*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**24** (2008), 179–189
www.emis.de/journals
ISSN 1786-0091

# NONLINEAR DYNAMICAL SYSTEMS AND KCC-THEORY

TAKAHIRO YAJIMA AND HIROYUKI NAGAHAMA

Abstract. Nonlinear dynamical systems can be uniquely investigated by a geometric theory (KCC-theory). The five KCC-invariants express intrinsic properties of the nonlinear dynamical systems. The second invariant as a curvature tensor determines the stability of the systems. The third invariant as a torsion tensor expresses the chaotic behavior. As an example, the KCC-theory is applied to a geodynamical system (the Rikitake system).

## 1. Introduction

Nonlinear dynamical systems can be seen in various fields. For example, the Lotka-Volterra system in ecology [20, 25], the Rikitake system in geodynamo theory [22], the Lorenz model in meteorology [19], the Belousov-Zhabotinskii reaction or the Rössler equation in chemistry [23, 24, 27]. These nonlinear dynamical systems can be unified into one expression:

$$\frac{d^2x^i}{d\tau^2} + 2G^i\left(x^j, \frac{dx^j}{d\tau}, \tau\right) = 0,\ i, j = 1, \cdots, n, \tag{1}$$

where $x^i$ is a coordinate of the configuration space, $\tau$ is a kind of parameter such as a time $t$ and $G^i$ is a smooth function. Under a coordinate transformation, the system of differential equations (1) will change to a system of differential equations in a new coordinate system. In this case, the objects which do not change are called invariants of the system. The study of the invariants of the

2000 *Mathematics Subject Classification.* 34A26, 34A34, 37N99, 46A63, 53B40, 53B50, 86A25.

*Key words and phrases.* Nonlinear dynamical systems, Rikitate system, KCC-theory, Finsler geometry, topological invariant.

One of the authors (T. Yajima) was financially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists and by the 21st Center-Of-Excellence program, "Advanced Science and Technology Center for the Dynamic Earth", of Tohoku University.

second order differential equations is called the KCC-theory (the general path-space theory of Kosambi [17], Cartan [7] and Chern [8]). Then, one can find five KCC-invariants for the system (1) under the coordinate transformation [1, 2, 4]. Therefore, the KCC-theory is useful for studying the intrinsic properties of the nonlinear dynamical systems.

In the following sections, geometric descriptions of the nonlinear dynamical systems are shown based on the KCC-theory. In section 2, the KCC-theory is briefly reviewed. In section 3, it is shown that nonlinear dynamical systems can be uniquely investigated by the KCC-theory. In section 4, the KCC-theory is applied to the geodynamical system (the Rikitake system) and the KCC-invariants are obtained. In section 5, the chaotic behavior of the Rikitake system is discussed by using the geometric invariants. It is remarked that our results are extension of some results of [26].

## 2. Geometric preliminary

In this section, the geometric background of this study is briefly explained based on the notations [1, 2, 3, 4]. Throughout this paper, Einstein's summation convention is used and Latin indices $i, j, k, \dots$ run from 1 to $n$.

Now, let $M$ be a real smooth $n$-dimensional manifold, and $(TM, \pi, M)$ be its tangent bundle (phase space), where $\pi\colon TM \to M$ is a projection from the total space $TM$ to the base manifold $M$. A point $x \in M$ has local coordinates $(x^i)$, where $i = 1, \dots n$. The local chart of a point $u \in TM$ is denoted by $(x^i, y^i)$, where $y^i = dx^i/dt$ and $t$ is a time. The time $t$ is regarded as an absolute invariant. Therefore, a non-singular coordinate transformation is given by

$$\tilde{t} = t, \quad \tilde{x}^i = \tilde{x}^i(x^1, x^2, \dots, x^n). \tag{2}$$

Let us consider an equation of a path $c(t) = (x^i(t))$ as

$$\frac{d^2x^i}{dt^2} + 2G^i(x^j, y^j, t) = 0, \tag{3}$$

where $G^i(x^j, y^j)$ is a smooth function. Then, under the non-singular coordinate transformations (2), the KCC-covariant differential of a vector field $\xi^i(t)$ along the path $c(t)$ is

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j \xi^j, \tag{4}$$

where $N^i_j$ is a coefficient of the nonlinear connection defined by $N^i_j = \partial G^i/\partial y^j$ [1]. Moreover, when $\xi^i = y^i$, the covariant differential is

$$\frac{Dy^i}{dt} = N^i_j y^j - 2G^i \equiv -\epsilon^i, \tag{5}$$

where the contravariant vector field $\epsilon^i$ is called the first KCC-invariant. When the first invariant does not vanish, the path of the second order differential equations does not coincide with the autoparallel curve of the nonlinear connection.

It is shown from expression (26) in section 4 that the first invariant means the external force of the system.

Next, let us consider that the trajectory $x^i(t)$ of the system (3) is changed into nearby ones according to

$$\bar{x}^i(t) = x^i(t) + \xi^i(t)\eta, \tag{6}$$

where $\eta$ is a small parameter. Substituting (6) in (3) and taking the limit $\eta \to 0$, one gets the variational equation

$$\frac{d^2\xi^i}{dt^2} + 2N^i_l \frac{d\xi^l}{dt} + 2\frac{\partial G^i}{\partial x^l}\xi^l = 0. \tag{7}$$

Using the covariant differential (4), one can rewrite (7) in the covariant form

$$\frac{D^2\xi^i}{dt^2} + B^i_j \xi^j = 0, \tag{8}$$

where

$$B^i_j = 2\frac{\partial G^i}{\partial x^j} + 2G^l G^i_{jl} - y^l \frac{\partial N^i_j}{\partial x^l} - N^i_l N^l_j - \frac{\partial N^i_j}{\partial t}. \tag{9}$$

Here, the $G^i_{jk} \equiv \partial N^i_j/\partial y^k$, it is a Finsler connection (Berwald connection). The $B^i_j$ is the second invariant or the deviation curvature tensor and gives the stability of whole trajectories (the Jacobi stability) [4]. In engineering field, this variational equation called the hunting equation has been introduced in order to study a stability of electrical machine systems in non-Riemannian space [15, 16, 18] and in Finsler space [11]. On the other hand, the linear stability theory is the theory on a local stability around a point on the tangent space [5]. This means that the behavior of the nonlinear dynamical systems is described on the tangent bundle $TM$. In this case, the equation (3) is a first order differential equation with respect to $y^i$ and the Jacobi stability equation (7) is reduced to the equation of a linear stability theory. Therefore, the Jacobi stability gives a more global stability than the linear stability.

The third invariant is defined by

$$B^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}, \tag{10}$$

where $\delta/\delta x^i$ is given by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}. \tag{11}$$

Because of the skew symmetry of lower indices $j$ and $k$, the third invariant is regarded as a torsion tensor. The third invariant is also called the curvature tensor of the nonlinear connection $N^i_j$ [1]. In this case, the tensor $B^i_{jk}$ plays a role of field strength and the nonlinear connection is interpreted as the potential. The fourth and fifth invariants are given by $B^i_{jkl} = \partial B^i_{kl}/\partial y^j$ and $D^i_{jkl} = \partial G^i_{jk}/\partial y^l$, respectively. These fourth and fifth invariants are curvature tensors. The fourth invariant expresses the rate of change of the torsion in the phase space. The fifth invariant $D^i_{jkl}$ is also called Douglas tensor [9]. By the definition, the fifth invariant vanishes if and only if the $G^i$ is a quadratic in $y^i$. This means that two variables interact each other for $D^i_{jkl} = 0$. In case of $D^i_{jkl} \neq 0$, there are more higher interactions. Therefore, the fifth invariant expresses a measure of interactions in the system.

## 3. A relation between nonlinear dynamical systems and the KCC-theory

As mentioned in the introduction, nonlinear dynamical systems generally take the form (1). In this section, it is shown that the KCC-theory can be applied to the various nonlinear dynamical systems.

In ecology, the Lotka-Volterra system is a simplified model of predator and prey population dynamics. The two species Lotka-Volterra system is expressed by the non-tensor form [20, 25]:

$$\frac{dN^1}{dt} = N^1(\tilde{a} - \tilde{b}N^2), \tag{12}$$

$$\frac{dN^2}{dt} = -N^2(\tilde{c} - \tilde{d}N^1), \tag{13}$$

where the $\tilde{a}, \tilde{b}, \tilde{c}$ and $\tilde{d}$ are constant parameters. The $N^i$ is a population species given by the production variable $x^i$, i.e. $N^i = dx^i/dt$. Therefore, this system can be regarded as the system of second order differential equations (1). Thus, the ecological system can be studied by the KCC-theory, and the geometrical properties have been obtained [2, 3, 4].

In the Earth sciences, the above equation (1) can describe some nonlinear geodynamical systems, the Lorenz model in meteorology and the Rikitake system in geodynamo theory.

The Rikitake system is a unified electro-mechanical system which describes the chaotic behavior of magnetic field (Figure 1). This system consists of two rotating conductive discs coupled with electrically each other. The equations of motion are derived from the Faraday's law [26] and written by [22]:

$$\begin{aligned} L^1_1\frac{dI^1}{dt} + R^1_1 I^1 &= M^1_{23} I^2 \omega^1, \quad & L^2_2\frac{dI^2}{dt} + R^2_2 I^2 &= M^2_{14} I^1 \omega^2, \\ J^3_3\frac{d\omega^1}{dt} &= F^3 - M^3_{12} I^1 I^2, \quad & J^4_4\frac{d\omega^2}{dt} &= F^4 - M^4_{12} I^1 I^2, \end{aligned} \tag{14}$$

where the variables, $I^i$ and $\omega^i$, represent the current and the angular velocity with the subscripts corresponding to the system number I or II, respectively. The coefficients ($L^1_1$, $L^2_2$), ($R^1_1$, $R^2_2$), ($J^3_3$, $J^4_4$) and ($F^3$, $F^4$) are the self-inductances, resistances, moments of inertia and driving couples, respectively. The indices of the coefficients 1 and 2 express the electrical part of system I and II, respectively. The indices of the coefficients 3 and 4 express the mechanical part of system I

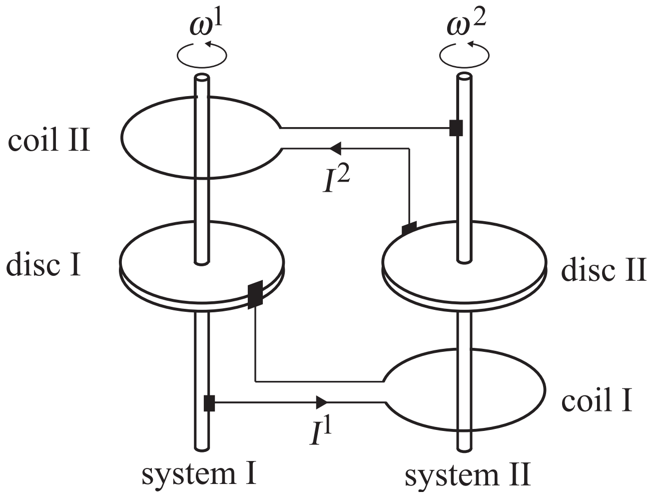


FIGURE 1. The Rikitake system (modified from [22])

and II, respectively. $M^i_{jk}$ is the mutual-inductance. For example, $M^1_{23}$ is the interaction between the current $I^2$ and the angular velocity $\omega^1$. These coefficients are all positive constants. Moreover, the current and the angular velocity are expressed by the differentials with respect to the electric charge and the angle of rotating discs, respectively. Thus, the Rikitake system can be regarded as the second order system (1) and has been geometrized based on the KCC-theory [26].

The Lorenz model simulates thermal convections in the atmosphere. The Lorenz equations are expressed by the following equations [19]:

$$\frac{dX}{dt} = \sigma(Y - X), \tag{15}$$

$$\frac{dY}{dt} = -XZ + \rho X - Y, \tag{16}$$

$$\frac{dZ}{dt} = -\lambda Z + XY, \tag{17}$$

where $X$ is a speed of motion of the air, $Y$ and $Z$ denote the temperature differences. The $\sigma$, $\rho$ and $\lambda$ are constant parameters. These equations can be rewritten in the form of one second order differential equation (1), when one variable is excluded. For example, after excluding the variable $Z$, the above equations are equivalent to the geodesic equation, $Y'' + F(X, Y, Y') = 0$, in Finsler space [10], where $Y' = dY/dX$. Therefore, the Lorenz model can be discussed in terms of the KCC-theory in Finsler space.

The nonlinear dynamical phenomena can be also found in chemistry. The Belousov-Zhabotinskii reaction is a nonlinear chemical system in a nonequilibrium state [27]. The equations of motion are expressed by:

$$\frac{d\mathcal{X}}{dt} = r(\mathcal{Y} - \mathcal{X}\mathcal{Y} + \mathcal{X} - q\mathcal{X}^2), \tag{18}$$

$$\frac{d\mathcal{Y}}{dt} = r^{-1}(-\mathcal{Y} - \mathcal{X}\mathcal{Y} + p\mathcal{Z}), \tag{19}$$

$$\frac{d\mathcal{Z}}{dt} = w(\mathcal{X} - \mathcal{Z}), \tag{20}$$

where the $p$, $q$, $r$ and $w$ are constant parameters. The $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ represent concentrations of the reactants. As well as the Lorenz model, after eliminating one of the variables $\mathcal{Z}$, these equations can be written in the form of the second order differential equation, $\mathcal{Y}'' + H(\mathcal{X}, \mathcal{Y}, \mathcal{Y}') = 0$. Moreover, motivated by the search for chemical chaos, it has been proposed the Rössler equation, which contains just one nonlinear term of second order [23, 24]. Since this equation is a simplification of the equation of the Belousov-Zhabotinskii reaction, the Rössler equation is also equivalent to the second order differential equation in Finsler space [6]. Hence, the nonlinear chemical systems are regarded as the second order system (1) and can be geometrically studied by the KCC-theory.

The equation (1) expresses nonlinear phenomena in physics, chemistry, earth science and ecology. Therefore, the nonlinear dynamical systems can connect with the geometrical concept via the KCC-theory and Finsler geometry. Then, the five KCC-invariants express the chaotic behavior of the various nonlinear dynamical systems (1).

## 4. Application of the KCC-theory to the Rikitake system

In this section, the KCC-theory applies to the Rikitake system, and the geometrical invariants are obtained.

The equations of motion of the Rikitake system are given by (14). In order to study geometrically, let us consider 4-dimensional manifold $M$ with a coordinate $(x^i) = (q^1, q^2, \theta^1, \theta^2)$, where $(x^1,\, x^2)$ and $(x^3,\, x^4)$ are the electric charge $(q^1, q^2)$ and the angle of rotating discs $(\theta^1, \theta^2)$, respectively. Then let $(x^i, y^i)$ denote natural coordinates in a local chart of the tangent bundle $TM$, where $(y^1, y^2, y^3, y^4) = (I^1, I^2, \omega^1, \omega^2)$. In the following, the Latin indices $i$, $j$, $k$, ... run 1 to 4.

At first, the equations of motion (14) and the nonlinear connection $N^i_j$ are expressed by

$$y^i = \frac{dx^i}{dt}, \quad \frac{dy^i}{dt} = -G^i_{jk} y^j y^k + \gamma^i_j y^j + f^i, \tag{21}$$

$$N^i_j = G^i_{kj} y^k - \frac{1}{2}\gamma^i_j. \tag{22}$$

Here, the coefficients are

$$\begin{cases} G^1_{23} &= G^1_{32} = -\dfrac{M^1_{23}}{2L^1_1},\ G^2_{14} = G^2_{41} = -\dfrac{M^2_{14}}{2L^2_2}, \\ G^3_{12} &= G^3_{21} = \dfrac{M^3_{12}}{2J^3_3},\ G^4_{12} = G^4_{21} = \dfrac{M^4_{12}}{2J^4_4}, \end{cases} \tag{23}$$

$$\gamma^1_1 = -2N^1_1 = -\frac{R^1_1}{L^1_1},\ \gamma^2_2 = -2N^2_2 = -\frac{R^2_2}{L^2_2},\ \gamma^3_3 = \gamma^4_4 = 0, \tag{24}$$

$$f^1 = f^2 = 0,\ f^3 = \frac{F^3}{J^3_3},\ f^4 = \frac{F^4}{J^4_4}. \tag{25}$$

The constant Berwald connection (23) plays an important role of interactions between the electrical system and the mechanical system. For example, the connection $G^3_{12} \propto M^3_{12}$ implies the interaction between the currents $I^1$ and $I^2$.

Next, let us consider the geometric invariants of the Rikitake system. The first invariant of the Rikitake system is given by (5)

$$\epsilon^1 = \frac{R^1_1}{2L^1_1} I^1,\ \epsilon^2 = \frac{R^2_2}{2L^2_2} I^2,\ \epsilon^3 = -\frac{F^3}{J^3_3},\ \epsilon^4 = -\frac{F^4}{J^4_4}. \tag{26}$$

The components of the first invariant consist of the voltages $R^1_1 I^1$, $R^2_2 I^2$ and the driving couples $F^3$, $F^4$. Therefore, the $\epsilon^i$ represents an external force.

The Douglas tensor $D^i_{jkl}$ as the fifth invariant vanishes because the equations of motion (14) are quadratic with the constant Berwald connection $G^i_{jk}$. This shows that the fourth invariant $B^i_{jkl}$ is reduced to $B^i_{jkl} = 2G^m_{j[k}G^i_{l]m}$, where $G^m_{j[k}G^i_{l]m} = (G^m_{jk}G^i_{lm} - G^m_{jl}G^i_{km})/2$.

The third invariant as the torsion tensor (10) is expressed by the nonlinear connection and the Berwald connection:

$$B^i_{kj} = G^i_{mj} N^m_k - G^i_{mk} N^m_j. \tag{27}$$

The nonlinear connection $N^i_j$ can be regarded as the projection from the $(y^i)$-field to the $(y^j)$-field. This shows that the torsion tensor $B^i_{jk}$ is characterized by the interactions between electrical system and mechanical system.

Finally, using the third invariant, the second invariant as the deviation curvature tensor is can be derived from the coefficient of the hunting equation (8):

$$\begin{aligned} \frac{D^2\xi^i}{dt^2} + \Bigg\{ & 2G^m_{j[k}G^i_{l]m} y^j y^k + \left( -\gamma^m_j G^i_{ml} + \frac{1}{2}\gamma^m_l G^i_{mj} + \frac{1}{2}\gamma^i_m G^m_{lj} \right) y^j \\ & -G^i_{lh} f^h - \frac{1}{4}\gamma^i_m \gamma^m_l \Bigg\} \xi^l = 0. \end{aligned} \tag{28}$$

It is known that the system is Jacobi stable if and only if the real parts of the eigenvalues of the second invariant $B^i_j$ are strictly positive everywhere, and Jacobi unstable otherwise [4]. In particular, eigenvalues of the second invariant at the origin $(x_0^i, y_0^i) = (0,0)$ are $\mu_1 = \mu_2 = 0$, $\mu_3 = (-R^2J+2FML)/4L^2J$ and $\mu_4 = -(R^2J + 2FML)/4L^2J$, where for the simplification, we put $M^i_{jk} \equiv M$, $L^i_j \equiv L$, $J^i_j \equiv J$, $R^i_j \equiv R$ and $F^i \equiv F$. Also, these coefficients are all positive. Therefore, the Rikitake system is Jacobi unstable at the origin for $\mu_4 < 0$. Hence, the stability of whole trajectories can be determined by this second invariant.

Generally, for the Berwald connection, the two curvature tensors $B^i_{jkl}$, $D^i_{jkl}$ and one torsion tensor $B^i_{jk}$ survive [3]. In the Rikitake system, the Douglas tensor $D^i_{jkl}$ disappears. Hence, the curvature $B^i_{jkl}$ and the torsion $B^i_{jk}$ can represent the behavior of the Rikitake system.

## 5. Discussion

The Rikitake system can be regarded as a unified system which is splitted into electrical and mechanical systems. On the other hand, Ikeda [13, 14] has discussed the unified field theory of electromagnetism and gravitation based on Lagrange geometry [21]. In this Ikeda's theory, the nonlinear connection $N^i_j$ is interpreted as an electromagnetic potential and the torsion tensor $B^i_{jk}$ means an electromagnetic field tensor. Therefore, the electrical system and the mechanical system in the Rikitake system correspond to the electromagnetic field and the gravitational field in the Ikeda's theory, respectively. Hence, the unified structures in the Rikitake system can be studied in the same framework of the Ikeda's geometrical unified theory [12] in the Lagrange space [21]. In the following, the aperiodic behavior of the Rikitake system is discussed based on the nonlinear connection and the torsion tensor.

Let us consider a projected trajectory in the $(I^1, I^2)$-plane. The trajectory is not closed because the Rikitake system behaves chaotically. In this case, the discrepancy $\Delta I^1$ along the trajectory $C$ is given by the nonlinear connection

$$\Delta I^1 = \oint_C dy^1 = -\oint_C N^1_\alpha dx^\alpha \neq 0, \tag{29}$$

where we use the following relation given by the equation (21):

$$\begin{aligned} dy^1 &= -N^1_\alpha dx^\alpha \\ &= -(N^1_j dx^j + N^1_0 dt). \end{aligned} \tag{30}$$

Here, the Greek indices run from 0 to 4 and we put $x^0 \equiv t$, $y^0 \equiv 1$ and $N^1_0 \equiv \epsilon^1$. Moreover, using the Stokes' theorem and the equation (30), the integral of the nonlinear connection is

$$\begin{aligned} -\oint_C N^1_\alpha dx^\alpha &= -\int_S dN^1_\alpha \wedge dx^\alpha \\ &= \frac{1}{2}\int_S B^1_{\alpha\beta} dx^\alpha \wedge dx^\beta, \end{aligned} \tag{31}$$

where $S$ is a region inside the trajectory $C$ and the $\wedge$ is an exterior product. Thus, the torsion tensor expresses the discrepancy $\Delta I^1$ given by the projection from the mechanical field to the electrical field:

$$\Delta I^1 = -\oint_C N^1_\alpha dx^\alpha = \frac{1}{2}\int_S B^1_{\alpha\beta} dx^\alpha \wedge dx^\beta \neq 0. \tag{32}$$

Similarly, a mechanical discrepancy $\Delta\omega^i$ can be obtained by a projection from the electrical field to the mechanical field. The above relation (32) shows that $N^1_\alpha$ and $B^1_{\alpha\beta}$ mean the potential and the field strength in the Rikitake system, respectively. With this correspondence, the discrepancy implies an electromagnetic phase difference around a trajectory in the Aharonov-Bohm effect [14]. Therefore, the aperiodic behavior due to the phase difference $\Delta I^1$ can be expressed by the torsion tensor $B^1_{\alpha\beta}$ as the interaction between the electrical system and the mechanical system. Moreover, topologically, the aperiodic behavior corresponds to a turbulent motion by the Chern-Simons number which expresses the interaction between an electromagnetic field and a hydrodynamic motion [26]. Hence, the discrepancy $\Delta I^1$ as the phase difference can be represented by the topological invariant (the Chern-Simons number).

## 6. Conclusion

Geometrically, the nonlinear dynamical systems in various fields can be uniquely expressed by the KCC-theory. Then, the geometric invariants for the nonlinear dynamical systems are obtained. As an example, the KCC-theory is applied to a geodynamo model (the Rikitake system). The first invariant of the Rikitake system means the external force. The second invariant as the curvature tensor determines the stability of the Rikitake system. The third invariant (the torsion tensor) implies the discrepancy given by a projection from the unified electromechanical field to the subspace as the mechanical field or the electrical field. Because this discrepancy relates to the topological invariant as the Chern-Simons number, the aperiodic behavior can be expressed by the topological invariant. The fourth invariant consists of the constant Berwald connection. The fifth invariant vanishes because the equations of motion of the Rikitake system are quadratic. As well as the Rikitake system, the intrinsic properties of other nonlinear dynamical systems can be represented by the KCC-theory.

## Acknowledgments

The authors would like to thank to the anonymous referee for the helpful and valuable comments on our manuscript.

## References

[1] P. L. Antonelli and I. Bucătaru. New results about the geometric invariants in KCC-theory. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)*, 47(2):405–420, 2001.

[2] P. L. Antonelli and I. Bucataru. Volterra-Hamilton production models with discounting: general theory and worked examples. *Nonlinear Anal. Real World Appl.*, 2(3):337–356, 2001.

[3] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto. *The theory of sprays and Finsler spaces with applications in physics and biology*, volume 58 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[4] P. L. Antonelli, S. F. Rutz, and V. S. Sabău. A transient-state analysis of Tyson's model for the cell division cycle by means of KCC-theory. *Open Syst. Inf. Dyn.*, 9(3):223–238, 2002.

[5] V. I. Arnol′d. *Ordinary differential equations.* Springer Textbook. Springer-Verlag, Berlin, 1992. Translated from the third Russian edition by Roger Cooke.

[6] L. A. Bordag and V. S. Dryuma. Investigation of dynamical systems using tools of the theory of invariants and projective geometry. *Z. Angew. Math. Phys.*, 48(5):725–743, 1997.

[7] E. Cartan. Observations sur le mémoire précédent. (Extrtait d'une lettre à M. D. D. Kosambi). *Math. Z.*, 37(1):619–622, 1933.

[8] S.-S. Chern. Sur la géométrie d'un système d'équations différentielles du second ordre. *Bull. Sci. Math.*, 63:206–212, 1939.

[9] J. Douglas. The general geometry of paths. *Ann. of Math. (2)*, 29(1-4):143–168, 1927/28.

[10] V. S. Dryuma. Geometrical properties of the multidimensional nonlinear differential equations and the Finsler metrics of phase spaces of dynamical systems. *Teoret. Mat. Fiz.*, 99(2):241–249, 1994.

[11] M. Fujinaka. On Finsler spaces and dynamics with special reference to equations of hunting. In *Proceedings of the Third Japan National Congress for Applied Mechanics, 1953*, pages 433–436, Tokyo, 1954. Science Council of Japan.

[12] S. Ikeda. A geometrical construction of the physical interaction field and its application to the rheological deformation field. *Tensor (N.S.)*, 24:60–68, 1972. Commemoration volumes for Prof. Dr. Akitsugu Kawaguchi's seventieth birthday, Vol. I.

[13] S. Ikeda. Some remarks on the Lagrangian theory of electromagnetism. *Tensor (N.S.)*, 49(2):204–208, 1990.

[14] S. Ikeda. On the theory of gravitational field in Finsler spaces. *Tensor (N.S.)*, 50(3):256–262, 1991.

[15] T. Kawaguchi. On the design variation of the simplified induction machine based on the theory of hyper-film space. *Tensor, N. S.*, 26:277–290, 1972.

[16] K. Kondo and Y. Ishizuka. Recapitulation of the geometrical aspects of Gabriel Kron's non-Riemannian electrodynamics. In *Memoirs of the unifying study of the basic problems in engineering sciences by means of geometry. Vol. I*, pages 185–239. Gakujutsu Bunken Fukyu-Kai, Tokyo, 1955.

[17] D. D. Kosambi. Parallelism and path-spaces. *Math. Z.*, 37(1):608–618, 1933.

[18] G. Kron. Non-Riemannian dynamics of rotating electrical machinery. *J. Math. Phys., Mass. Inst. Techn.*, 13:103–194, 1934.

[19] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci.*, 20(2):130–141, 1963.

[20] A. J. Lotka. *Elements of physical biology.* Williams and Wilkins Company, Baltimore, 1925.

[21] R. Miron and M. Anastasiei. *The geometry of Lagrange spaces: theory and applications*, volume 59 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 1994.

[22] T. Rikitake. Oscillations of a system of disk dynamos. *Proc. Cambridge Philos. Soc.*, 54:89–105, 1958.

[23] O. E. Rössler. An equation for continuous chaos. *Phys. Lett.*, 57A(5):397–398, 1976.

[24] O. E. Rössler and K. Wegmann. Chaos in the Zhabotinskii reaction. *Natrure*, 271:89–90, 1978.

[25] V. Volterra. Variations and fluctuations of the number of individuals in animal species living together. *J. Cons. Int. Explor. Mer.*, 3(1):3–51, 1928.

[26] T. Yajima and H. Nagahama. KCC-theory and geometry of the Rikitake system. *J. Phys. A*, 40(11):2755–2772, 2007.
[27] A. M. Zhabotinskii. Periodic oxidizing reactions in the liquid phase (in russian). *Dokl. Akad. Nauk. SSSR*, 157(2):392–395, 1964.

Department of Geoenvironmental Sciences,
Graduate School of Science,
Tohoku University, 6-3 Aoba-ku, Sendai 980-8578, Japan
*E-mail address*: yajima@dges.tohoku.ac.jp