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RUSCHEWEYH DIFFERENTIAL OPERATOR SETS OF BASIC SETS OF POLYNOMIALS OF SEVERAL COMPLEX VARIABLES IN HYPERELLIPTICAL REGIONS

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ABSTRACT. In this paper we study the effectiveness of Ruscheweyh differential operator sum and product sets of basic sets of polynomials of several complex variables in hyperelliptical regions. These results extend and improve the existing relevant results of Ruscheweyh differential operator sets in hyperspherical regions.

1. INTRODUCTION

The idea of the basic sets of polynomials of one complex variable appeared in 1930's by Whittaker [33,34,35] who laid down the definition of a basic sets and their effectiveness. The study of the basic sets of polynomials of several complex variables was initiated by Mursi and Makar [25,26], Nassif [27], Kishka and others [13,14,17,18,19], where the representation in polycylindrical and hyperspherical regions was considered. Also, there are studies on basic sets of polynomials such as in Clifford Analysis [1,2,3,4,5,6,7,8] and in Faber regions [11,28,31,32]. The problem of derived and integrated sets of basic sets of polynomials in one and two complex variables was studied by many authors [9,10,20,21,23,24], where they considered the unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$, circles and hyperspherical regions. Recently, in [12], the author studied this problem in a new region which is called hyperelliptical region. The purpose of this paper is to establish the effectiveness of Ruscheweyh differential operator sum and product sets of basic sets of polynomials of several complex variables in an open hyperellipse, in a closed hyperellipse and in the regions $D(\overline{E}_{[\mathbf{r}]})$ which means unspecified domain containing the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$. These results extend my results concerning the effectiveness in hyperspherical regions found in [15].

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Let $\mathbb{C}$ represent the filed of complex variables and let $\mathbf{z} = (z_1, z_2, \dots, z_k)$ be an element of $\mathbb{C}^k$, the space of several complex variables. To avoid lengthy scripts, the following notations are adopted throughout this paper:

$$\mathbf{m} = (m_1, m_2, \dots, m_k), \qquad a\mathbf{m} = (am_1, am_2, \dots, am_k),$$
$$\langle \mathbf{m} \rangle = m_1 + m_2 + \cdots + m_k, \qquad \mathbf{h} = (h_1, h_2, \dots, h_k),$$
$$|\mathbf{z}| = \{\sum_{s=1}^{k} |z_s|^2\}^{\frac{1}{2}}; \qquad \mathbf{z}^{\mathbf{m}} = \prod_{s=1}^{k} z_s^{m_s}$$
$$\overline{E}_{[\mathbf{r}]} = \overline{E}_{[r_1, r_2, \dots, r_k]}, \qquad D(\overline{E}_{[\mathbf{r}]}) = D(\overline{E}_{[r_1, r_2, \dots, r_k]}),$$
$$\mathbf{r} = r_1, r_2, \dots, r_k, \qquad [\mathbf{r}_i] = [r_i^{(1)}, r_i^{(2)}, \dots, r_i^{(k)}],$$
$$\mathbf{t}^{\mathbf{m}} = \prod_{s=1}^{k} t_s^{m_s},$$
$$\alpha([\mathbf{r}],\ [\mathbf{R}]) = \max\{r_1 \prod_{s=2}^{k} R_s,\ r_\nu \prod_{s=1, s\neq\nu}^{k} R_s,\ r_k \prod_{s=1}^{k-1} R_s\},$$

where $m_1, m_2, \dots, m_k,\ \ h_1, h_2, \dots, h_k$ are non-negative integers and

$$\nu = \{2, 3, 4, \dots, k-1\}.$$

In $\mathbb{C}^k$, $E_{[\mathbf{r}]}$ denotes an open hyperellipse region $\sum_{s=1}^{k} \frac{|z_s|^2}{r_s^2} < 1$ and by $\overline{E}_{[\mathbf{r}]}$ its closure, where $r_s,\ s \in I = \{1, 2, \dots, k\}$ are positive numbers. Thus these regions satisfy the following inequalities [14]:

$$D(\overline{E}_{[\mathbf{r}]}) = \{\mathbf{w}' : |\mathbf{w}| \leq 1\}, \tag{1.1}$$
$$E_{[\mathbf{r}]} = \{\mathbf{w} : |\mathbf{w}| < 1\}, \tag{1.2}$$
$$\overline{E}_{[\mathbf{r}]} = \{\mathbf{w} : |\mathbf{w}| \leq 1\}, \tag{1.3}$$

where $\mathbf{w} = (w_1, w_2, \dots, w_k);\ w_s = \frac{z_s}{r_s}$ and $\mathbf{w}' = (w_1', w_2', \dots, w_k');\ w_s' = \frac{z_s}{r_s^+}$; $s \in I$.

Thus the function $f(\mathbf{z})$, which is regular in $\overline{E}_{[\mathbf{r}]}$ can be represented by the power series

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}} = \sum_{(m_1, m_2, \dots, m_k)=0}^{\infty} a_{m_1, m_2, \dots, m_k} z_1^{m_1} z_2^{m_2} \dots z_k^{m_k}. \tag{1.4}$$

For the function $f(\mathbf{z})$, we have from [13,14] that

$$M(f,\ \overline{E}_{[\mathbf{r}]}) = \sup_{\overline{E}_{[\mathbf{r}]}} |f(\mathbf{z})|, \tag{1.5}$$

then it follows that

$$\lim_{\langle \mathbf{m}\rangle\to\infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod\limits_{s=1}^{k} \{r_s\}^{\langle \mathbf{m}\rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m}\rangle}} \leq 1/\prod_{s=1}^{k} r_s,$$

where

$$\sigma_{\mathbf{m}} = \inf_{|\mathbf{t}|=1} \frac{1}{\mathbf{t}^{\mathbf{m}}} = \frac{\{\langle \mathbf{m}\rangle\}^{(1/2)\langle \mathbf{m}\rangle}}{\prod\limits_{s=1}^{k} m_s^{(1/2)m_s}}, \text{ (see [14] and [27])}$$

$1 \leq \sigma_{\mathbf{m}} \leq \left(\sqrt{k}\right)^{\langle \mathbf{m}\rangle}$ on the assumption $m_s^{(1/2)m_s} = 1$, whenever $m_s = 0, s \in I$.

Definition 1.1 ([25,26])**.** A set of polynomials

$$\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0[\mathbf{z}],\ P_1[\mathbf{z}], \ldots,\ P_n[\mathbf{z}], \ldots\}$$

is said to be basic when every polynomial in the complex variables $z_s,\ s \in I$, can be uniquely expressed as a finite linear combination of the elements of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to [25, Th.5] the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and only if there exists a unique row-finite matrix $\overline{P}$ such that

$$P\overline{P} = P\overline{P} = I, \tag{1.6}$$

where $P = \{P_{\mathbf{m},\mathbf{h}}\}$ is the matrix of coefficients, $\overline{P} = \{\overline{P}_{\mathbf{m},\mathbf{h}}\}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and I is the infinite unit matrix.

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}, \tag{1.7}$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \overline{P}_{\mathbf{h}}[\mathbf{z}], \tag{1.8}$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}. \tag{1.9}$$

Thus, (1.4) becomes

$$f(\mathbf{z}) = \sum_{\mathbf{m}} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}],$$

where

$$\Pi_{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h},\mathbf{m}}\ a_{\mathbf{h}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h},\mathbf{m}} \frac{f^{(\mathbf{h})}(\mathbf{0})}{h_s!},$$

where $h! = h(h-1)(h-2)\cdots 3\cdot 2\cdot 1$. The series $\sum\limits_{\mathbf{m}} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}]$ is the associated basic series of $f(\mathbf{z})$.

Definition 1.2 ([13,14])**.** The associated basic series $\sum_{\mathbf{m}} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}]$ is said to represent $f(\mathbf{z})$ in $\overline{E}_{[\mathbf{r}]}$ (resp. $E_{[\mathbf{r}]}$, $D(\overline{E}_{[\mathbf{r}]})$) when $\sum_{\mathbf{m}} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}]$ converges uniformly to $f(\mathbf{z})$ in $\overline{E}_{[\mathbf{r}]}$ (resp. $E_{[\mathbf{r}]}$, some hyperellipse surrounding the hyperellipse $\overline{E}_{[\mathbf{r}]}$, not necessarily the former hyperellipse).

Definition 1.3 ([13,14])**.** The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be effective in $\overline{E}_{[\mathbf{r}]}$ (resp. $E_{[\mathbf{r}]}$, $D(\overline{E}_{[\mathbf{r}]})$) when the associated basic series represents in $\overline{E}_{[\mathbf{r}]}$ (resp. $E_{[\mathbf{r}]}$, some hyperellipse surrounding the hyperellipse $\overline{E}_{[\mathbf{r}]}$, not necessarily the former hyperellipse) every function which is regular there.

To study the convergence properties of such basic sets of polynomials in hyperelliptical regions we consider the following notations,

$$M(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}) = \sup_{\overline{E}_{[\mathbf{r}]}} |P_{\mathbf{m}}[\mathbf{z}]|\,, \tag{1.10}$$

$$G(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) = \sum_{\mathbf{h}} \left|\overline{P}_{\mathbf{m},\mathbf{h}}\right| M(P_{\mathbf{h}},\ \overline{E}_{[\mathbf{r}]}), \tag{1.11}$$

$$\Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) = \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_s\}^{\langle\mathbf{m}\rangle - m_s}\ G(P_{\mathbf{h}},\ \overline{E}_{[\mathbf{r}]}), \tag{1.12}$$

where $\Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})$ is called the Cannon sum of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for the hyperellipse $\overline{E}_{[\mathbf{r}]}$ (see [13,14]).

Also, the Cannon function for the basic sets of polynomials in hyperelliptical regions [13,14] were defined as follows:

$$\Omega(P,\ \overline{E}_{[\mathbf{r}]}) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup\ \{\Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})\}^{\frac{1}{\langle\mathbf{m}\rangle}}. \tag{1.13}$$

Let $N_{\mathbf{m}} = \ N_{m_1,m_2,\ldots,m_k}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m},\mathbf{h}}$ in the representation (1.8). A basic set satisfying the condition

$$\lim_{\langle\mathbf{m}\rangle\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle\mathbf{m}\rangle}} = a, \quad a > 1, \tag{1.14}$$

is called general basic set and if $a = 1$, then the basic set is called Cannon set [25,26]

Now, let $\mathbf{D}_{\mathbf{m}} = \mathbf{D}_{m_1,m_2,\ldots,m_k}$ be the degree of the polynomial of the highest degree in the representation (1.8). That is to say, if $\mathbf{D}_{\mathbf{h}} = \mathbf{D}_{h_1,h_2,\ldots,h_k}$ is the degree of the polynomial $P_{\mathbf{h}}[\mathbf{z}]$, then $\mathbf{D}_{\mathbf{h}} \le \mathbf{D}_{\mathbf{m}}\ \ \forall\ h_s \le m_s; s \in I$, and since the elements of the basic set are linearly independent, then

$$N_{\mathbf{m}} \le 1 + 2 + \cdots + (\mathbf{D}_{\mathbf{m}} + 1) \le \lambda_1 \mathbf{D}_{\mathbf{m}}^2,$$

where λ_1 be a constant.

Therefore, the condition (1.14) for a basic set to be Cannon set implies the following condition:

$$\lim_{\langle\mathbf{m}\rangle\to\infty}\{\mathbf{D}_{\mathbf{m}}\}^{\frac{1}{\langle\mathbf{m}\rangle}} = 1\ (see[30]) \tag{1.15}$$

Results on the effectiveness of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in hyperelliptical regions are:

Theorem 1.1 ([13,14])**.** *The necessary and sufficient condition for a Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$ is that $\Omega(P,\ \overline{E}_{[\mathbf{r}]}) = \prod_{s=1}^{k} r_s$.*

Theorem 1.2 ([13,14])**.** *The necessary and sufficient condition for a Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the open hyperellipse $E_{[\mathbf{r}]}$ is that $\Omega(P,\ \overline{E}_{[\mathbf{R}]}) < \alpha([\mathbf{r}],[\mathbf{R}])$.*

Theorem 1.3 ([13,14])**.** *The Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables will be effective in $D(\overline{E}_{[\mathbf{r}]})$, if and only if,*

$$\Omega(P,\ D(\overline{E}_{[\mathbf{r}]})) = \prod_{s=1}^{k} r_s.$$

For more information about the study of basic sets of polynomials, we refer to [3,4,7,13,15,16,19,22,31,32].

2. Ruscheweyh differential operator sum sets of polynomials of several complex variables

Ruscheweyh differential operator $D^{\mathbf{n}}$ [15] acting on the monomial $\mathbf{z}^{\mathbf{m}}$ is defined by:

$$D^{\mathbf{n}}\mathbf{z}^{\mathbf{m}} = \begin{cases} [\sum_{s=1}^{k} D_{z_s}^{n_s}]\mathbf{z}^{\mathbf{m}}; & \mathbf{m}\neq 0,\\ 1; & \mathbf{m}=0 \end{cases}$$

where

$$D_{z_s}^{n_s}\ z_s^{m_s} = \frac{z_s}{n_s!}\ (z_s^{n_s+m_s-1})^{(n_s)},\ (\text{see [29]}) \tag{2.1}$$

the derivatives are repeated n_s-times; $s\in I$. Thus,

$$D^{\mathbf{n}}\mathbf{z}^{\mathbf{m}} = \begin{cases} \sum_{s=1}^{k} \binom{n_s+m_s-1}{n_s}\ \mathbf{z}^{\mathbf{m}}; & \mathbf{m}\neq 0\\ 1; & \mathbf{m}=0. \end{cases} \tag{2.2}$$

Applying $D^{\mathbf{n}}$ into (1.8) we get

$$\begin{cases} \sum_{s=1}^{k} \binom{n_s+m_s-1}{n_s} \mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}^{(\mathbf{n})}[\mathbf{z}]; & \mathbf{m}\neq 0\\ 1 = \sum_{\mathbf{h}} \overline{P}_{\mathbf{0},\mathbf{h}} P_{\mathbf{h}}^{(\mathbf{n})}[\mathbf{z}]; & \mathbf{m}=0 \end{cases}$$

where

$$\begin{aligned}
P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}] &= D^{\mathbf{n}}\ P_{\mathbf{m}}[\mathbf{z}] \\
&= P_{\mathbf{m},\mathbf{0}} + \sum_{\mathbf{h}\geq 1} P_{\mathbf{m},\mathbf{h}} \sum_{s=1}^{k} \binom{n_s+h_s-1}{n_s} \mathbf{z}^{\mathbf{h}} \\
&= \sum_{\mathbf{h}} \alpha_{\mathbf{n},\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}
\end{aligned}$$

and

$$\alpha_{\mathbf{n},\mathbf{h}} = \begin{cases} \sum\limits_{s=1}^{k} \binom{n_s+h_s-1}{n_s}; & \mathbf{h}\neq 0 \\ 1; & \mathbf{h}=0 \end{cases}$$

The set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ is said to be Ruscheweyh differential operator sum set of polynomials of several complex variables.

Now, consider the next question: If we apply the operator $D^{\mathbf{n}}$ on a basic set of polynomials of several complex variables $\{P_{\mathbf{m}}[\mathbf{z}]\}$ then can we say that $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ is still basic? In fact the aim of the following section is to give an answer to this question.

To construct the basic property of the set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ we write

$$D^{\mathbf{n}} P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \alpha_{\mathbf{n},\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}} = \sum_{\mathbf{h}} P^{(\mathbf{n})}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}.$$

The matrix of coefficients $P^{(\mathbf{n})}$ of this set are

$$P^{(\mathbf{n})} = (\alpha_{\mathbf{n},\mathbf{h}}\ P_{\mathbf{m},\mathbf{h}}).$$

Also the matrix of operators $\overline{P}^{(\mathbf{n})}$ follows from the representation

$$\mathbf{z}^{\mathbf{m}} = \frac{1}{\alpha_{\mathbf{n},\mathbf{m}}} \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}}\ P^{(\mathbf{n})}_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}^{(\mathbf{n})}_{\mathbf{m},\mathbf{h}} P^{(\mathbf{n})}_{\mathbf{h}}[\mathbf{z}],$$

that is to say

$$\overline{P}^{(\mathbf{n})} = \left(\frac{1}{\alpha_{\mathbf{n},\mathbf{m}}}\ \overline{P}_{\mathbf{m},\mathbf{h}} \right).$$

Therefore

$$\begin{aligned}
P^{(\mathbf{n})}\ \overline{P}^{(\mathbf{n})} &= \left(\sum_{\mathbf{h}} P^{(\mathbf{n})}_{\mathbf{m},\mathbf{h}} \overline{P}^{(\mathbf{n})}_{\mathbf{h},\mathbf{k}} \right) \\
&= \left(\sum_{\mathbf{h}} \alpha_{\mathbf{n},\mathbf{h}}\ P_{\mathbf{m},\mathbf{h}}\ \frac{1}{\alpha_{\mathbf{n},\mathbf{h}}}\ \overline{P}_{\mathbf{h},\mathbf{k}} \right) \\
&= P\ \overline{P} = I.
\end{aligned}$$

where I is the infinite unit matrix.

Also,

$$\begin{aligned}\overline{P}^{(\mathbf{n})}\ P^{(\mathbf{n})} &= \left(\sum_{\mathbf{h}} \overline{P}^{(\mathbf{n})}_{\mathbf{m},\mathbf{h}}\ P^{(\mathbf{n})}_{\mathbf{h},\mathbf{k}}\right)\\ &= \left(\frac{\alpha_{\mathbf{n},\mathbf{k}}}{\alpha_{\mathbf{n},\mathbf{m}}}\sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}}\ P_{\mathbf{h},\mathbf{k}}\right)\\ &= \left(\frac{\alpha_{\mathbf{n},\mathbf{k}}}{\alpha_{\mathbf{n},\mathbf{m}}}\ \delta^{\mathbf{m}}_{\mathbf{k}}\right) = I.\end{aligned}$$

Hence the basic property of Ruscheweyh differential operator sum set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ follows directly from Theorem 5 of [25].

3. Effectiveness of Ruscheweyh differential operator sum set of polynomials in closed hyperellipse

Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is a basic set of polynomials of several complex variables and $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ is the Ruscheweyh differential operator sum set. Consider the next question: If the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in closed hyperellipse $\overline{E}_{[\mathbf{r}]}$, do the set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ still effective in $\overline{E}_{[\mathbf{r}]}$? In this section we will give the answer of this question.

Let $\Omega(P^{(\mathbf{n})}_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})$ be the Cannon sum of the set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ for the hyperellipse $\overline{E}_{[\mathbf{r}]}$, then

$$\begin{aligned}\Omega(P^{(\mathbf{n})}_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) &= \sigma_{\mathbf{m}}\prod_{s=1}^{k}\{r_s\}^{\langle\mathbf{m}\rangle-m_s}\sum_{\mathbf{h}}\left|\overline{P}^{(\mathbf{n})}_{\mathbf{m},\mathbf{h}}\right| M(P^{(\mathbf{n})}_{\mathbf{h}},\ \overline{E}_{[\mathbf{r}]})\\ &= \frac{\sigma_{\mathbf{m}}}{\alpha_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{k}\{r_s\}^{\langle\mathbf{m}\rangle-m_s}\sum_{\mathbf{h}}\left|\overline{P}_{\mathbf{m},\mathbf{h}}\right|\ M(P^{(\mathbf{n})}_{\mathbf{h}},\ \overline{E}_{[\mathbf{r}]})\end{aligned} \tag{3.1}$$

where,

$$M(P^{(\mathbf{n})}_{\mathbf{h}},\ \overline{E}_{[\mathbf{r}]}) = \max_{\overline{E}_{[\mathbf{r}]}}\left|P^{(\mathbf{n})}_{\mathbf{h}}[\mathbf{z}]\right|.$$

Now, we let, $\mathbf{D_m}$ be the degree of the polynomial of the highest degree in the representation (1.8). Hence by Cauchy's inequality we see that

$$
\begin{aligned}
M(P^{(\mathbf{n})}_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}) &= \max_{\overline{E}_{[\mathbf{r}]}} \left|P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\right| \leq \sum_{\mathbf{h}} \left|P^{(\mathbf{n})}_{\mathbf{m},\mathbf{h}}\right| \frac{\prod_{s=1}^{k}\{r_s\}^{h_s}}{\sigma_{\mathbf{h}}} \\
&= \sum_{\mathbf{h}} \alpha_{\mathbf{n},\mathbf{h}} \left|P_{\mathbf{m},\mathbf{h}}\right| \frac{\prod_{s=1}^{k}\{r_s\}^{h_s}}{\sigma_{\mathbf{h}}} \leq \ M(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) \sum_{\mathbf{h}} \alpha_{\mathbf{n},\mathbf{h}} \\
&= M(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) \left[1 + \sum_{\mathbf{h}\geq 1} \alpha_{\mathbf{n},\mathbf{h}}\right] \\
&= M(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) \ \left[1 + \sum_{\mathbf{h}\geq 1} \sum_{s=1}^{k} \binom{n_s + h_s - 1}{n_s}\right] \\
&\leq K N_{\mathbf{m}} \mathbf{D}^{n}_{\mathbf{m}} M(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) \\
&\leq K \ \mathbf{D}^{n+2}_{\mathbf{m}} M(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})
\end{aligned} \tag{3.2}
$$

where K be a constant and the power n here because we differentiated n_s-times.

Thus the relation between the Cannon sums of the two sets $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and $\{P^{(n)}_{\mathbf{m}}[\mathbf{z}]\}$ can be obtained from the relations (3.1) and (3.2) as follows

$$
\Omega(P^{(\mathbf{n})}_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}) \leq K \ \frac{\mathbf{D}^{n+2}_{\mathbf{m}}}{\alpha_{\mathbf{n},\mathbf{m}}} \ \ \Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}). \tag{3.3}
$$

Consider condition (1.15), we obtain that

$$
\begin{aligned}
\Omega(P^{(\mathbf{n})}, \overline{E}_{[\mathbf{r}]}) &= \lim_{\langle \mathbf{m}\rangle\to\infty} \sup\{\Omega(P^{(\mathbf{n})}_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})\}^{\frac{1}{\langle \mathbf{m}\rangle}} \\
&\leq \lim_{\langle \mathbf{m}\rangle\to\infty} \sup\{K \ \frac{\mathbf{D}^{n+2}_{\mathbf{m}}}{\alpha_{\mathbf{n},\mathbf{m}}} \ \ \Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})\}^{\frac{1}{\langle \mathbf{m}\rangle}} \\
&\leq \Omega(P,\ \overline{E}_{[\mathbf{r}]}) = \prod_{s=1}^{k} r_s.
\end{aligned} \tag{3.4}
$$

But

$$
\Omega(P^{(\mathbf{n})},\ \overline{E}_{[\mathbf{r}]}) \geq \prod_{s=1}^{k} \ r_s \ \text{(see [14]).}
$$

Then,

$$
\Omega(P^{(\mathbf{n})},\ \overline{E}_{[\mathbf{r}]}) = \prod_{s=1}^{k} r_s. \tag{3.5}
$$

Therefore, according to (3.5) and using Theorem 1.1, we deduce that the effectiveness of the original set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in $\overline{E}_{[\mathbf{r}]}$ implies the effectiveness of Ruscheweyh differential operator sum set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ in $\overline{E}_{[\mathbf{r}]}$.

Hence, we obtain the following theorem

Theorem 3.1. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s, $s \in I$ for which the condition (1.15) is satisfied, is effective in the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$, then the Ruscheweyh differential operator sum set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $\overline{E}_{[\mathbf{r}]}$.*

If, condition (1.15) is not satisfied then the set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ can not be effective in $\overline{E}_{[\mathbf{r}]}$, where the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\overline{E}_{[\mathbf{r}]}$. To ensure this, we give the following example.

Example 1. Consider the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables $z_s \in I$ given by

$$\begin{cases} P_{\mathbf{m}}[\mathbf{z}] = \sigma_{\mathbf{m}} \prod_{s=1}^{k} z_s^{m_s} + \sigma_{a\mathbf{m}} \prod_{s=1}^{k} z_s^{am_s}, & \mathbf{m} \neq 0, \\ P_{\mathbf{m}}[\mathbf{z}] = \sigma_{\mathbf{m}} \prod_{s=1}^{k} z_s^{m_s}, & \text{otherwise}, \end{cases} \tag{3.6}$$

where $a = b^{\langle \mathbf{m} \rangle}, b > 1$, then

$$\prod_{s=1}^{k} z_s^{m_s} = \mathbf{z}^{\mathbf{m}} = \frac{1}{\sigma_{\mathbf{m}}}\, [P_{\mathbf{m}}[\mathbf{z}] - P_{a\mathbf{m}}[\mathbf{z}]],$$

and the Cannon sum $\Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]})$ will given by

$$\Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}]}) = \prod_{s=1}^{k} [r_s^{\langle \mathbf{m} \rangle} + 2\ r_s^{\langle \mathbf{m} \rangle + (a-1)m_s}],$$

which leads to

$$\Omega(P, \overline{E}_{[\mathbf{1}]}) \leq \lim_{\langle \mathbf{m} \rangle \to \infty} \sup\{\Omega(P_{\mathbf{m}}, \overline{E}_{[1]})\}^{\frac{1}{\langle \mathbf{m} \rangle}} = 1;$$

i.e. the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\overline{E}_{[\mathbf{r}]}$ for $r_s = 1,\ s \in I$.

Now, construct Ruscheweyh differential operator sum set $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ as follows;

$$\begin{cases} P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}] = \sigma_{\mathbf{m}} \alpha_{\mathbf{n},\mathbf{m}}\ \mathbf{z}^{\mathbf{m}} + \sigma_{a\mathbf{m}}\ \alpha_{\mathbf{n},a\mathbf{m}} \prod_{s=1}^{k} z_s^{am_s}; & (\mathbf{m}) \neq 0 \\ P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}] = \sigma_{\mathbf{m}} \alpha_{\mathbf{n},\mathbf{m}} \mathbf{z}^{\mathbf{m}} & \text{otherwise.} \end{cases}$$

Thus it follows that,

$$\mathbf{z}^{\mathbf{m}} = \frac{1}{\sigma_{\mathbf{m}} \alpha_{\mathbf{n},\mathbf{m}}}\ [P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\ - P^{(\mathbf{n})}_{a\mathbf{m}}[\mathbf{z}]]$$

and the Cannon sum $\Omega(P_{\mathbf{m}}^{(\mathbf{n})},\ \overline{E}_{[\mathbf{r}]})$ will given by

$$\begin{aligned}\Omega(P_{\mathbf{m}}^{(\mathbf{n})},\ \overline{E}_{[\mathbf{r}]}) &= \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_s\}^{\langle \mathbf{m}\rangle - m_s} \sum_{\mathbf{h}} \left| \overline{P}_{\mathbf{m},\mathbf{h}}^{(\mathbf{n})} \right| \ M(P_{\mathbf{h}}^{(\mathbf{n})},\ \overline{E}_{[\mathbf{r}]}) \\ &= \frac{1}{\alpha_{\mathbf{n},\mathbf{m}}} \ [\alpha_{\mathbf{n},\mathbf{m}} \prod_{s=1}^{k} r_s^{\langle \mathbf{m}\rangle} + 2\alpha_{\mathbf{n},a\mathbf{m}} \prod_{s=1}^{k} r_s^{\langle \mathbf{m}\rangle + (a-1)m_s}] \\ &= \prod_{s=1}^{k} r_s^{\langle \mathbf{m}\rangle} + \zeta(a) \prod_{s=1}^{k} r_s^{\langle \mathbf{m}\rangle + (a-1)m_s},\end{aligned}$$

where $\zeta(a) > 1$ is a constant depending only on a and

$$\Omega(P^{(\mathbf{n})}, \overline{E}_{[\mathbf{r}]}) = 1 + \zeta(a) > 1,$$

that is to say that the Ruscheweyh differential operator sum set $\{P_{\mathbf{m}}^{(\mathbf{n})}[\mathbf{z}]\}$ is not effective in $\overline{E}_{[\mathbf{r}]}$ for $r_s = 1, s \in I$, although the original set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\overline{E}_{[\mathbf{r}]}$. The reason for this, obviously, that condition (1.15) is not satisfied by the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ as required.

4. Effectiveness of Ruscheweyh differential operator sum set of polynomials in open hyperellipse and the region $D(\overline{E}_{[\mathbf{r}]})$

Now, we establish the effectiveness property for Ruscheweyh differential operator sum set $\{P_{\mathbf{m}}^{(\mathbf{n})}[\mathbf{z}]\}$ in open hyperellipse $\overline{E}_{[\mathbf{r}]}$ and the region $D(\overline{E}_{[\mathbf{r}]})$.

Suppose that the Cannon set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $E_{[\mathbf{r}]}$. Then from the properties of Cannon functions, it follows from Theorem 1.1 of [14], that

$$\Omega(P,\ \overline{E}_{[\mathbf{R}]}) < \alpha([\mathbf{r}],\ [\mathbf{R}]) \text{ for all } 0 < R_s < r_s,\ s \in I. \tag{4.1}$$

Construct the sets of numbers $\{r_i^{(s)},\ s \in I\}$, (cf. [14]), in such a way that $0 < r_0^{(s)} < r_s,\ s \in I$ and

$$\frac{r_0^{(s)}}{r_0^{(j)}} = \frac{r_s}{r_j};\ s, j \in I, \tag{4.2}$$

$$r_{i+1}^{(s)} = \frac{1}{2}(r_s + r_i^{(s)});\ s \in I;\ i \geq 0. \tag{4.3}$$

It follows, easily, from (4.2) and (4.3) that

$$\frac{r_i^{(s)}}{r_i^{(j)}} = \frac{r_s}{r_j};\ s, j \in I;\ i \geq 0. \tag{4.4}$$

Therefore it follows that

$$R_s < r_i^{(s)} < r_s;\ s \in I;\ i \geq 0.$$

Now, since the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ accord to (4.1), in view of (1.12) and (1.13), then corresponding to the numbers $r_i^{(s)},\ s \in I$, there exists a constant $K \geq 1$ such that

$$\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_i^{(s)}\}^{\langle \mathbf{m} \rangle - m_s}\ G(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}_i]}) < K\{r_{i+1}^{(1)}\ \prod_{s=2}^{k} r_i^{(s)}\}^{\langle \mathbf{m} \rangle},$$

from which we get, in view of (4.4), the following inequality

$$\begin{aligned} G(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}_i]}) &< \frac{K}{\sigma_{\mathbf{m}}} \{\frac{r_{i+1}^{(1)}}{r_i^{(1)}}\}^{\langle \mathbf{m} \rangle}\ \prod_{s=1}^{k} \{r_i^{(s)}\}^{m_s} \\ &= \frac{K}{\sigma_{\mathbf{m}}}\ \prod_{s=1}^{k} \{\frac{r_{i+1}^{(1)}}{r_i^{(1)}}\ r_i^{(s)}\}^{m_s} \\ &= \frac{K}{\sigma_{\mathbf{m}}}\ \prod_{s=1}^{k} \{\frac{r_{i+1}^{(s)}}{r_i^{(s)}}\ r_i^{(s)}\}^{m_s} \\ &= \frac{K}{\sigma_{\mathbf{m}}}\ \prod_{s=1}^{k} \{r_{i+1}^{(s)}\}^{m_s};\ (m_s \geq 0;\ s \in I). \end{aligned} \tag{4.5}$$

Now, for the numbers $r_s, R_s; s \in I$, we have at least one of the following cases:

(i) $\frac{R_1}{R_s} \leq \frac{r_1}{r_s}; s \in I$ or
(ii) $\frac{R_\nu}{R_s} \leq \frac{r_\nu}{r_s};\ s \in I,\ \nu = 2$ or 3 or $\ldots$ or $k-1$ or
(iii) $\frac{R_k}{R_s} \leq \frac{r_k}{r_s}; s \in I$.

Suppose now, that relation (i) is satisfied, then from the construction of the sets $\{r_i^{(s)},\ s \in I\}$, we see that

$$\frac{R_1}{R_s} \leq \frac{r_1}{r_s} = \frac{r_{i+1}^{(1)}}{r_{i+1}^{(s)}};\ s \in I. \tag{4.6}$$

Thus the Cannon sum of the set $\{P^{(n)}_{\mathbf{m}}[\mathbf{z}]\}$ for the hyperellipse $\overline{E}_{[\mathbf{R}]}$, in view of (4.5) and (4.6) lead to

$$
\begin{aligned}
\Omega(P^{(n)}_{\mathbf{m}},\ \overline{E}_{[\mathbf{R}]}) &= \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m}\rangle - m_s} \sum_h \left|\overline{P}^{(n)}_{m,h}\right| \ M(P^{(n)}_h,\ \overline{E}_{[\mathbf{R}]}) \\
&= \frac{\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m}\rangle - m_s}}{\alpha_{n,m}} \sum_h \left|\overline{P}_{m,h}\right| \ M(P^{(n)}_h,\ \overline{E}_{[\mathbf{R}]}) \\
&< L \frac{\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m}\rangle - m_s}}{\alpha_{n,m}} \sum_h \left|\overline{P}_{m,h}\right| \ M(P_h,\ \overline{E}_{[\mathbf{r}_i]}) \\
&= L \ \frac{\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m}\rangle - m_s}}{\alpha_{n,m}} G(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{r}_i]}) \\
&< \frac{KL}{\alpha_{n,m}} \ \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m}\rangle - m_s} \ \{r^{(s)}_{i+1}\}^{m_s} \\
&= \frac{KL}{\alpha_{n,m}} \ \prod_{s=1}^{k} \{r^{(s)}_{i+1}\}^{m_s} \{\frac{R_1}{R_s}\}^{m_s} \ \prod_{s=2}^{k} \{R_s\}^{\langle \mathbf{m}\rangle} \\
&\le \frac{KL}{\alpha_{n,m}} \ \prod_{s=1}^{k} \{r^{(s)}_{i+1}\}^{m_s} \{\frac{r_1}{r_s}\}^{m_s} \ \prod_{s=2}^{k} \{R_s\}^{\langle \mathbf{m}\rangle} \\
&= \frac{KL}{\alpha_{n,m}} \ \prod_{s=1}^{k} \{r^{(s)}_{i+1}\}^{m_s} \{\frac{r^{(1)}_{i+1}}{r^{(s)}_{i+1}}\}^{m_s} \ \prod_{s=2}^{k} \{R_s\}^{\langle \mathbf{m}\rangle} \\
&= \frac{KL}{\alpha_{n,m}} \ \{r^{(1)}_{i+1} \ \prod_{s=2}^{k} R_s\}^{\langle \mathbf{m}\rangle};
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Omega(P^{(\mathbf{n})},\ \overline{E}_{[\mathbf{R}]}) &= \lim_{\langle \mathbf{m}\rangle\to\infty} \sup\ \{\Omega(P_{\mathbf{m}},\ \overline{E}_{[\mathbf{R}]})\}^{\frac{1}{\langle \mathbf{m}\rangle}} \\
&\le r^{(1)}_{i+1} \ \prod_{s=2}^{k} R_s < r_1 \prod_{s=2}^{k} R_s,
\end{aligned} \tag{4.7}
$$

where

$$
L = 1 + \sum_{(h)\ge 1} \ \sum_{s=1}^{k} \binom{n_s + h_s - 1}{n_s} \prod_{s=1}^{k} \{\frac{R^{(s)}_i}{r^{(s)}_i}\}^{h_s} \quad \forall\ 0 < R_s < r_s;\ s \in I.
$$

Also, if relation (ii) is satisfied for $\nu = 2$ *or* 3 *or* $\ldots$ *or* $k-1$, then we shall have

$$\frac{R_\nu}{R_s} \le \frac{r_\nu}{r_s} = \frac{r_{i+1}^{(\nu)}}{r_{i+1}^{(s)}};\ s \in I. \tag{4.8}$$

Thus (4.5) and (4.8) leads

$$\begin{aligned}
\Omega(P_{\mathbf{m}}^{(\mathbf{n})},\ \overline{E}_{[\mathbf{R}]}) &< \frac{KL}{\alpha_{\mathbf{n},\mathbf{m}}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m}\rangle - m_s}\ \{r_{i+1}^{(s)}\}^{m_s} \\
&= \frac{KL}{\alpha_{n,m}} \prod_{s=1}^{k} \{r_{i+1}^{(s)}\}^{m_s} \{\frac{R_\nu}{R_s}\}^{m_s}\ \{\prod_{s=1,s\neq\nu}^{k} R_s\}^{\langle \mathbf{m}\rangle} \\
&\le \frac{KL}{\alpha_{n,m}} \prod_{s=1}^{k} \{r_{i+1}^{(s)}\}^{m_s} \{\frac{r_\nu}{r_s}\}^{m_s}\ \{\prod_{s=1,s\neq\nu}^{k} R_s\}^{\langle \mathbf{m}\rangle} \\
&= \frac{KL}{\alpha_{n,m}} \prod_{s=1}^{k} \{r_{i+1}^{(s)}\}^{m_s} \{\frac{r_{i+1}^{(\nu)}}{r_{i+1}^{(s)}}\}^{m_s}\ \{\prod_{s=1,s\neq\nu}^{k} R_s\}^{\langle \mathbf{m}\rangle} \\
&= \frac{KL}{\alpha_{n,m}}\ \{r_{i+1}^{(\nu)} \prod_{s=1,s\neq\nu}^{k} R_s\}^{\langle \mathbf{m}\rangle}
\end{aligned}$$

Therefore

$$\Omega(P^{\mathbf{n}},\ \overline{E}_{[\mathbf{R}]}) \le r_{i+1}^{(\nu)} \prod_{s=1,s\neq\nu}^{k} R_s < r_\nu \prod_{s=1,s\neq\nu}^{k} R_s \tag{4.9}$$

where $\nu = 2$ *or* 3 *or* $\ldots$ *or* $k-1$. Similarly if relation (iii) is satisfied, we proceed similarly as above to show,

$$\Omega(P^{(\mathbf{n})},\ \overline{E}_{[\mathbf{R}]}) < r_k \prod_{s=1}^{k-1} R_s. \tag{4.10}$$

Thus, it follows in view of (4.7), (4.9) and (4.10) that

$$\Omega(P^{(\mathbf{n})},\ \overline{E}_{[\mathbf{R}]}) < \alpha([\mathbf{r}],[\mathbf{R}]). \tag{4.11}$$

Therefore, according to (4.11) and using Theorem 1.2 the Ruscheweyh differential operator sum set $\{P_{\mathbf{m}}^{(\mathbf{n})}[\mathbf{z}]\}$ is effective in the open hyperellipse $E_{[\mathbf{r}]}$ when the original set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $E_{[\mathbf{r}]}$.

Hence, we obtain the following theorem:

Theorem 4.1. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s, $s \in I$ is effective in the open hyperellipse $E_{[\mathbf{r}]}$, then the Ruscheweyh differential operator sum set $\{P_{\mathbf{m}}^{(\mathbf{n})}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $E_{[\mathbf{r}]}$.*

Now, using a similar proof as done to Theorem 4.1, the following relation follows

$$\Omega[P^{(\mathbf{n})},\ D(\overline{E}_{[\mathbf{r}]}] = \prod_{s=1}^{k} r_s \text{ when } \Omega[P,\ D(\overline{E}_{[\mathbf{r}]}] = \prod_{s=1}^{k} r_s$$

Therefore, by using Theorem 1.3, we obtain the following theorem

Theorem 4.2. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables $z_s,\ s \in I$ is effective in the region $D(\overline{E}_{[\mathbf{r}]})$, then the Ruscheweyh differential operator sum set $\{P_{\mathbf{m}}^{(\mathbf{n})}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $D(\overline{E}_{[\mathbf{r}]})$.*

Now, we define the Ruscheweyh differential operator $D^{*\mathbf{n}}$ acting on the monomial $\mathbf{z}^{\mathbf{m}}$, such that

$$D^{*\mathbf{n}}\mathbf{z}^{\mathbf{m}} = \begin{cases} \left[\prod\limits_{s=1}^{k} D_{z_s}^{n_s}\right] \mathbf{z}^{\mathbf{m}}; & \mathbf{m} \neq 0, \\ 1; & \mathbf{m} = 0, \end{cases}$$

where $D_{z_s}^{n_s}$ is defined as in (2.1). Thus,

$$D^{*\mathbf{n}}\mathbf{z}^{\mathbf{m}} = \begin{cases} \prod\limits_{s=1}^{k} \binom{n_s + m_s - 1}{n_s} \mathbf{z}^{\mathbf{m}}; & \mathbf{m} \neq 0 \\ 1; & \mathbf{m} = 0. \end{cases} \tag{4.12}$$

Thus inserting the operator $D^{*\mathbf{n}}$ into (1.8) we get

$$\begin{cases} \prod\limits_{s=1}^{k} \binom{n_s + m_s - 1}{n_s} \mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}}\ P_{\mathbf{h}}^{(\mathbf{n})}[\mathbf{z}]; & \mathbf{m} \neq 0 \\ 1 = \sum_{\mathbf{h}} \overline{P}_{\mathbf{0},\mathbf{h}}\ P_{\mathbf{h}}^{(\mathbf{n})}[\mathbf{z}]; & \mathbf{m} = 0, \end{cases}$$

where

$$\begin{aligned} P_{\mathbf{m}}^{*(\mathbf{n})}[\mathbf{z}] &= D^{*\mathbf{n}}\ P_{\mathbf{m}}[\mathbf{z}] \\ &= P_{\mathbf{m},\mathbf{0}} + \sum_{\mathbf{h} \geq 1} P_{\mathbf{m},\mathbf{h}} \prod_{s=1}^{k} \binom{n_s + h_s - 1}{n_s} \mathbf{z}^{\mathbf{h}} \\ &= \sum_{\mathbf{h}} \gamma_{\mathbf{n},\mathbf{h}}\ P_{\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}} \end{aligned}$$

where,

$$\gamma_{\mathbf{n},\mathbf{h}} = \begin{cases} \prod\limits_{s=1}^{k} \binom{n_s + h_s - 1}{n_s}; & \mathbf{h} \neq 0 \\ 1; & \mathbf{h} = 0 \end{cases}$$

The set $\{P_{\mathbf{m}}^{*(\mathbf{n})}[\mathbf{z}]\}$ is said be Ruscheweyh differential operator product set of polynomials of several complex variables.

Similarly we may proceed as in Section 2 to prove the basic property of this set $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$, such that

$$P^{*(\mathbf{n})}\ \overline{P}^{*(\mathbf{n})} = \overline{P}^{*(\mathbf{n})}\ P^{*(\mathbf{n})} = I.$$

For $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$, we can proceed very similar as in Theorem 3.1,Theorem 4.1 and Theorem 4.2 to prove the following theorems:

Theorem 4.3. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s, $s \in I$ for which the condition (1.15) is satisfied, is effective in the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$, then the Ruscheweyh differential operator product set $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $\overline{E}_{[\mathbf{r}]}$.*

Theorem 4.4. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s, $s \in I$ is effective in the open hyperellipse $E_{[\mathbf{r}]}$, then the Ruscheweyh differential operator product sum set $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $E_{[\mathbf{r}]}$.*

Theorem 4.5. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s, $s \in I$ is effective in the region $D(\overline{E}_{[\mathbf{r}]})$, then the Ruscheweyh differential operator product set $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $D(\overline{E}_{[\mathbf{r}]})$.*

Before obtaining effectiveness conditions of basic sets of polynomials in hyperspherical regions from our results, we introduce the following notations:

The open hypersphere is defined by:

$$S_r = \{\mathbf{z} \in \mathbb{C}^k : \left(\sum_{s=1}^{k} |z_s|^2\right)^{\frac{1}{2}} < r\},$$

the closed hypersphere is defined by:

$$\overline{S}_r = \{\mathbf{z} \in \mathbb{C}^k : \left(\sum_{s=1}^{k} |z_s|^2\right)^{\frac{1}{2}} \leq r\}.$$

The region $D[\overline{S}_r]$ means unspecified domain containing the closed hypersphere $\overline{S}_r$.

To get the results concerning the effectiveness in hyperspherical regions (cf. [15]) as special cases from the results concerning effectiveness in hyperelliptical regions, put $r_s = r, s \in I$ in Theorem 3.1, Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4 and Theorem 4.5 we can arrive to the following result

Corollary 4.1. *The effectiveness of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in the equiellipse*

1. $\overline{E}_{[\mathbf{r}]^*}$ yields the effectiveness of the sets $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ and $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ in the hypersphere $\overline{S}_r$

2. $E_{[\mathbf{r}]^*}$ *yields the effectiveness of the sets* $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ *and* $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ *in the hypersphere* S_r

3. $D(\overline{E}_{[\mathbf{r}]^*})$ *yields the effectiveness of the sets* $\{P^{(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ *and* $\{P^{*(\mathbf{n})}_{\mathbf{m}}[\mathbf{z}]\}$ *in the region* $D(\overline{S}_r)$ *where* $[\mathbf{r}]^* = (r, r, r, \ldots, r)$*,* r *is repeated* k*-times.*

Now, suppose that $J_N(D^{\mathbf{n}})$ is a polynomial of the operator $D^{\mathbf{n}}$ as given in (2.2) and $J_N(D^{*\mathbf{n}})$ is a polynomial of the operator $D^{*\mathbf{n}}$ as given in (4.12) such that $J_N(D^{\mathbf{n}}) = \sum\limits_{j=1}^{N} \lambda_j (D^{\mathbf{n}})^j$ and $J_N(D^{*\mathbf{n}}) = \sum_{j=1}^{N} \lambda_j (D^{*\mathbf{n}})^j$, where $(D^{\mathbf{n}})^j = (D^{\mathbf{n}})^{j-1} D^{\mathbf{n}}$, $(D^{*\mathbf{n}})^j = (D^{*\mathbf{n}})^{j-1} D^{*\mathbf{n}}$, j be a finite positive integer and λ_j are constants $\neq 0$.

It is worthy to ensure that Theorem 3.1, Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 4.5 and Corollary 4.1 will be still true when we replace the set $\{D^{\mathbf{n}} P_{\mathbf{m}}(\mathbf{z})\}$ and $\{D^{*\mathbf{n}} P_{\mathbf{m}}(\mathbf{z})\}$ by the sets $\{J_N(D^{\mathbf{n}}) P_{\mathbf{m}}(\mathbf{z})\}$ and $\{J_N(D^{*\mathbf{n}}) P_{\mathbf{m}}(\mathbf{z})\}$, respectively.

Remark 4.1. Similar results for the sets

$$\{D^{\mathbf{n}} P_{\mathbf{m}}(\mathbf{z})\}, \{D^{*\mathbf{n}} P_{\mathbf{m}}(\mathbf{z})\}, \{J_N(D^{\mathbf{n}}) P_{\mathbf{m}}(\mathbf{z})\}$$

and $\{J_N(D^{*\mathbf{n}}) P_{\mathbf{m}}(\mathbf{z})\}$ in hyperelliptical regions can be obtained when the original set $\{P_{\mathbf{m}}(\mathbf{z})\}$ is general basic set.

Remark 4.2. The effectiveness of Ruscheweyh differential operator sets of polynomials of several complex variables in complete Reinhardt domains was studied in [15].

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