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SEMI-INVARIANT PRODUCTS OF A NEARLY SASAKIAN MANIFOLD

WAN YONG AND RUIZHI GUO

ABSTRACT. In this paper, we give some sufficient conditions for semi–invariant product of a nearly Sasakian manifold, and generalize Bejancu's result.

1. INTRODUCTION

Let $\overline{M}$ be a real $(2n+1)$-dimensional almost contact metric manifold with the structure tensors (Φ, ξ, η, g), then

$$\Phi\xi = 0, \eta(\xi) = 1, \Phi^2 = -I + \eta \otimes \xi, \eta(X) = g(X, \xi), \tag{1.1}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta \circ \Phi = 0, \tag{1.2}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.1 ([1])**.** The Nijenhuis tensor field of Φ on an almost contact metric manifold is defined by

$$[\Phi, \Phi](X, Y) = [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y], \tag{1.3}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([1])**.** An almost contact metric manifold $\overline{M}$ is called a nearly Sasakian manifold, if we have

$$(\overline{\nabla}_X \Phi)Y + (\overline{\nabla}_Y \Phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y, \tag{1.4}$$

for any $X, Y \in \Gamma(T\overline{M})$.

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Definition 1.3 ([1])**.** An almost contact metric manifold $\overline{M}$ is called a Sasakian manifold, if we have

$$(\overline{\nabla}_X\Phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{1.5}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Obviously, a Sasakian manifold is a nearly Sasakian manifold.

Let M be an m-dimensional submanifold of an n-dimensional almost contact metric manifold $\overline{M}$. We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M, and denote by $\nabla^{\perp}$ the normal connection on M. Thus, for any $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.6}$$

where $h : \Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM^{\perp})$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.6) is called the Gauss formula and h is called the second fundamental form of M.

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, we denote by $-A_V X$ and $\nabla^{\perp}_X V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V. \tag{1.7}$$

Thus, for any $V \in \Gamma(TM^{\perp})$, we have a linear operator, satisfying

$$g(A_V X, Y) = g(X, A_V Y) = g(h(X,Y), V). \tag{1.8}$$

The equation (1.7) is called the Weingarten formula.

An m-dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an m-dimensional linear subspace D_x of $T_x\overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$ there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of x. From now on, all distribution are supposed to be differentiable of class C^{∞}.

The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X,Y] \in \Gamma(D)$. A submanifold M of $\overline{M}$ is said to be an integral manifold of D if for every point $x \in M$, D_x coincides with the tangent space to M at x. If there exists no integral manifold of D which contains M, then M is called a maximal integral manifold or a leaf of D. The distribution D is said to be integrable if for every $x \in \overline{M}$ there exists an integral manifold of D containing x.

Definition 1.4 ([1])**.** Let M be a real $(2m+1)$-dimensional submanifold of a real $(2n+1)$-dimensional almost contact metric manifold $\overline{M}$ with the structure tensors (Φ, ξ, η, g). We assume that the structure tensor ξ is tangent to M, and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M. Then M is called a semi-invariant submanifold of $\overline{M}$, if there exist two differentiable distributions D and $D^{\perp}$ on M, satisfying

(1) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where D, $D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other;
(2) the distribution D is invariant by Φ, that is, $\Phi(D_x) = D_x$, for each $x \in M$;
(3) the distribution $D^{\perp}$ is anti-invariant by Φ, that is, $\Phi(D_x^{\perp}) \subset T_xM^{\perp}$, for each $x \in M$.

Let M be a semi-invariant submanifold of a nearly Sasakian Manifold $\overline{M}$. Then the normal bundle to M has the orthogonal decomposition

$$TM^{\perp} = JD^{\perp} \oplus \nu. \tag{1.9}$$

For each vector field X tangent to M, we put

$$\Phi X = \psi X + \omega X, \tag{1.10}$$

where ψX and ωX are respectively the tangent part and the normal part of ΦX. Also, for each vector field V normal to M, we put

$$\Phi V = BV + CV, \tag{1.11}$$

where BV and CV are respectively the tangent part and the normal part of ΦV.

We denote by P and Q the projection morphisms of TM on D and $D^{\perp}$ respectively. Then

$$X = PX + QX + \eta(X)\xi, \tag{1.12}$$

for any $X \in \Gamma(TM)$.

Definition 1.5 ([1])**.** Let M be a semi-invariant submanifold of a nearly Sasakian Manifold $\overline{M}$. We say that M is a semi-invariant product if the distribution $D \oplus \{\xi\}$ and the distribution $D^{\perp}$ are integrable and locally M is a Riemannian product $M_1 \times M_2$, where M_1(resp. M_2) is a leaf of $D \oplus \{\xi\}$(resp. $D^{\perp}$)

In the references [1], we know that a distribution D on M is integrable if and only if $[X, Y] \in \Gamma(D)$ for all vector fields $X, Y \in \Gamma(D)$.

2. Main results

Lemma 2.1. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then*

$$\begin{aligned} &\nabla_X \psi Y - A_{\omega Y}X + \nabla_Y \psi X - A_{\omega X}Y = \\ &\quad = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y + \psi\nabla_X Y + \psi\nabla_Y X + 2Bh(X,Y), \end{aligned} \tag{2.1}$$

$$\eta(\nabla_X \psi Y) + \eta(\nabla_Y \psi X) = 2g(\Phi X, \Phi Y) + \eta(A_{\omega Y}X) + \eta(A_{\omega X}Y), \tag{2.2}$$

$$h(X, \psi Y) + \nabla_X^{\perp}\omega Y + h(Y, \psi X) + \nabla_Y^{\perp}\omega X = \omega\nabla_X Y + \omega\nabla_Y X + 2Ch(X,Y), \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Taking $X, Y \in \Gamma(TM)$ and using (1.4), (1.5), (1.7) and (1.8) we obtain

$$(2.4)\quad \begin{aligned}(\overline{\nabla}_X\Phi)Y &= \overline{\nabla}_X\Phi Y - \Phi\overline{\nabla}_X Y\\ &= \nabla_X\psi Y + h(X,\psi Y) - A_{\omega Y}X + \nabla^{\perp}_X\omega Y - \psi\nabla_X Y\\ &- \omega\nabla_X Y - Bh(X,Y) - Ch(X,Y).\end{aligned}$$

Replacing X and Y we get

$$\begin{aligned}(\overline{\nabla}_Y\Phi)X = \nabla_Y\psi X + h(Y,\psi X) - A_{\omega X}Y + \nabla^{\perp}_Y\omega X - \psi\nabla_Y X\\ - \omega\nabla_Y X - Bh(Y,X) - Ch(Y,X).\end{aligned}$$

By using (2.4), (2.5), (1.2) and (1.4) we have

$$(2.5)\quad \begin{aligned}&2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\\ &= \nabla_X\psi Y + h(X,\psi Y) - A_{\omega Y}X + \nabla^{\perp}_X\omega Y -\\ &- \psi\nabla_X Y - \omega\nabla_X Y - Bh(X,Y) - Ch(X,Y) + \nabla_Y\psi X + h(Y,\psi X)\\ &- A_{\omega X}Y + \nabla^{\perp}_Y\omega X - \psi\nabla_Y X - \omega\nabla_Y X - Bh(Y,X) - Ch(Y,X).\end{aligned}$$

By comparing to tangent part and normal part of (2.6) we obtain (2.1), (2.2) and (2.3). □

Lemma 2.2. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then*

$$(2.6)\quad \psi\nabla_X\xi - P\nabla_\xi\Phi X + \psi\nabla_\xi X = X,$$

$$(2.7)\quad 2Bh(X,\xi) = Q\nabla_\xi\Phi X,$$

$$(2.8)\quad \omega\nabla_X\xi + \omega\nabla_\xi X + 2Ch(X,\xi) = h(\xi,\Phi X),$$

for any $X \in \Gamma(D)$.

Proof. By using (1.1) and (1.3) we obtain

$$(2.9)\quad (\overline{\nabla}_X\Phi)Y + (\overline{\nabla}_Y\Phi)X = -X$$

for any $X \in \Gamma(D)$. On the other hand, by using (1.4), (1.6), (1.8), (1.10), (1.11) and (1.12) we have

$$(2.10)\quad \begin{aligned}(\overline{\nabla}_X\Phi)Y + (\overline{\nabla}_Y\Phi)X = -\psi\nabla_X\xi - \omega\nabla_X\xi - 2Bh(X,\xi) - 2Ch(X,\xi)\\ + P\nabla_\xi\Phi X + Q\nabla_\xi\Phi X + h(\xi,\Phi X) - \psi\nabla_\xi X - \omega\nabla_\xi X.\end{aligned}$$

From (2.10) and (2.11) we get

$$(2.11)\quad \begin{aligned}&-X = -\psi\nabla_X\xi - \omega\nabla_X\xi - 2Bh(X,\xi)\\ &- 2Ch(X,\xi) + P\nabla_\xi\Phi X + Q\nabla_\xi\Phi X + h(\xi,\Phi X) - \psi\nabla_\xi X - \omega\nabla_\xi X.\end{aligned}$$

By comparing to tangent part and normal part of (2.12) we obtain (2.7), (2.8) and (2.9). □

Lemma 2.3. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. If*

$$h(X,\xi)\in\Gamma(\nu), \tag{2.12}$$

for any $X\in\Gamma(D)$, then

$$\nabla_\xi X\in\Gamma(D\oplus\{\xi\}), \tag{2.13}$$

$$\nabla_X\xi\in\Gamma(D\oplus\{\xi\}). \tag{2.14}$$

Proof. For any $X\in\Gamma(D)$, $Z\in\Gamma(D^\perp)$, by using (1.1), (1.2), (1.12), (2.8) and (2.13) we obtain

$$\begin{aligned} g(\nabla_\xi\Phi X,Z)&=g(Q\nabla_\xi\Phi X,Z)=2g(Bh(X,\xi),Z)\\ &=2g(\Phi^2h(X,\xi),\Phi Z)=-2g(h(X,\xi),\Phi Z)=0. \end{aligned} \tag{2.15}$$

From (2.16) we get (2.14).

On the other hand, by using (1.1), (1.2), (1.12), (2.9), (2.13) and (2.14) we have

$$g(\nabla_X\xi,Z)=g(\Phi\nabla_X\xi,\Phi Z)=g(\omega\nabla_X\xi,\Phi Z)=0, \tag{2.16}$$

From (2.17) we obtain (2.15). □

Lemma 2.4 (see [6])**.** *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$g(A_{\Phi U}V-A_{\Phi V}U-2\Phi\nabla_UV,\Phi Z)=\eta([U,V])\eta(Z), \tag{2.17}$$

for any $U,V\in\Gamma(D^\perp)$ and $Z\in\Gamma(D\oplus\{\xi\})$.

Theorem 2.1. *Let M be a semi-invariant submanifold a nearly Sasakian manifold $\overline{M}$, and let condition (2.13) and condition (2.18) be satisfied. If*

$$\nabla_YX\in\Gamma(D\oplus\{\xi\}) \tag{2.18}$$

for any $Y\in\Gamma(TM)$ and $X\in\Gamma(D)$, then M is a semi-invariant product of nearly Sasakian manifold $\overline{M}$.

Proof. For any $X,Y\in\Gamma(D)$, from (2.13), (2.14), (2.15) and (2.19) we have

$$[X,Y]=\nabla_XY-\nabla_YX\in\Gamma(D\oplus\{\xi\}) \tag{2.19}$$

$$[X,\xi]=\nabla_X\xi-\nabla_\xi X\in\Gamma(D\oplus\{\xi\}) \tag{2.20}$$

From (2.20) and (2.21) we obtain

$$[X,Y]\in\Gamma(D\oplus\{\xi\}) \tag{2.21}$$

for any $X\in\Gamma(D)$ and $Y\in\Gamma(D\oplus\xi)$.

On the other hand, thinking about $[\xi,\xi]=0$ and (2.21), we have

$$[Y,\xi]\in\Gamma(D\oplus\{\xi\}) \tag{2.22}$$

for any $Y\in\Gamma(D\oplus\xi)$.

From (2.22) and (2.23) we get that $[X,Y] \in \Gamma(D \oplus \{\xi\})$, for any $X, Y \in \Gamma(D \oplus \xi)$. Hence the distribution $D \oplus \{\xi\}$ is integrable. Moreover, if M_1 is a leaf of $(D \oplus \{\xi\})$, then from (2.19) and the Gauss formula for the immersion of M_1 in M it follows that M_1 is totally geodesic in M. For any $Y \in \Gamma(TM)$, $X \in \Gamma(D)$ and $U \in \Gamma(D^{\perp})$, from (2.19) we have

$$g(\nabla_Y U, X) = -g(U, \nabla_Y X) = 0. \tag{2.23}$$

From (2.24) it follows that $\nabla_Y U \in \Gamma(D^{\perp})$ for any $Y \in \Gamma(TM)$ and $U \in \Gamma(D^{\perp})$. By using the equation of Gauss for a leaf M_2 of $D^{\perp}$ we obtain that M_2 is totally geodesic in M. The proof is complete. □

In the references [1], if M is a semi-invariant submanifold of a Sasakian manifold $\overline{M}$,then the distribution $D^{\perp}$ is integrable and the condition (2.13) is automatically satisfied (p.102). So we have

Corollary 2.1 ([1])**.** *Let M be a semi-invariant submanifold a Sasakian manifold $\overline{M}$. If the condition (2.19) holds, then M is a semi-invariant product of Sasakian manifold $\overline{M}$.*

Theorem 2.2. *Let M be a semi-invariant submanifold a nearly Sasakian manifold $\overline{M}$, and let condition (2.13) and condition (2.18) be satisfied. If*

$$2Bh(X,Y) = \nabla_X \psi Y - A_{\omega Y} X \tag{2.24}$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$, then M is a semi-invariant product of nearly Sasakian manifold $\overline{M}$.

Proof. From (2.1) we have

$$\begin{aligned} \nabla_Y \Phi X = 2g(X,Y)\xi - \eta(Y)X + \psi\nabla_X Y \\ + \psi\nabla_Y X + 2Bh(X,Y) - \nabla_X \psi Y + A_{\omega Y} X, \end{aligned} \tag{2.25}$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$. Clearly, (2.25) and (2.26) imply that (2.19) holds. Thus, the proof is complete. □

Theorem 2.3. *Let M be a semi-invariant submanifold a nearly Sasakian manifold $\overline{M}$, and let condition (2.13) and condition (2.18) be satisfied. If*

$$h(X, \psi Y) + \nabla_X^{\perp} \omega Y + h(Y, \Phi X) = \omega \nabla_X Y + 2Ch(X,Y), \tag{2.26}$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$, then M is a semi-invariant product of nearly Sasakian manifold $\overline{M}$.

Proof. From (2.3) we have

$$\omega\nabla_Y X = h(X, \psi Y) + \nabla_X^{\perp} \omega Y + h(Y, \Phi X) - \omega \nabla_X Y - 2Ch(X,Y), \tag{2.27}$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$. Clearly, (2.27) and (2.28) imply that (2.19) holds. Thus, the proof is complete. □

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Wan Yong,
Department of Mathematics and Computing Science,
Changsha University of Science and Technology,
Changsha, Hunan,
P. R. China.
E-mail address: wany@csust.edu.cn

RUIZHI GUO,
College of Mathematics and Computer Science,
Hunan Normal University,
Changsha, Hunan,
P. R. China.
E-mail address: Guorz6279@sohu.com