{k=n}^{\infty} (k+1) \left|a_{k+1} - b_{k+1}\right| \leq \delta\right\}. \tag{1.8}$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in A(n) : g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \wedge \sum_{k=n}^{\infty} (k+1) \left| b_{k+1} \right| \leq \delta \right\}. \tag{1.9}$$

We can write the following equalities for the functions $f(z)$ belong to the class $A(n)$

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z) = z - \sum_{k=n}^{\infty} (k+1) a_{k+1} z^{k+1},$$

$$D^2 f(z) = D(Df(z)) = z - \sum_{k=n}^{\infty} (k+1)^2 a_{k+1} z^{k+1},$$

$$\ldots$$

$$D^{\Omega} f(z) = D(D^{\Omega-1} f(z)) = z - \sum_{k=n}^{\infty} (k+1)^{\Omega} a_{k+1} z^{k+1},\ (\Omega \in \mathrm{N} \cup \{0\})\ [7].$$

Let $S_n(\gamma, \lambda, \beta, \Omega)$ denote the subclass of $A(n)$consisting of functions $f(z)$ which satisfy the inequality

$$\left| \frac{1}{\gamma} \left(\frac{(1-\lambda) z (D^{\Omega} f(z))' + \lambda z (D^{\Omega+1} f(z))'}{(1-\lambda) D^{\Omega} f(z) + \lambda D^{\Omega+1} f(z)} - 1 \right) \right| < \beta \tag{1.10}$$

$$(z \in \Delta;\ \gamma \in \mathrm{C} - \{0\}\,; 0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \Omega \in \mathrm{N} \cup \{0\}).$$

Also let $\Re_n(\gamma, \lambda, \beta, \Omega)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$\left| \frac{1}{\gamma} \left((1-\lambda)(D^{\Omega} f(z))' + \lambda (D^{\Omega+1} f(z))' - 1 \right) \right| < \beta, \tag{1.11}$$

$$(z \in \Delta;\ \gamma \in \mathrm{C} - \{0\}\,; 0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \Omega \in \mathrm{N} \cup \{0\}).$$

Various further subclasses of the classes $S_n(\gamma, \lambda, \beta, 0)$ and $\Re_n(\gamma, \lambda, \beta, 0)$ with $\gamma = 1$ were studied in many earlier works (cf., e.g., [1, 12] ; see also the references cited in these earlier works). Clearly, we have

$$S_n(\gamma, 0, 1, 0) \subset S_n^*(\gamma) \ \text{ and } \Re_n(\gamma, 0, 1, 0) \subset C_n(\gamma),$$

$$(n \in \mathrm{N},\ \gamma \in \mathrm{C} - \{0\}).$$

The main object of the present paper is to investigate the (n, δ)-neighborhoods of the following $S_n(\gamma, \lambda, \beta, \Omega)$ and $\Re_n(\gamma, \lambda, \beta, \Omega)$ subclasses of the class $A(n)$ of normalized analytic functions in Δ with negative coefficients.

2. A set of inclusion relations involving $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we need the following Lemma 1 and Lemma 2.

Lemma 1. *Let the function $f \in A(n)$ be defined by (1.7), then $f(z)$ is in the class $S_n(\gamma, \lambda, \beta, \Omega)$ if and only if*

$$\sum_{k=n}^{\infty} (k+1)^{\Omega} (\lambda k + 1)(k + \beta \left| \gamma \right|) a_{k+1} \leq \beta \left| \gamma \right| \tag{2.1}$$

Proof. We suppose that $f \in S_n(\gamma, \lambda, \beta, \Omega)$. Then, by appealing to condition (1.10), we readily get

$$Re\left\{\frac{(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)} - 1\right\} > -\beta\,|\gamma| \tag{2.2}$$

or equivalently,

$$Re\left\{\frac{-\sum\limits_{k=n}^{\infty}(k+1)^{\Omega}k(\lambda k+1)a_{k+1}z^{k+1}}{z-\sum\limits_{k=n}^{\infty}(k+1)^{\Omega}(\lambda k+1)a_{k+1}z^{k+1}}\right\} > -\beta\,|\gamma|\,,\ (z \in \Delta) \tag{2.3}$$

where we have made use of definition (1.7).

Now choose values of z on the real axis and let $z \to 1^-$ through real values. Then inequality (2.3) immediately yields the wanted condition (2.1).

Conversely, by applying hypothesis (2.1) and letting $|z| = 1$, we find that

$$\begin{aligned}
&\left|\frac{(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)} - 1\right| \\
&\qquad = \left|\frac{\sum\limits_{k=n}^{\infty}(k+1)^{\Omega}k(\lambda k+1)a_{k+1}z^{k+1}}{z-\sum\limits_{k=n}^{\infty}(k+1)^{\Omega}(\lambda k+1)a_{k+1}z^{k+1}}\right| \\
&\qquad \le \frac{\beta\,|\gamma|\left\{1-\sum\limits_{k=n}^{\infty}(k+1)^{\Omega}(\lambda k+1)a_{k+1}\right\}}{1-\sum\limits_{k=n}^{\infty}(k+1)^{\Omega}(\lambda k+1)a_{k+1}} \\
&\qquad \le \beta\,|\gamma|\,.
\end{aligned} \tag{2.4}$$

Hence, by maximum modulus theorem, we have $f \in S_n(\gamma, \lambda, \beta, \Omega)$, which evidently completes the proof of lemma 1. ☐

Similarly, we can prove the following.

Lemma 2. *Let the function $f \in A(n)$ be defined by (1.7), then $f(z)$ is in the class $\Re_n(\gamma, \lambda, \beta, \Omega)$ if and only if*

$$\sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)a_{k+1} \le \beta\,|\gamma|. \tag{2.5}$$

Remark 1. A special case of Lemma 1 when $\gamma = 1$, $\Omega = 0$ and $\beta = 1-\alpha$ $(0 \le \alpha < 1)$ was given earlier by Altıntaş [1, p. 489, Theorem 1].

Our first inclusion relation involving $N_{n,\delta}(e)$ is given by the following.

Theorem 1. *Let*

$$\delta = \frac{\beta\,|\gamma|}{(n+1)^{\Omega-1}(\lambda n+1)(n+\beta\,|\gamma|)} \quad (|\gamma| < 1), \tag{2.6}$$

then

$$S_n(\gamma, \lambda, \beta, \Omega) \subset N_{n,\delta}(e). \tag{2.7}$$

Proof. For $f \in S_n(\gamma, \lambda, \beta, \Omega)$, lemma 1 immediately yields

$$(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)\sum_{k=n}^{\infty} a_{k+1} \leq \beta|\gamma|$$

so that

$$\sum_{k=n}^{\infty} a_{k+1} \leq \frac{\beta|\gamma|}{(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)}. \tag{2.8}$$

On the other hand, we also find from (2.1) and (2.8) that

$$\begin{aligned}
&\sum_{k=n}^{\infty}(k+1)^{\Omega}(\lambda k+1)(k+\beta|\gamma|)a_{k+1} \leq \beta|\gamma| \\
&\quad\Rightarrow \sum_{k=n}^{\infty}(k+1)^{\Omega}(\lambda k+1)(k+1-1+\beta|\gamma|)a_{k+1} \leq \beta|\gamma| \\
&\quad\Rightarrow (n+1)^{\Omega}(\lambda n+1)\sum_{k=n}^{\infty}(k+1)a_{k+1} \\
&\quad\leq \beta|\gamma| + (1-\beta|\gamma|)(n+1)^{\Omega}(\lambda n+1)\sum_{k=n}^{\infty} a_{k+1} \\
&\quad\leq \beta|\gamma| + (1-\beta|\gamma|)(n+1)^{\Omega}(\lambda n+1)\frac{\beta|\gamma|}{(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)} \\
&\quad= \beta|\gamma| + (1-\beta|\gamma|)\frac{\beta|\gamma|}{n+\beta|\gamma|} = \frac{(n+1)\beta|\gamma|}{n+\beta|\gamma|}.
\end{aligned}$$

Thus

$$\sum_{k=n}^{\infty}(k+1)a_{k+1} \leq \frac{\beta|\gamma|}{(n+1)^{\Omega-1}(\lambda n+1)(n+\beta|\gamma|)} \quad (|\gamma|<1),$$

that is

$$\sum_{k=n}^{\infty}(k+1)a_{k+1} \leq \frac{\beta|\gamma|}{(n+1)^{\Omega-1}(\lambda n+1)(n+\beta|\gamma|)} = \delta, \tag{2.9}$$

which, in view of definition (1.9), proves Theorem 1. □

By similarly, applying Lemma 2 instead of Lemma 1, we can prove the following.

Theorem 2. *Let*

$$\delta = \frac{\beta|\gamma|}{(\lambda n+1)(n+1)^{\Omega}} \tag{2.10}$$

then

$$\Re_n(\gamma, \lambda, \beta, \Omega) \subset N_{n,\delta}(e). \tag{2.11}$$

Remark 2. A special case of Theorem 1 and Theorem 2 when $\Omega = 0$ was proven recently by Altıntaş, Özkan and Srivastava [3].

Remark 3. A special case of Theorem 1 when

$$\gamma = 1-\alpha, \ (0 \leq \alpha < 1), \ \lambda = \Omega = 0 \quad \text{and} \quad \beta = 1 \tag{2.12}$$

was proven recently by Altıntaş and Owa [2, p. 798, Theorem 2.1].

3. Neighborhoods for the classes $S_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ and $R_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$

In this section, we determine the neighborhood for each of the classes

$$S_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega) \text{ and } \Re_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega),$$

which we define as follows. A function $f \in A(n)$ be defined by (1.7) is said to be in the class $S_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ if there exist a function $g \in S_n(\gamma,\lambda,\beta,\Omega)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha, \ (z \in \Delta;\ 0 \le \alpha < 1). \tag{3.1}$$

Analogously, a function $f \in A(n)$ be defined by (1.7) is said to be in the class $\Re_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ if there exists a function $g \in \Re_n(\gamma,\lambda,\beta,\Omega)$ such that inequality (3.1) holds true.

Theorem 3. *If $g \in S_n(\gamma,\lambda,\beta,\Omega)$ and*

$$\alpha = 1 - \frac{\delta(n+1)^{\Omega-1}(\lambda n+1)(n+\beta\left|\gamma\right|)}{(n+1)^{\Omega}(\lambda n+1)(n+\beta\left|\gamma\right|) - \beta\left|\gamma\right|} \tag{3.2}$$

then

$$N_{n,\delta}(g) \subset S_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega). \tag{3.3}$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from (1.8) that

$$\sum_{k=n}^{\infty}(k+1)\left|a_{k+1} - b_{k+1}\right| \le \delta \tag{3.4}$$

which readily implies the coefficients inequality

$$\sum_{k=n}^{\infty}\left|a_{k+1} - b_{k+1}\right| \le \frac{\delta}{n+1}, \quad (n \in \mathrm{N}). \tag{3.5}$$

Next, since $g \in S_n(\gamma,\lambda,\beta,\Omega)$, we have (cf. Equation (2.8))

$$\sum_{k=n}^{\infty} b_{k+1} \le \frac{\beta\left|\gamma\right|}{(n+1)^{\Omega}(\lambda n+1)(n+\beta\left|\gamma\right|)}, \tag{3.6}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=n}^{\infty}\left|a_{k+1} - b_{k+1}\right|}{1 - \sum\limits_{k=n}^{\infty} b_{k+1}} \\ &\le \frac{\delta}{n+1}.\frac{1}{1 - \frac{\beta|\gamma|}{(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)}} \\ &= \frac{\delta(n+1)^{\Omega-1}(\lambda n+1)(n+\beta\left|\gamma\right|)}{(n+1)^{\Omega}(\lambda n+1)(n+\beta\left|\gamma\right|) - \beta\left|\gamma\right|} \\ &= 1-\alpha,\end{aligned} \tag{3.7}$$

provided that α is given precisely by (3.2). Thus, by definition, $f \in S_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ for α given by (3.2), which evidently completes our proof of Theorem 3. □

Our proof of Theorem 4 below is much similar to that of Theorem 3.

Theorem 4. *If* $g \in \Re_n(\gamma, \lambda, \beta, \Omega)$ *and*

$$\alpha = 1 - \frac{\delta(n+1)^{\Omega}(\lambda n+1)}{(n+1)^{\Omega+1}(\lambda n+1) - \beta|\gamma|} \tag{3.8}$$

then

$$N_{n,\delta}(g) \subset \Re_n^{(\alpha)}(\gamma, \lambda, \beta, \Omega). \tag{3.9}$$

Remark 4. A special case of Theorem 3 and Theorem 4 when $\Omega = 0$ was proven recently by Altıntaş, Özkan and Srivastava [3].

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