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CONTINUITY FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR ON BESOV SPACES

TAN LU, CHUANGXIA HUANG AND LIU LANZHE

Abstract. In this paper, we prove the the continuity for the multilinear commutator associated to the Littlewood-Paley operator on the Besov spaces.

1. Introduction

As the development of the singular integral operators, their commutators have been well studied (see [1, 2, 3, 14]). From [2, 3, 9, 13], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce the multilinear commutator associated to the Littlewood-Paley operator and prove the continuity properties for the multilinear commutator on the Besov spaces.

2. Preliminaries and Theorem

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that(see [15, 16])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

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For $\beta \geq 0$, the Besov space $\dot{\Lambda}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator(see [13]).

For $b_j \in \dot{\Lambda}_\beta(R^n)$ $(j = 1, \cdots, m)$, set

$$||\vec{b}||_{\dot{\Lambda}_\beta} = \prod_{j=1}^{m} ||b_j||_{\dot{\Lambda}_\beta}.$$

Given some functions b_j $(j = 1, \ldots, m)$ and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\dot{\Lambda}_\beta} = ||b_{\sigma(1)}||_{\dot{\Lambda}_\beta} \cdots ||b_{\sigma(j)}||_{\dot{\Lambda}_\beta}$.

Definition 1. Let $0 < p, q \leq \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,\, p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,\, p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,\, p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,\, p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,\, p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,\, p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 2. Let $1 \leq q < \infty$, $\alpha \in R$. The central Campanato space is defined by (see [17])

$$CL_{\alpha,\, q}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{CL_{\alpha,\, q}} < \infty\},$$

where

$$||f||_{CL_{\alpha,\, q}} = \sup_{r>0} |B(0, r)|^{-\alpha} \left(\frac{1}{|B(0, r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx\right)^{1/q}.$$

Definition 3. Fix $\delta > 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$;
(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1-\delta)}$;
(3) $|\psi(x+y) - \psi(x)| \leq C|y|^{\epsilon}(1+|x|)^{-(n+\epsilon-\delta)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear commutator is defined by

$$S_\delta^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(z))\psi_t(y-z)f(z)dz,$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. We also define that

$$S_\delta(f)(x) = \left(\int\int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [16]).

Let H be the space $H = h : ||h|| = (\int\int_{R_+^{n+1}} |h(y,t)|^2 \frac{dydt}{t^(n+1)})^{1/2} < \infty$, then, for each fixed $x \in R_n$, $F_t^{\vec{b}}(f)(x,y)$ may be viewed as a mapping from $[0,+\infty)$ to H, and it is clear that

$$S_\delta(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(x)||$$

and

$$S_\delta^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x,y)||.$$

Note that when $b_1 = \cdots = b_m$, $S_\delta^{\vec{b}}$ is just the commutator of order m. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 4, 7, 6, 5, 8, 9, 10, 13, 14]). Our main purpose is to study the boundedness properties for the multilinear commutator on Besov spaces.

Now, we state our theorems as following.

Theorem 1. *Let* $0 < \delta < n$, $1 < r < n/\delta$, $1/s = 1/r - \delta/n$, $0 < \beta < 1/2m$ *and* $b_j \in \dot{\Lambda}_\beta(R^n)$ *for* $j = 1, \cdots, m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $\dot{\Lambda}_{(\delta+m\beta)-n/p}(R^n)$ *for any* $n/(\delta+m\beta) \leq p \leq n/\delta$.

Theorem 2. *Let* $0 < \delta < n$, $0 < \beta < 1/2m$, $1 < q_1 < n/(\delta+m\beta)$, $1/q_2 = 1/q_1 - (\delta+m\beta)/n$, $-n/q_2 - 1/2 < \alpha \leq -n/q_2$ *and* $b_j \in \dot{\Lambda}_\beta(R^n)$ *for* $j = 1, \cdots, m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $\dot{K}_{q_1}^{\alpha,\infty}(R^n)$ *to* $CL_{-\alpha/n-1/q_2,q_2}(R^n)$.

Remark 1. Theorem 2 also hold for the nonhomogeneous Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1 (see [13]). *For $0 < \beta < 1, 1 \le p \le \infty$, we have*

$$||b||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx$$
$$\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p}$$
$$\approx \sup_Q \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - c| dx$$
$$\approx \sup_Q \inf_c \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - c|^p dx \right)^{1/p}.$$

Lemma 2 (see [12]). *For $\alpha < 0, 0 < q < \infty$, we have*

$$||f||_{\dot{K}_q^{\alpha,\infty}} \approx \sup_{\mu \in Z} 2^{\mu\alpha} ||f\chi_{B_\mu}||_{L^q}.$$

Lemma 3. *Let $0 < \eta < n$, $1 < p < n/\eta$. Suppose $b \in \dot{\Lambda}_\beta(R^n)$, then*

$$|b_{2^{k+1}B} - b_B| \le C||b||_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n} \text{ for } k \ge 1.$$

Proof.

$$|b_{2^{k+1}B} - b_B| \le \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^jB}|$$
$$\le \sum_{j=0}^{k} \frac{1}{|2^jB|} \int_{2^jB} |b(y) - b_{2^{j+1}B}| dy$$
$$\le C \sum_{j=0}^{k} \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^p dy \right)^{1/p}$$
$$\le C||b||_{\dot{\Lambda}_\beta} \sum_{j=0}^{k} |2^{j+1}B|^{\beta/n} \le C||b||_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n}.$$

□

Lemma 4 (see [16]). *Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_δ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Lemma 5 (see [7]). *Let $0 \le \eta < n$, $1 < r < n/\eta$, $1/r - 1/s = \eta/n$ and $b_j \in \dot{\Lambda}_\beta(R^n)$ for $j = 1, \cdots, m$. Then $S_\delta^{\vec{b}}$ is bounded from $L^r(R^n)$ to $L^s(R^n)$.*

Proof of Theorem 1. It is only to prove that there exists a constant C_0 such that

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |S_\delta^{\vec{b}}(f)(x) - C_0| dx \le C||f||_{L^p}.$$

Fix a cube Q, $Q = Q(x_0, d)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q,\ f_2 = f\chi_{(R^n\setminus Q)}$.

When $m = 1$, for $C_0 = S_\delta(((b_1)_Q - b_1)f_2)(x_0)$, we have

$$S_\delta^{b_1}(f)(x) = (b_1(x) - (b_1)_Q)S_\delta(f)(x) - S_\delta((b_1 - (b_1)_Q)f_1)(x) - S_\delta((b_1 - (b_1)_Q)f_2)(x).$$

Then

$$\begin{aligned}
&|S_\delta^{b_1}(f)(x) - S_\delta(((b_1)_Q - b_1)f_2)(x_0)| = \\
&= \big| ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x,y)|| - ||\chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1)f_2)(y)|| \big| \\
&\le ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x,y) - \chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1)f_2)(y)|| \\
&\le ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q)F_t(f)(y)|| + ||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q)f_1)(y)|| \\
&\quad + ||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q)f_2)(y) - \chi_{\Gamma(x_0)} F_t((b_1 - (b_1)_Q)f_2)(y)|| \\
&= A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, for $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundness of S_δ from $L^p(R^n)$ to $L^q(R^n)$, by Hölder's inequality with exponent$1/q + 1/q' = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |A(x)|dx \le \\
&\le C\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \left(\int_Q |(b_1(x) - (b_1)_Q)|^{q'}dx\right)^{1/q'} \left(\int_Q |S_\delta(f)(x)|^q dx\right)^{1/q} \\
&\le C\frac{|Q|^{\beta/n+1/q'}}{|Q|^{1+(\delta+\beta)/n-1/p}} \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|}\int_Q |(b_1(x) - (b_1)_Q)|^{q'}dx\right)^{1/q'} \left(\int_Q |f(x)|^p dx\right)^{1/p} \\
&\le C\frac{|Q|^{1+(\delta+\beta)/n-1/p}}{|Q|^{1+(\delta+\beta)/n-1/p}} ||b_1||_{\dot{\Lambda}_\beta} ||f||_{L^p} \\
&\le C||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

For $B(x)$, denoting $p = rt$, $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of S_δ from $L^r(R^n)$ to $L^s(R^n)$, by Hölder's inequality with exponent $1/t + 1/t' = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |B(x)|dx \le \\
&\le C\frac{1}{|Q|^{(\delta+\beta)/n-1/p}} \left(\frac{1}{|Q|}\int_{R^n} |S_\delta(b_1(x) - (b_1)_Q)f\chi_Q(x)|^s dx\right)^{1/s} \\
&\le C\frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \left(\int_Q |(b_1(x) - (b_1)_Q)f(x)|^r dx\right)^{1/r} \\
&\le C\frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \left(\int_Q |b_1(x) - (b_1)_Q|^{rt'}dx\right)^{1/rt'} \left(\int_Q |f(x)|^{rt}dx\right)^{1/rt}
\end{aligned}$$

$$\leq C\frac{|Q|^{\beta/n+1/rt'}}{|Q|^{(\delta+\beta)/n-1/p+1/s}}\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_Q|^{rt'}dx\right)^{1/rt'}\left(\int_Q|f(x)|^{rt}dx\right)^{1/rt}$$
$$\leq C\frac{|Q|^{(\delta+\beta)/n-1/p+1/s}}{|Q|^{(\delta+\beta)/n-1/p+1/s}}||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}$$
$$\leq C||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

For $C(x)$, by the Minkowski's inequality, we have

$$C(x)\leq$$
$$\leq\left[\int\int_{R_+^{n+1}}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}||b_1(z)-(b_1)_Q||\psi_t(y-z)||f(z)|\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$
$$\leq C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|$$
$$\times\left|\int\int_{|x-y|\leq t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}-\int\int_{|x_0-y|\leq t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right|^{1/2}dz$$
$$\leq\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\times$$
$$\times\left(\iint\limits_{|y|\leq t,|x+y-z|\leq t}\left|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}}-\frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\right|\frac{dydt}{t^{n-1}}\right)^{1/2}dz$$
$$\leq\int_{Q^c}|b_1(z)-(b_1)_{2Q}||f(z)|\left(\int\int_{|y|\leq t,|x+y-z|\leq t}\frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}}dydt\right)^{1/2}dz,$$

by $2t+|x+y-z|\geq 2t+|x-z|-|y|\geq t+|x-z|$, for $|y|\leq t$ and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+3-2\delta}}=C|x-z|^{-2n-1+2\delta}.$$

Thus, for $x\in Q$, by Hölder's inequality with exponent $1/p+1/p'=1$ and Lemma 1, 3, we have

$$C(x)\leq$$
$$\leq C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\left(\int\int_{|y|\leq t}\frac{2^{2n+3}|x_0-x|t^{1-n}dydt}{(2t+2|x+y-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$
$$\leq C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\int\int_{|y|\leq t}\frac{t^{1-n}dydt}{(2t+|x+y-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$
$$\leq C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\int\int_{|y|\leq t}\frac{t^{1-n}dydt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$

$$\leq C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$

$$\leq C\sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-2\delta)}|b_1(z)-(b_1)_Q||f(z)|dz$$

$$\leq C\sum_{k=0}^\infty 2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(z)|^p dz\right)^{1/p}\times$$

$$\times\left(\int_{2^{k+1}Q}|b_1(z)-(b_1)_{2Q}|^{p'}dz\right)^{1/p'}$$

$$\leq C\sum_{k=0}^\infty 2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(z)|^p dz\right)^{1/p}\times$$

$$\times\left[\int_{2^{k+1}Q}(|b_1(z)-(b_1)_{2^{k+1}Q}|^{p'}+|(b_1)_{2^{k+1}Q}-(b_1)_{2Q}|^{p'})dz\right]^{1/p'}$$

$$\leq C\sum_{k=0}^\infty 2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(z)|^p dz\right)^{1/p}\times$$

$$\times\Bigg[|2^{k+1}Q|^{\beta/n+1/p'}\frac{1}{|2^{k+1}Q|^{\beta/n}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_1(z)-(b_1)_{2^{k+1}Q})|^{p'}dz\right)^{1/p'}$$

$$+|(b_1)_{2^{k+1}Q}-(b_1)_Q||2^{k+1}Q|^{1/p'}\Bigg]$$

$$\leq C\sum_{k=0}^\infty 2^{-k/2}\frac{|2^{k+1}Q|^{\beta/n+1/p'}||b_1||_{\dot\Lambda_\beta}+k|2^{k+1}Q|^{\beta/n+1/p'}||b_1||_{\dot\Lambda_\beta}}{|2^{k+1}Q|^{1-\delta/n}}||f||_{L^p}$$

$$\leq C\sum_{k=0}^\infty k2^{-k/2}|2^{k+1}Q|^{\beta/n+1/p'-1+\delta/n}||b_1||_{\dot\Lambda_\beta}||f||_{L^p}$$

$$\leq C\sum_{k=0}^\infty k2^{k(-1/2+\delta+\beta-n/p)}|Q|^{\beta/n+1/p'-1+\delta/n}||b_1||_{\dot\Lambda_\beta}||f||_{L^p}$$

$$\leq C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot\Lambda_\beta}||f||_{L^p}.$$

Thus,

$$\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q|C(x)|dx\leq C\frac{||b_1||_{\dot\Lambda_\beta}||f||_{L^p}}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q|Q|^{(\delta+\beta)/n-1/p}dx$$
$$\leq C||b_1||_{\dot\Lambda_\beta}||f||_{L^p}.$$

This completes the case $m=1$.

Now, we consider the *Case* $m\geq 2$. we have, for $b=(b_1,\cdots,b_m)$,

$$F_t^{\vec{b}}(f)(x,y)=$$

$$= \int_{R^n} \prod_{j=1}^{m} \left[(b_j(x) - (b_j)_Q) - (b_j(z) - (b_j)_Q)\right] \psi_t(y-z) f(z) dz$$

$$= \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_Q)_\sigma \int_{R^n} (b(z) - (b)_Q)_{\sigma^c} \psi_t(y-z) f(z) dz$$

$$= \prod_{j=1}^{m} (b_j(x) - (b_j)_Q) F_t(f)(y) + (-1)^m F_t(\prod_{j=1}^{m} (b_j - (b_j)_Q)) f)(y)$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_Q)_\sigma \int_{R^n} (b(z) - b(x))_{\sigma^c} \psi_t(y-z) f(z) dz$$

$$= \prod_{j=1}^{m} (b_j(x) - (b_j)_Q) F_t(f)(y) + (-1)^m F_t(\prod_{j=1}^{m} (b_j - (b_j)_Q)) f)(y)$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b(x) - (b)_Q)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x, y).$$

Thus, set $C_0 = S_\delta(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)$,

$$|S_\delta^{\vec{b}}(f)(x) - S_\delta(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)| \leq$$

$$\leq ||\chi_{\Gamma(x)} \prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q}) F_t(f)(x)||$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ||\chi_{\Gamma(x)} (b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||$$

$$+ ||\chi_{\Gamma(x)} F_t(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_1)(x)||$$

$$+ ||\chi_{\Gamma(x)} F_t(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x) - \chi_{\Gamma(x_0)} F_t(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)||$$

$$= I_1(x) + I_2(x) + I_3(x) + I_4(x).$$

For $I_1(x)$, for $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundness of S_δ from $L^p(R^n)$ to $L^q(R^n)$, by Hölder's inequality with exponent$1/q'_1 + \cdots + 1/q'_m + 1/q = 1$ and Lemma 1, we have

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_1(x)| dx =$$

$$= \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_Q) S_\delta(f)(x)| dx$$

$$\leq C \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \prod_{j=1}^m \left(\int_Q |(b_j(x) - (b_j)_Q)|^{q'_j} dx \right)^{1/q'_j} \left(\int_Q |S_\delta(f)(x)|^q dx \right)^{1/q}$$

$$\leq C \frac{|Q|^{m\beta/n+1/q'_1+\cdots q'_m}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \prod_{j=1}^m \frac{1}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b_1(x) - (b_1)_Q)|^{q'_j} dx \right)^{1/q'_j} \times$$

$$\times \left(\int_Q |f(x)|^p dx \right)^{1/p}$$

$$\leq C \frac{|Q|^{1+(\delta+m\beta)/n-1/p}}{|Q|^{1+(\delta+m\beta)/n-1/p}} ||\vec{b}||_{\dot{\Lambda}_\beta} ||f||_{L^p}$$

$$\leq C ||\vec{b}||_{\dot{\Lambda}_\beta} ||f||_{L^p}.$$

For $I_2(x)$, denoting $p = rt$, $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of S_δ from $L^r(R^n)$ to $L^s(R^n)$, by Hölder's inequality with $1/s' + 1/s = 1$, $1/t' + 1/t = 1$ and Lemma 1, we have

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_2(x)| dx =$$

$$= \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |(b(x) - b_Q)_\sigma S_\delta((b - b_Q)_{\sigma^c} f)(x)| dx$$

$$\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \left(\int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'} \times$$

$$\times \left(\int_Q S_\delta((b - b_Q)_{\sigma^c} f)(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \frac{1}{|Q|^{|\sigma|\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'}$$

$$\times \left(\int_Q |(b - b_Q)_{\sigma^c} f(x)|^r dx \right)^{1/r}$$

$$\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+m\beta/n-1/p}} \frac{1}{|Q|^{|\sigma|\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'}$$

$$\times \left(\int_Q |(b(x) - b_Q)_{\sigma^c}|^{rt'} dx \right)^{1/rt'} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}\frac{1}{|Q|^{|\sigma|\beta/n}}\left(\frac{1}{|Q|}\int_Q|(b(x)-b_Q)_\sigma|^{s'}dx\right)^{1/s'}$$

$$\times|Q|^{|\sigma^c|\beta/n+1/rt'}\frac{1}{|Q|^{|\sigma^c|\beta/n}}\left(\frac{1}{|Q|}\int_Q|(b(x)-b_Q)_{\sigma^c}|^{rt'}dx\right)^{1/rt'}||f||_{L^p}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|Q|^{m\beta/n+1/s'+1/rt'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}||\vec{b}_\sigma||_{\dot{\Lambda}_\beta}||\vec{b}_{\sigma^c}||_{\dot{\Lambda}_\beta}||f||_{L^p}$$

$$\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

For $I_3(x)$, for $1<r<s<n/\delta$, $1/s=1/r-\delta/n$, by the boundness of S_δ from $L^r(R^n)$ to $L^s(R^n)$, taking $1<r<p<\infty$, $p=rt$, $1/t_1+\cdots+1/t_m+1/t=1$, by Hölder's inequality and Lemma 1, we have

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q|I_3(x)|dx$$

$$\leq\frac{1}{|Q|^{(\delta+m\beta)/n-1/p}}\left(\frac{1}{|Q|}\int_{R^n}|S_\delta(\prod_{j=1}^m(b_j-(b_j)_Q))f\chi_B(x)|^sdx\right)^{1/s}$$

$$\leq\frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}}\left(\int_Q|(\prod_{j=1}^m(b_j-(b_j)_Q)f(x)|^rdx\right)^{1/r}$$

$$\leq\frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}}\prod_{j=1}^m\left(\int_Q|b_j(x)-(b_j)_Q|^{rt_j}dx\right)^{1/rt_j}\left(\int_Q|f(x)|^{rt}dx\right)^{1/rt}$$

$$\leq\frac{|Q|^{m\beta/n+1/rt_1+\cdots 1/rt_m}}{|Q|^{(\delta+m\beta)/n-1/p+1/s}}\prod_{j=1}^m\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q|b_j(x)-(b_j)_Q|^{rt_j}dx\right)^{1/rt_j}||f||_{L^p}$$

$$\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

For $I_4(x)$, similar to the proof of $C(x)$ in the case $m=1$, we get

$$I_4(x)\leq C\int_{(2Q)^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)||f(z)|dz.$$

Thus, we choose $1<p_j<\infty, j=1,\cdots,m$, $1/p_1+\cdots+1/p_m+1/p=1$, by Hölder's inequality and Lemma 1, 3, we have

$$I_4(x)\leq$$

$$\leq C\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)||f(z)|dz$$

$$\leq C\sum_{k=0}^\infty 2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(z)|^pdz\right)^{1/p}\times$$

$$\times \prod_{j=1}^{m} \left(\int_{2^{k+1}Q} |(b_j(z) - (b_j)_Q)|^{p'_j} dz \right)^{1/p'_j}$$

$$\leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p}$$

$$\times \prod_{j=1}^{m} \left[\int_{2^{k+1}Q} (|b_j(z) - (b_j)_{2^{k+1}Q}|^{p'_j} + |(b_j)_{2^{k+1}Q} - (b_j)_{2Q}|^{p'_j}) dz \right]^{1/p'_j}$$

$$\leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p}$$

$$\times \prod_{j=1}^{m} \left[|2^{k+1}Q|^{m\beta/n+1/p'} \frac{1}{|2^{k+1}Q|^{m\beta/n}} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b_j(z) - (b_j)_{2^{k+1}Q})|^{p'} dz \right)^{1/p'} \right.$$

$$\left. + |(b_j)_{2^{k+1}Q} - (b_j)_Q| |2^{k+1}Q|^{1/p'} \right]$$

$$\leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \times$$

$$\times \prod_{j=1}^{m} \left[|2^{k+1}Q|^{m\beta/n+1/p'} ||b_j||_{\dot{\wedge}_\beta} + k|2^{k+1}Q|^{m\beta/n+1/p'} ||b_j||_{\dot{\wedge}_\beta} \right] ||f||_{L^p}$$

$$\leq C \sum_{k=0}^{\infty} k 2^{k(-1/2+\delta+m\beta-n/p)} |Q|^{m\beta/n+1/p'-1+\delta/n} ||\vec{b}||_{\dot{\wedge}_\beta} ||f||_{L^p}$$

$$\leq C|Q|^{(\delta+m\beta)/n-1/p} |||\vec{b}||_{\dot{\wedge}_\beta} ||f||_{L^p}.$$

Thus,

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_4(x)| dx \leq$$

$$\leq C||\vec{b}||_{\dot{\wedge}_\beta} ||f||_{L^p} \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |Q|^{(\delta+m\beta)/n-1/p} dx \leq C||\vec{b}||_{\dot{\wedge}_\beta} ||f||_{L^p}.$$

□

Proof of Theorem 2. Fix a ball $B = B(0,l)$, there exists $\epsilon_0 \in \mathbf{Z}$ such that $2^{\epsilon_0 - 1} \leq l < 2^{\epsilon_0}$. We choose x_0 such that $2l < |x_0| < 3l$. It is only to prove that

$$2^{\epsilon_0(\alpha+n/q_2)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x|<2^{\epsilon_0}} |S_\delta^{\vec{b}}(f)(x) - S_\delta^{\vec{b}}(f_2)(x_0)|^{q_2} dx \right)^{1/q_2} \leq C||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.$$

We write, for $f_1 = f\chi_{4B_{\epsilon_0}}$ and $f_2 = f\chi_{R^n \setminus 4B_{\epsilon_0}}$, then

$$|S_\delta^{\vec{b}}(f)(x) - S_\delta^{\vec{b}}(f_2)(x_0)| \leq |S_\delta^{\vec{b}}(f_1)(x)| + |S_\delta^{\vec{b}}(f_2)(x) - S_\delta^{\vec{b}}(f_2)(x_0)|.$$

So

$$
\begin{aligned}
&2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|S_\delta^{\vec{b}}(f)(x)-S_\delta^{\vec{b}}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\\
&\le 2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|S_\delta^{\vec{b}}(f_1)(x)|^{q_2}dx\right)^{1/q_2}\\
&\quad+2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|S_\delta^{\vec{b}}(f_2)(x)-S_\delta^{\vec{b}}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\\
&= J_1+J_2.
\end{aligned}
$$

For J_1, by the (L^{q_1}, L^{q_2})-boundedness of $S_\delta^{\vec{b}}$ (see Lemma 4) and Lemma 2, we get

$$
\begin{aligned}
J_1 &\le C2^{\epsilon_0(\alpha+n/q_2)}2^{-\epsilon_0 n/q_2}\left(\int_{R^n}|f_1(x)|^{q_1}dx\right)^{1/q_1}\\
&\le C2^{\epsilon_0\alpha}||f\chi_{B_{\epsilon_0}}||_{L^{q_1}}\\
&\le C||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.
\end{aligned}
$$

For J_2, similar to the estimates of Theorem 1, set $1/v_1+\cdots+1/v_m+1/q_1=1$, by Hölder's inequality and recall that $-1/q_2<\alpha$, $1/q_2=1/q_1-(\delta+m\beta)/n$, we obtain

$$
\begin{aligned}
&|S_\delta^{\vec{b}}(f_2)(x)-S_\delta^{\vec{b}}(f_2)(x_0)|\le\\
&\le |S_\delta(\prod_{j=1}^m(b_j-(b_j)_B)(f_2)(x)-S_\delta(\prod_{j=1}^m(b_j-(b_j)_B)(f_2)(x_0)|\\
&\quad+|\prod_{j=1}^m(b_j(x)-(b_j)_B)||S_\delta(f_2)(x)-S_\delta(f_2)(x_0)|=W_1(x)+W_2(x).
\end{aligned}
$$

For $W_1(x)$, similar to the proof of $C(x)$ in Theorem 1, by the Minkowski's inequality, we have

$$
\begin{aligned}
W_1(x) &\le C\sum_{k=1}^\infty\int_{B_{\epsilon_0+k}}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_B)||f(z)|dz\\
&\le C\sum_{k=1}^\infty\int_{B_{\epsilon_0+k}}\frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}}|\prod_{j=1}^m(b_j(z)-(b_j)_B)||f(z)|dz\\
&\le C\sum_{k=1}^\infty\frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}}\prod_{j=1}^m\left(\int_{B_{\epsilon_0+k}}|b_j(z)-(b_j)_B|^{v_j}dy\right)^{1/v_j}\times\\
&\times\left(\int_{B_{\epsilon_0+k}}|f(z)^{q_1}dz\right)^{1/q_1}
\end{aligned}
$$

$$\leq C\sum_{k=1}^{\infty}\frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}}2^{(\epsilon_0+k)(m\beta+n/v_1+\cdots n/v_m)}$$

$$\times\prod_{j=1}^{m}\frac{1}{|B_{\epsilon_0+k}|^{m\beta/n}}\left(\frac{1}{|B_{\epsilon_0+k}|}\int_{B_{\epsilon_0+k}}|b_j(z)-(b_j)_B|^{v_j}dz\right)^{1/v_j}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$

$$\leq C\sum_{k=1}^{\infty}2^{k(m\beta+n-n/q_1-n+\delta-1/2)}2^{\epsilon_0(m\beta+n-n/q_1-n+\delta)}||\vec{b}||_{\dot{\Lambda}_\beta}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$

$$\leq C\sum_{k=1}^{\infty}2^{k(\delta+m\beta-n/q_1-\alpha-1/2)}2^{\epsilon_0(\delta+m\beta-n/q_1-\alpha)}||\vec{b}||_{\dot{\Lambda}_\beta}2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$

$$\leq C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

For $W_2(x)$, by Hölder's inequality with $1/q_1'+1/q_1=1$, we have,

$$W_2(x) \leq C|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}||x-z|^{-n+\delta}-|x_0-z|^{-n+\delta}||f(z)|dz$$

$$\leq C|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}|x-z|^{-n+\delta-1}|x-x_0||f(z)|dz$$

$$\leq C|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|\sum_{k=1}^{\infty}\frac{2^{\epsilon_0}}{2^{(\epsilon_0+k)(n+1-\delta)}}2^{(\epsilon_0+k)(n-n/q_1)}\left(\int_{B_{\epsilon_0+k}}|f(z)|^{q_1}dz\right)^{1/q_1}$$

$$\leq C|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|\sum_{k=1}^{\infty}2^{k(\delta-n-1-\alpha+n-n/q_1)}2^{\epsilon_0(\delta-n-\alpha+n-n/q_1)}\times$$

$$\times 2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$

$$\leq C2^{\epsilon_0(\delta-\alpha-n/q_1)}|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

Thus, by Hölder's inequality with $1/v_1+\cdots+1/v_m=1$, we have

$$2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|W_2(x)|^{q_2}dx\right)^{1/q_2}$$

$$\leq C2^{\epsilon_0(\alpha+n/q_2)}2^{\epsilon_0(\delta-\alpha-n/q_1)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|^{q_2}dx\right)^{1/q_2}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$$

$$\leq C2^{\epsilon_0(\alpha+n/q_2)}2^{\epsilon_0(\delta-\alpha-n/q_1)}2^{\epsilon_0(m\beta+n/q_2(1/v_1+\cdots+1/v_m))}$$

$$\times\left(\frac{1}{2^{\epsilon_0 n}}\frac{1}{|B|^{m\beta/n}}\prod_{j=1}^{m}\left(\frac{1}{|B|}\int_{|x|<2^{\epsilon_0}}|(b_j(x)-(b_j)_B)|^{q_2 v_j}dx\right)^{1/q_2 v_j}\right)||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$$

$$\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

Thus,

$$J_2 \leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.$$

□

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College of Mathematics,
Changsha University of Science and Technology,
Changsha 410077, P. R. of China
E-mail address: lanzheliu@163.com