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GEODESICS ON NON–COMPLETE FINSLER MANIFOLDS

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Abstract. In this note based on paper [3] we deal with domains D (i.e. connected open subsets) of a Finsler manifold (M, F). At first we carry out a comparison between different notions of convexity for their boundaries. Then a careful application of variational methods to the geodesic problem yields that the convexity of ∂D is equivalent to the existence of a minimal geodesic for each pair of points of D. Furthermore multiplicity of connecting geodesics can be obtained if D is not contractible.

1. Introduction

Let M be a smooth connected finite dimensional manifold and g_R a Riemannian metric on it. Let us denote by ℓ_R and d_R respectively the length functional and the distance associated to g_R; then a geodesic γ joining two given points $p, q \in M$ is a *minimal* one if $d_R(p, q) = \ell_R(\gamma)$. By the Hopf–Rinow Theorem geodesic and metric completeness are equivalent notions, thus we can simply refer to *complete* Riemannian manifolds. Moreover, this classical result asserts that g_R-convexity (which means that any couple of points can be joined by a minimal geodesic) is ensured by g_R-completeness. We briefly recall how such result can be proved by using critical point theory and, what is more, how a simple topological assumption on M implies a multiplicity result for connecting geodesics.

Let us recall that usually, when one seeks for the critical points of a functional on non–compact/ infinite dimensional manifolds, the *Palais–Smale condition* is needed; this means that, given a Riemannian manifold $\mathcal{M}$ and a C^1 functional f on it, every sequence $(x_m)_m$ such that $\big(f(x_m)\big)_m$ is bounded and $\mathrm{d}f(x_m) \to 0$, as $m \to +\infty$, admits a converging subsequence. With the Palais–Smale condition in

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hands one can get existence (cf. [27, Theorem 2.7]) and sometimes multiplicity of critical points of a bounded from below functional, for instance using the Lusternik–Schnirelman theory (cf. [29], [26] and [1, Chapter 2]). We recall that, given a topological space X, the Lusternik–Schnirelman category of $A \subset X$, denoted by $\operatorname{cat}_X A$, is defined as the minimum number of closed contractible subsets of X needed to cover A. By definition $\operatorname{cat}_X A = +\infty$ if the covering cannot be realized by a finite number of subsets.

Theorem 1.1. *Let $\mathcal{M}$ be a Riemannian manifold and $f \in C^1(\mathcal{M}, \mathbb{R})$. Assume that $\mathcal{M}$ is complete or that the sublevels of f, i.e. the subsets*

$$f^c = \{x \in \mathcal{M} \mid f(x) \le c\}$$

with $c \in \mathbb{R}$, are complete.

1. *If f is bounded from below and it satisfies the Palais–Smale condition, then f attains its infimum.*
2. *Furthermore, set for any $m \in \mathbb{N}$*

$$c_m = \inf_{A \in \Gamma_m} \sup_{x \in A} f(x), \quad \textit{with } \Gamma_m = \{A \subset \mathcal{M} \mid \operatorname{cat}_{\mathcal{M}} A \ge m\},$$

if Γ_m is not empty and $c_m \in \mathbb{R}$, then c_m is a critical value of f.

From a variational viewpoint it is well–known that fixed two points $p, q \in (M, g_R)$, a curve γ, say parametrized in the interval $[0,1]$, is a geodesic joining them if and only if it is a (smooth) critical curve on the infinite dimensional manifold (cf. [21] for details)

$$\Omega(p, q; M) = \{y \in H^1([0,1], M) \mid y(0) = p, y(1) = q\}$$

of the C^2 energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 g_R(\gamma)[\dot\gamma, \dot\gamma] \, \mathrm{d}s.$$

If (M, g_R) is complete, then $\Omega(p, q; M)$ is complete as well and, as a consequence of Theorem 1.1 (cf. also [24, Lemma 12.1] and [7, Lemma 2.1]), the following theorem can be stated.

Theorem 1.2. *Let (M, g_R) be a complete Riemannian manifold. Then*

1. *M is convex;*
2. *furthermore, if M is not contractible, each couple $p, q \in M$ can be joined by infinitely many geodesics with diverging lengths.*

The second part of the theorem is consequence of the fact that the category of the based loop space $\Omega(p, q; M)$ is $+\infty$ when M is not contractible, as proved in [16].

The infinite dimensional setting for the energy functional and the variational theory for geodesics was extended by F. Mercuri [23] to Finsler manifolds. Let us now recall a few basic notions in Finsler Geometry, following [2]. A Finsler structure on a smooth finite dimensional manifold M is a function $F\colon TM \to [0,+\infty)$ continuous on TM, C^∞ on $TM\setminus 0$, vanishing only on the zero section, fiberwise positively homogeneous of degree one, i.e. $F(x,\lambda y)=\lambda F(x,y)$ for all $x\in M$, $y\in T_xM$ and $\lambda>0$, and having fiberwise strictly convex square i.e. the matrix

$$g(x,y)=\left[\frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i\partial y^j}(x,y)\right]$$

is positively defined for any $(x,y)\in TM\setminus 0$. The length of a piecewise smooth curve $\gamma\colon[0,1]\to M$ with respect to the Finsler structure F is defined by

$$\ell_F(\gamma)=\int_0^1 F(\gamma,\dot\gamma)\,\mathrm{d}s$$

hence the distance between two arbitrary points $p,q\in M$ is given by

$$\mathrm{d}_F(p,q)=\inf_{\gamma\in\mathcal{P}(p,q;M)}\ell_F(\gamma),$$

being $\mathcal{P}(p,q;M)$ the set of all continuous piecewise smooth curves $\gamma\colon[0,1]\to M$ s.t. $\gamma(0)=p$ and $\gamma(1)=q$. The distance function is non–negative and satisfies the triangle inequality, but it is not symmetric since F is only positively homogeneous of degree one in y. As a consequence, for each $p\in M$ and $r>0$, two different balls centered at p of radius r can be defined: the *forward ball* $B^+(p,r)=\{q\in M\mid \mathrm{d}_F(p,q)<r\}$ and the *backward* one $B^-(p,r)=\{q\in M\mid \mathrm{d}_F(q,p)<r\}$. Analogously, it makes sense to give two different notions of Cauchy sequences and then of completeness. Indeed a sequence $(x_n)_n\subset M$ is a *forward* (resp. *backward*) *Cauchy sequence* if for all $\varepsilon>0$ an index $\nu\in\mathbb{N}$ exists such that for all $m\ge n\ge\nu$ it is $\mathrm{d}_F(x_n,x_m)<\varepsilon$ (resp. $\mathrm{d}_F(x_m,x_n)<\varepsilon$). A Finsler manifold is so *forward complete* (resp. *backward complete*) if every forward (resp. backward) Cauchy sequence converges.

The Hopf–Rinow Theorem has a backward and a forward version in the Finsler case (cf. [2, Theorem 6.6.1]). In particular $F-$completeness (backward or forward) of M implies $F-$convexity of M.

Geodesics can be defined in more than one way using different connections defined on the bundle π^*TM, $\pi\colon TM\to M$, (cf. [2, Chapter 2]) or as critical points of the length functional (cf. [2, Proposition 5.1.1]). As in Riemannian Geometry constant speed geodesics $\gamma\colon[0,1]\to(M,F)$ on a Finsler manifold satisfy a variational principle: indeed, fixed $p,q\in M$, they are the critical points of the energy functional

$$E(\gamma)=\frac{1}{2}\int_0^1 F^2(\gamma,\dot\gamma)\,\mathrm{d}s$$

on $\Omega(p,q;M)$, cf. e.g. [13, Proposition 2.3]. The lack of regularity of a Finsler metric on the zero section makes that the energy functional is only C^1 with locally Lipschitz differential.

Our references start with the already quoted paper by Mercuri, where closed geodesics on compact manifolds are studied. The main difficulty in applying Theorem 1.1 relies in the proof of the Palais–Smale condition. Later on L. Kozma, A. Kristály, C. Varga obtained in [22] existence and multiplicity results for geodesics joining two submanifolds of (M, F) when F is a reversible Finsler metric controlled from below by a complete Riemannian one. Recently E. Caponio, M. A. Javaloyes and A. Masiello proved in [13] analogous existence and multiplicity results in the general case of a non–reversible backward or forward complete Finsler metric, removing both the assumption of reversibility and the control of F. The key point in the proof of the Palais–Smale condition is that, by the completeness of the metric, it can be proved that the supports of any Palais–Smale sequence are contained in a compact subset. Nevertheless, in order to have this it is enough to work under the Heine–Borel property for (M, d_s), where d_s denotes the symmetric distance:

$$\mathrm{d}_s(p,q) := \frac{1}{2}\left(\mathrm{d}_F(p,q) + \mathrm{d}_F(q,p)\right),$$

i.e. for all $x \in M, r > 0$ the closed balls $\overline{B}_s(x,r)$ are compact (or equivalently the subsets $\overline{B}^+(x,r_1) \cap \overline{B}^-(y,r_2)$ are compact for any $x, y \in M, r_1, r_2 > 0$). Hence in [14] convexity and multiplicity results are proved for geodesics on any Finsler manifold satisfying this property. It is worth to stress that the Hopf–Rinow Theorem in general does not hold for the metric d_s since it is not a length metric.

Classical results in critical point theory aren't directly appliable without the completeness assumption, which at any case is not necessary for geodesic connectedness; but, roughly speaking, its lack imposes the use of suitable convexity assumptions on the boundary of an open subset, which, as we see below, characterize the convexity of the subset itself. In what follows firstly we recall how variational methods successfully apply to the geodesic problem on non–complete Riemannian manifolds under appropriate (almost equivalent) definitions of convexity, remainding to papers [5], [28] for details; then we study an analogous problem in the Finsler setting, stressing the differences and the novelties in the proof. Apart from the analytic and geometric interest, it is worth to notice that convexity of Finsler domains is also stimulating in relation with the existence of lightlike geodesics and timelike with fixed enery ones joining a point with a line in an open region of a standard stationary spacetime, cf. [13, Section 4] and Section 4.

From now on we consider domains D of M, that is connected open subsets of M. W. B. Gordon stressed in [20] that looking for criteria about the g_R–connectivity of D is important also because, via Jacobi metrics, reparametrizations of geodesics are trajectories of fixed energy for a Lagrangian system. In the joint paper [5] with A. Germinario and M. Sánchez we established in the Riemannian context connectedness and sometimes also convexity under very general conditions, including the case of non–smooth boundaries, hence it is quite natural to wonder whether analogous results keep holding in the Finsler setting or not. Partial answers to these questions are contained in the paper [3] with E. Caponio, A. Germinario and M. Sánchez. More precisely, we ask:

1. which relations among different notions of *convexity* hold (see Section 2);
2. if suitable convexity assumptions on (even smooth) ∂D imply F-convexity (and existence of multiple connecting geodesics) (see Section 3).

2. Boundary of domains

Let us consider a C^2 domain D of (M, g_R) or (M, F). We denote by $\tilde{F}$ the reversed metric of F, i.e. $\tilde{F}(x, y) = F(x, -y)$.

The boundary ∂D is

(L) *locally* convex if for each $x \in \partial D$ a neighbourhood $U \subset M$ of x exists such that all geodesics in U starting from x and tangent to ∂D lie in $M \setminus D$.

If M is a Riemannian manifold, this means that each $x \in \partial D$ admits a neighborhood U in M s.t. the exponential map restricted to the tangent space to ∂D has no points in the intersection between U and D; on the other hand, if M is a Finsler manifold, condition above must hold for both the exponential maps associated to the metrics F and $\tilde{F}$, hence

$$\exp_x (T_x \partial D) \cap (U \cap D) = \emptyset$$

and the same holds for $\widetilde{\exp}$, which denotes the exponential map associated to $\tilde{F}$. This definition implies a condition which simply states that all geodesics with endpoints in D are contained in D. Then we say that the boundary ∂D is

(G) *geometrically convex* if for every $p, q \in D$ the range of any geodesic $\gamma\colon [0, 1] \to \overline{D}$ s.t. $\gamma(0) = p$ and $\gamma(1) = q$ satisfies $\gamma([0, 1]) \subset D$.

Still this notion implies another one, which fits the application of variational methods. The boundary ∂D is

(I) *infinitesimally convex* if, called ϕ a differentiable function on M s.t.

$$\begin{cases} \phi^{-1}(0) = \partial D & \\ \phi > 0 & \text{on } D \\ \mathrm{d}\phi(x) \neq 0 & \text{for every } x \in \partial D, \end{cases}$$

then

$$H_\phi(x, y)[y, y] \leq 0 \quad \forall x \in \partial D, y \in T_p \partial D$$

(in the Riemannian case $H_\phi(x)[y, y] \leq 0$.)

Equivalently this condition can be expressed requiring respectively on (M, g_R) and (M, F) that the second fundamental form with respect to the interior normal is positive semidefinite (cf. e.g. [11, p. 198]) and that the normal curvature is non–negative (if defined up to the sign as in cf. [30, Chapter 14]).

Let us also remark that it is equivalent assuming any one of previous notions of convexity for the Finsler metric F or its reversed metric $\tilde{F}$.

2.1. **On Riemannian manifolds.** A celebrated result due to R.L. Bishop in [10] ensures that, at least for C^4 domains of Riemannian manifolds, if (I) holds in a neighbourhood of a point x, then ∂D is (L) at this point. Using different techniques A. Germinario directly proved in [17] that, for C^3 domains of a complete (M, g_R), (I) implies (G), thus in this case we simply speak of convex boundary. Combining the results in [20] and [8] it can be proved that, if M is complete, the convexity of the boundary is, as in the Euclidean case, equivalent to the convexity of the domain itself. Indeed the following theorem holds.

Theorem 2.1. *Let $D \subset (M, g_R)$ be a C^3 domain and assume M g_R-complete. Then*

1. *D is convex $\Leftrightarrow$ ∂D is convex;*
2. *furthermore, if D is not contractible, each couple $p, q \in D$ can be joined by infinitely many geodesics with diverging lengths.*

2.2. **On Finsler manifolds.** We saw that also in the Finsler case the implication (L)$\Rightarrow$(I) keeps being true (cf. e.g. [30]), but, as observed in [12], the converse is known only for Berwald spaces. Indeed we can borrow M.P. do Carmo's words (who proved with F.W. Warner in [15] the statement in the costant curvature case) who pointed out to Bishop that "the general case is not as obvious as it sounds". We also observe that Bishop himself thought that C^4 was a redundant hipotesis and that the problems arising in the Finsler case suggest that it would be better to think over a new proof based on the interplay with variational methods. Here we focus only on the equivalence between (I) and (G) (cf. [3, Corollary 1.2]); we refer to [3, Theorem 1.1] for a definitive answer to the problem of extending Bishop's Theorem to Finsler manifolds, improving also in the Riemannian setting the typical requirements of differentiability.

Theorem 2.2. *Let D be a $C^{2,1}_{\mathrm{loc}}$ domain (i.e. an open subset of M whose boundary is locally defined as a level set of a $C^{2,1}$ function) in a manifold endowed with a Finsler metric whose fundamental tensor is $C^{1,1}_{\mathrm{loc}}$ on $TM \setminus 0$. Then ∂D is (I) $\Leftrightarrow$ ∂D is (G).*

Sketch of the proof. The idea of the proof is the following: arguing by contradiction we can consider a geodesic $\gamma\colon [0,1] \to \overline{D}$ s.t. $F(\gamma(s), \dot{\gamma}(s)) = 1$ on $[0,1]$, $\gamma(0), \gamma(1) \in D$ and assume that the subset $K := \{s \in [0,1] \mid \gamma(s) \in \partial D\}$ is not empty. Then, denoted by $s_M := \max K \in]0,1[$ its maximum, we claim that $\sigma > 0$ exists s.t. $\gamma([s_M, s_M + \sigma]) \subset \partial D$. Indeed, considering the curves of maximal slope of ϕ, γ can be projected on ∂D; called $\gamma_\pi\colon [s_M, s_M + \sigma] \to \partial D$ such projection it results

$$H_\phi(\gamma_\pi(s), \dot{\gamma}_\pi(s))[\dot{\gamma}_\pi(s), \dot{\gamma}_\pi(s)] \leq 0 \text{ on } [s_M, s_M + \sigma].$$

Then let us set $\rho(s) := \phi(\gamma(s))$ and express in local coordinates the Hessian of ϕ; it results on $[s_M, s_M + \sigma]$

$$\ddot{\rho}(s) \le (H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) - (H_\phi)_{ij}(\gamma_\pi(s), \dot{\gamma}_\pi(s))\dot{\gamma}^i_\pi(s)\dot{\gamma}^j_\pi(s) =$$
$$(H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) - (H_\phi)_{ij}(\gamma_\pi(s), \dot{\gamma}_\pi(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) +$$
$$(H_\phi)_{ij}(\gamma_\pi(s), \dot{\gamma}_\pi(s))(\dot{\gamma}^i(s) + \dot{\gamma}^i_\pi(s))(\dot{\gamma}^j(s) - \dot{\gamma}^j_\pi(s)).$$

Moreover we obtain the following bounds:

$$\big[(H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s)) - (H_\phi)_{ij}(\gamma_\pi(s), \dot{\gamma}_\pi(s))\big]\dot{\gamma}^i(s)\dot{\gamma}^j(s)$$
$$\le c_1(|\gamma(s) - \gamma_\pi(s)| + |\dot{\gamma}(s) - \dot{\gamma}_\pi(s)|)$$
$$(H_\phi)_{ij}(\gamma_\pi(s), \dot{\gamma}_\pi(s))(\dot{\gamma}^i(s) + \dot{\gamma}^i_\pi(s))(\dot{\gamma}^j(s) - \dot{\gamma}^j_\pi(s)) \le c_2|\dot{\gamma}(s) - \dot{\gamma}_\pi(s)|$$

where c_1, c_2 are suitable positive constants. Moreover it can be proved that

$$|\gamma(s) - \gamma_\pi(s)| \le c_3\rho(s)$$

and

$$|\dot{\gamma}(s) - \dot{\gamma}_\pi(s)| \le c_4\rho(s) + c_5|\dot{\rho}(s)|,$$

thus we get

$$\ddot{\rho}(s) \le c_6(\rho(s) + |\dot{\rho}(s)|).$$

Then, since $\rho \ge 0$, $\rho(s_M) = \dot{\rho}(s_M) = 0$, we get by [3, Lemma 3.1] $\rho \equiv 0$ on $[s_M, s_M + \sigma]$, getting a contradiction. □

3. The penalization method

In this section we state our result about convexity. Let us point out that for domains of $\mathbb{R}^N$ endowed with a Finsler metric, cf. e.g. [19], arguments which rely on the structure of vector space of $\mathbb{R}^N$ are used. It is worth to notice that we work under the assumption of compactness of the intersection of the closed symmetrized balls with the closure of D, which replaces in some sense the one involving the completeness of $\overline{D}$, or equivalently of M, in the Riemannian setting. All details about the proof can be found in [3, Theorem 1.3].

Theorem 3.1. *Let D be a $C^{2,1}_{\rm loc}$ domain of a smooth manifold M endowed with a Finsler metric F having $C^{1,1}_{\rm loc}$ fundamental tensor and such that $\overline{B}_s(p, r) \cap \overline{D}$ is compact for all $p \in M, r > 0$. Then*

1. *D is convex $\Leftrightarrow$ ∂D is convex;*
2. *furthermore, if D is not contractible, each couple $p, q \in D$ can be joined by infinitely many geodesics with diverging lengths.*

Sketch of the proof. We perform a proof based on a penalization argument at first introduced in [20]; at any case there are striking differences with respect to the Riemannian case. As we work with open subsets, the energy functional E does not satisfy the Palais–Smale condition, hence we perturbe it by a term depending on the function ϕ which defines the boundary and becomes infinite near to it. Thus let us consider on $\Omega(p,q;D)$ the family of functionals $(E_\varepsilon)_{\varepsilon>0}$ defined as follows:

$$E_\varepsilon(\gamma) = E(\gamma) + \int_0^1 \frac{\varepsilon}{\phi^2(\gamma)}\,\mathrm{d}s.$$

It can be proved ([3, Proposition 4.3]) that these functionals have complete sublevels and, adapting the proof for the complete case presented in [13], that they satisfy the Palais–Smale condition, hence for each $\varepsilon > 0$ we find a minimum γ_ε. Then we easily find a bound for the critical levels just obtained:

$$\exists k > 0 \text{ s.t. } E_\varepsilon(\gamma_\varepsilon) \leq k, \quad \text{for all } \varepsilon > 0$$

and moreover the following inequality holds on $[0,1]$:

$$\frac{1}{2}F^2(\gamma_\varepsilon, \dot\gamma_\varepsilon) \leq k + \frac{\varepsilon}{\phi^2(\gamma_\varepsilon)} \quad \text{for all } \varepsilon \in]0,1],$$

(cf. [3, Remark 4.4]). These critical points γ_ε loose regularity in spite of what happens in the Riemannian setting (compare with [18, Lemma 4.1]) or in the Finsler case without boundary: indeed, due to the presence of the penalization term, we find ([3, Lemma 4.1]) that they are C^1 and, for any $\bar s \in [0,1]$ such that $\dot\gamma_\varepsilon(\bar s) \neq 0$, a neighbourhood $I(\bar s)$ exists where γ_ε is twice differentiable and

$$\ddot\gamma_\varepsilon^i + \Gamma^i_{jk}(\gamma_\varepsilon, \dot\gamma_\varepsilon)\dot\gamma_\varepsilon^j\dot\gamma_\varepsilon^k = -\frac{2\varepsilon}{\phi^3(\gamma_\varepsilon)}\partial_{x_k}\phi(\gamma_\varepsilon)g^{ki}(\gamma_\varepsilon, \dot\gamma_\varepsilon),$$

where the Γ^i_{jk} are the components of the Chern connection and the g^{ij} are the components of the inverse of the fundamental tensor (cf. [2]). At any case we can carry out the limit process; the crucial point in order to finish relies on the estimate of the norms of the multipliers in previous equation: in fact, set

$$\lambda_\varepsilon(s) = \frac{2\varepsilon}{\phi^3(\gamma_\varepsilon(s))} \quad \text{for all } \varepsilon \in]0,1], s \in [0,1],$$

$\varepsilon_0 \in]0,1]$ exists s.t.

$$(\|\lambda_\varepsilon\|_\infty)_{\varepsilon \in]0,\varepsilon_0]} \quad \text{is bounded.}$$

The proof of this bound ([3, Lemma 4.5]) is delicate and based on the fact that the set where each $\dot\gamma_\varepsilon$ vanishes - and hence γ_ε is not twice differentiable - has empty interior (compare with [18, Lemma 4.4]). Then we can take the limit γ of the approximating solutions and establish the weak equation locally satisfied by it around the points where the vector velocity field is not zero; again a difficulty arises since we can not proceed as in previous references on the topic where thanks to the Nash Theorem M was always considered as a closed subset of $\mathbb{R}^N$ (cf. [25]), but we can skip this problem (([3, Proposition 4.6]) by a local representation of the energy functional, following [6]. At last, exploiting the convexity of the boundary we get that the range of the curve is contained in D. Quite standard arguments allow us to obtain convexity and multiplicity results. The converse easily follows by the infinitesimal convexity of ∂D. $\square$

4. Open problems

We already observed that in [5] Riemannian domains with non–smooth boundary were studied. Such study requires suitable generalizations of the notion of convexity and a delicate penalization process. It seems interesting to make an analogous analysis on Finsler manifolds as well, also for possible applications to spacetimes, as we are going to describe. A standard stationary spacetime is a Lorentzian manifold (L, g_L) endowed with a complete timelike Killing vector field Y and admitting a splitting $L = S \times \mathbb{R}$. The metric g_L in such a splitting is given as $g_L = g_R + 2\omega \mathrm{d}t - \beta \mathrm{d}t^2$, where g_R, ω and β are respectively a Riemannian metric, a one–form and a positive function on S; t is the natural coordinate in $\mathbb{R}$ and $\partial_t = Y$. The future–pointing lightlike geodesic flow of $(S \times \mathbb{R}, g_L)$ projects on the geodesic flow of the Randers metric $R = (g_R/\beta + (\omega/\beta)^2)^{1/2} + \omega/\beta$ on S (cf. [13, 14]). Hence the existence of future–pointing lightlike geodesics in a region $D \times \mathbb{R}$ in L connecting a point with an integral line of the field Y can be reduced to the study of the convexity of D as a subset of the Randers space (S, R). Our result in Section 3 allow us to obtain necessary and sufficient conditions for these lightlike connecting geodesics in the case of regions with smooth boundaries, but it is well–known that in many physical interesting examples the boundaries are non–smooth.

We conclude stressing that variational methods successfully apply to a classical problem in differential geometry: the existence of closed geodesics on (M, g_R). V. Benci and F. Giannoni in [9] used a refined Morse Theory to get a non–trivial closed geodesic on a non–compact Riemannian manifold. We refer to [4] for an extension of such result to domains. Up to our knowledge similar results on non–compact Finsler manifold are not known and may be argument of future investigation.

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